Closed affine manifolds with partially hyperbolic linear holonomy (preliminary)

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Abstract

- We will try to show that closed manifolds of negative curvature do not admit complete special affine structures whose linear parts are partially hyperbolic in the dynamical sense.
- Furthermore, a closed complete special affine manifold is not P-Anosov for some parabolic group P with index J depending on its linear holonomy. (corrected after the talk)

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- Furthermore, a closed complete special affine manifold is not P-Anosov for some parabolic group P with index J depending on its linear holonomy. (corrected after the talk)
- We will present an attempt to show that closed affine manifolds cannot have partially hyperbolic linear holonomy (without negative curvature condition). However, the fundamental group is now hyperbolic by the condition, and so is the universal cover.
- Partially a joint work with Kapovich.

Affine manifolds

- Let $\mathbb{A}^n$ be a complete affine space. Let $\mathbf{Aff}(\mathbb{A}^n)$ denote the group of affine transformations of $\mathbb{A}^n$ whose elements are of form:
$$x \mapsto Ax + \mathbf{v}$$
for a vector $\mathbf{v} \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n, \mathbb{R})$.
- Let $\mathcal{L} : \mathbf{Aff}(\mathbb{A}^n) \to \mathrm{GL}(n, \mathbb{R})$ denote map sending elements of $\mathbf{Aff}(\mathbb{A}^n)$ to its linear part in $\mathrm{GL}(n, \mathbb{R})$.

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- An affine n-manifold is an n-manifold equipped with an atlas of charts to $\mathbb{A}^n$ with affine transition maps.
  - There is a homomophism $\rho' : \pi_1(M) \to \Gamma \subset \textbf{Aff}(\mathbb{A}^n)$ called a *holonomy homomorphism*.
  - There is an immersion $\textbf{dev} : \tilde{M} \to \mathbb{A}^n$, called a *developing map*, so that

$$\textbf{dev} \circ \gamma = \rho'(\gamma) \circ \textbf{dev} \text{ for each deck transformation } \gamma \in \pi_1(M).$$

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- An affine n-manifold is *special* if $\mathcal{L}(\Gamma) \subset \mathrm{SL}_{\pm}(n, \mathbb{R})$.
- A *complete affine n-manifold* is an n-manifold M of form $\mathbb{A}^n/\Gamma$. **dev** is a diffeomorphism if and only if the affine n-manifold M is complete.

- Denote by $\tilde{M}$ the universal cover of M with the covering map p_M with the deck transformation group $\pi_1(M)$.
- Let $\pi_M : \mathbf{U}M \to M$ denote the fibration and $\tilde{\pi}_M : \mathbf{U}\tilde{M} \to \tilde{M}$ the induced fibration.
- There is a covering $\mathbf{U}p_M : \mathbf{U}\tilde{M} \to \mathbf{U}M$ from the unit tangent bundle $\mathbf{U}\tilde{M}$ of $\tilde{M}$. The deck transformation group of $\mathbf{U}p_M$ is $\pi_1(M)$.

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- (Affine bundle): For an affine representation $\rho' : \pi_1(M) \to \mathbf{Aff}(\mathbb{A}^n)$, define $\mathbb{A}^n_{\rho'} := (\mathbf{U}\tilde{M} \times \mathbb{A}^n)/\pi_1(M)$ with the diagonal action.
- (Vector bundle): We define $\mathbb{R}^n_\rho := (\mathbf{U}\tilde{M} \times \mathbb{R}^n)/\pi_1(M)$ for $\rho = \mathcal{L} \circ \rho'$.

Flows lifted to the bundle

- $\hat{\phi}_t : \mathbf{U}M \to \mathbf{U}M$ denote the geodesic flow. and let $\phi_t : \mathbf{U}\tilde{M} \to \mathbf{U}\tilde{M}$ denote the flow lifted from $\hat{\phi}_t$.
- There exists a flow $\Phi_t, t \in \mathbb{R}$, on $\mathbb{A}_{\rho'}$ acting as the geodesic flow ϕ_t on $\mathbf{U}M$ and acting trivially on $\mathbb{A}^n$ lifted.
- Also, there is a flow $D\Phi_t, t \in \mathbb{R}$, on $\mathbb{R}^n_\rho$ taking the linear part of Φ_t fiberwise acting as the geodesic flow on $\mathbf{U}M$ and acting trivially on $\mathbb{R}^n$ lifted.

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We have fiber-wise norm $\|\cdot\|_{\mathbb{A}^n_{\rho'}}$ on $\mathbb{A}^n_{\rho'}$, and a norm $\|\cdot\|_{\mathbb{R}^n_\rho}$ on $\mathbb{R}^n_\rho$ using partition of unity.

Partial hyperbolicity in the bundle sense.

- A representation $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbb{R})$ is *partially hyperbolic in a bundle sense* if the following hold:
  - (i) There exist nontrivial C^0-subbundles $\mathbb{V}_+, \mathbb{V}_0,$ and $\mathbb{V}_-$ in $\mathbb{R}^n_\rho$ invariant under the flow $D\Phi_t$.
  - (ii) $\mathbb{V}_+, \mathbb{V}_0$ and $\mathbb{V}_-$ are independent and their bundle sum equals $\mathbb{V}$.

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  (ii) $\mathbb{V}_+, \mathbb{V}_0$ and $\mathbb{V}_-$ are independent and their bundle sum equals $\mathbb{V}$.
  (iii) For any fiber-wise metric on $\mathbb{R}^n_\rho$ over $\mathbf{U}M$, the lifted action of $D\Phi_t$ on $\mathbb{V}_+$ (resp. $\mathbb{V}_-$) is dilating (resp. contracting): i.e., there are coefficients $A > 0$, $a > 0$, $A' > 0$:
    1. $||D\Phi_{-t}(\mathbf{v})||_{\mathbb{R}^n_\rho, \Phi_{-t}(m)} \leq A \exp(-at) \, ||\mathbf{v}||_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_+(m)$ as $t \to \infty$.
    2. $||D\Phi_t(\mathbf{v})||_{\mathbb{R}^n_\rho, \Phi_t(m)} \leq A \exp(-at) \, ||\mathbf{v}||_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_-(m))$ as $t \to \infty$.

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    2. $||D\Phi_t(\mathbf{v})||_{\mathbb{R}^n_\rho, \Phi_t(m)} \le A\exp(-at)\, ||\mathbf{v}||_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_-(m))$ as $t \to \infty$.
    3. (A dominance property)

$$\frac{||D\Phi_t(\mathbf{w})||_{\mathbb{R}^n_\rho, \phi_t(m)}}{||D\Phi_t(\mathbf{v})||_{\mathbb{R}^n_\rho, \phi_t(m)}} \le A' \exp(-a' t) \frac{||\mathbf{w}||_{\mathbb{R}^n_\rho, m}}{||\mathbf{v}||_{\mathbb{R}^n_\rho, m}} \begin{cases} \text{for } \mathbf{v} \in \mathbb{V}_+(m), \mathbf{w} \in \mathbb{V}_0(m) \text{ as } t \to \infty, \\ \text{or for } \mathbf{v} \in \mathbb{V}_0(m), \mathbf{w} \in \mathbb{V}_-(m) \text{ as } t \to \infty. \end{cases} \quad (1)$$

- Here $\dim \mathbb{V}_+$ is a *partial hyperbolicity index* of ρ.
- We assume that $\dim \mathbb{V}_+ = \dim \mathbb{V}_- \geq 1$. Also, $\mathbb{V}_0$ is said to be the *neutral subbundle* of $\mathbb{V}$. Often we will be in cases $\dim \mathbb{V}_0 > 0$.
- A related dynamical system is “partially hyperbolic system" as in Bonatti, Diaz, Viana [1] or Crovisier and Potrie [2]. (Related to Bochi-Sambarino and see Definition 1.5 of [2].)

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Theorem 1 (Negative curvature case)

Let M be a closed complete special affine n-manifold. Suppose that M admits a negatively curved Riemannian metric. Then the linear part of a holonomy homomorphism ρ is not an partially hyperbolic representation in a bundle sense.

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- Closed complete affine *n*-manifolds have virtually solvable groups. (Auslander conjecture: Fried-Goldman 83 ($n = 3$), Abels-Margulis-Soifer for $n \leq 6$)
- Linear holonomy in SO(p, q) implies the virtually solvable fundamental group. (Goldman-Kamishima 84 ($p = n - 1$), Abels-Margulis-Soifer other cases.)

Corollaries for P-Anosov.

We list the singular values of g $a_1(g), \dots, a_n(g)$ in a non-increasing order.

Corollary 1 (P-Anosov(corrected after the talk))

Let M be a closed complete special affine n-manifold with a fundamental group $\pi_1(M)$. *Suppose that M admits a negatively curved Riemannian metric. Let* $\rho : \pi_1(M) \to \mathrm{SL}(n, \mathbb{R})$ *is a linear part of the holonomy homomorphism. Then the linear part of the holonomy homomorphism* ρ *is not P-Anosov for any parabolic group P of index* $\leq J - 1$ *for* $J = \min\{i | a_i(g) = 1, g \in \rho(\pi_1(M))\}$.

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Corollary 2 (Special Lie groups)

Let M be a closed complete special affine n-manifold with a fundamental group $\pi_1(M)$ with linear holonomy in $\mathrm{SO}(k, n-k)$ for each integer k, $0 \leq k \leq n$ or in $\mathrm{SP}(m, \mathbb{R})$ for $n = 2m$. Suppose that M admits a negatively curved Riemannian metric. Then the linear part of the holonomy homomorphism ρ is not P-Anosov for any parabolic group P of $\mathrm{SO}(k, n-k)$.

Related work

Existence of actions

- Margulis, Drumm
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Nonexistence of actions

- Danciger and Zhang [3] showed that when M is a surface, there is no proper action on $\mathbb{R}^n$ by an affine representation with linear part in a Hitchin component.
- Ghosh [4] obtained some generalization to hyperbolic groups with affine representations with Anosov representation.
- Tsouvalas: some cases must virtually be free or be a surface group.

However, these work do not have our dimension conditions.

Developing sections

- We begin the proof Theorem 1.
- There is a projection $\tilde{\Pi}_{\mathbb{A}^n} : \mathbf{U}\tilde{M} \times \mathbb{A}^n \to \mathbb{A}^n$ inducing a bundle map

$$\Pi_{\mathbb{A}^n} : \mathbb{A}^n_{\rho'} := (\mathbf{U}\tilde{M} \times \mathbb{A}^n)/\pi_1(M) \to \mathbb{A}^n/\Gamma$$

  and $\tilde{\pi}_{\mathbf{U}M} : \mathbf{U}\tilde{M} \times \mathbb{A}^n \to \mathbf{U}\tilde{M}$ inducing a bundle map

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- We define a section $\tilde{s} : \mathbf{U}\tilde{M} \to \mathbf{U}\tilde{M} \times \mathbb{A}^n$ where

$$\tilde{s}((x, \vec{v})) = ((x, \vec{v}), \mathbf{dev}(x)), x \in \tilde{M}. \quad (2)$$

- $\tilde{s}$ induces a section $s : \mathbf{U}M \to \mathbb{A}^n_{\rho'}$, called the *developing section*. (See Goldman [5])

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- $\tilde{s}$ induces a section $s : \mathbf{U}M \to \mathbb{A}^n_{\rho'}$, called the *developing section*. (See Goldman [5])
- Since $M = \mathbb{A}^n/\Gamma$ has a complete affine structure, **dev** induces the map

$$\mathcal{I} := \Pi_{\mathbb{A}^n} \circ s : \mathbf{U}M \to \mathbb{A}^n/\Gamma.$$

Neutralizing the sections

Proposition 2

There is a section s_∞ homotopic to the developing section s in the C^0-topology with the following conditions:

- $\nabla_\phi s_\infty$ *is in* $V_0(x)$ *for each* $x \in \mathbf{U}M$.
- $\mathcal{I}_\infty := \Pi_{\mathbb{A}^n} \circ s_\infty$ *is onto.*
- $d_{\mathbb{A}^n_{\rho'}}(s(x), s_\infty(x))$ *is uniformly bounded for* $x \in \mathbf{U}M$.

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Proof.

We project to flat connections $\nabla^+, \nabla^-, \nabla^0$ respectively on $\mathbb{V}_+, \mathbb{V}_0, \mathbb{V}_-$ respectively.
We define $s_\infty := s + \int_0^\infty (D\Phi_t)_*(\nabla_\phi^- s)dt - \int_0^\infty (D\Phi_{-t})_*(\nabla_\phi^+ s)dt$. Then it is homotopic to s since we can replace ∞ by T, $T > 0$ and let $T \to \infty$. (See Section 8 of Goldman-Labourie-Margulis [6].)
Since M is compact and the norms of the integrand decreases exponentially, the integral is uniformly bounded above.

□

Corollary 3

$\tilde{\mathcal{I}_\infty} := \tilde{\Pi}_{\mathbb{A}^n} \circ \tilde{s}_\infty$ *restricted to each oriented geodesic* $\vec{l}$ *on* $\mathbf{U}\tilde{M}$ *lies on a neutral affine subspace parallel to* $V_0(\vec{l})$*.*

- Let $l_y := \{\phi_t(y) | t \geq 0\}$ for $y \in K$.
- The image $\tilde{\mathcal{I}_\infty}(l_y)$ is in a neutral affine subspace denoted it by A_y^0 or $A_{l_y}^0$.
- We choose l_y so that an infinite-order deck-transformation γ acts on the axis containing l_y.

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- We choose l_y so that an infinite-order deck-transformation γ acts on the axis containing l_y.
- $$\tilde{s}_\infty \circ \gamma = \rho'(\gamma) \circ \tilde{s}_\infty, \gamma \in \pi_1(M) \text{ implies} \tag{3}$$
  $$\rho'(\gamma)(A^0_y) = A^0_{\gamma(y)} = \rho'(\gamma)(A^0_{l_y}) = A^0_{\gamma(l_z)}. \tag{4}$$

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  In particular, γ acts on the axis containing l_y and on A^o_y.
- Finally since s_∞ is continuous, $x \mapsto A^0_x$ is a continuous function. Hence,
  $$A^0_{z_i} \to A^0_z \text{ if } z_i \to z \in \mathbf{U}\tilde{M}. \tag{5}$$

- Denote by $\mathbb{V}_\pm(y)$ be the vector subspace parallel to the lift of $\mathbb{V}_\pm$ at y. The C^0-decomposition property also implies

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- Let $p \in \partial_\infty \tilde{M}$ be a point of the Gromov boundary of $\tilde{M}$. We define $\mathcal{R}_p$ as the set

$$\{\vec{u} \in \mathbf{U}_x\tilde{M} | \vec{u} \text{ is tangent to a complete geodesic ending at } p\}.$$

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$\tilde{\mathcal{I}_\infty}(\mathcal{R}_p)$ equals $\mathbb{A}^n$.

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Definition 4

A_p^{0-}: the affine subspace containing A_p^0 and all points in directions of $\mathbb{V}^-(p)$ from points of A_p^0.

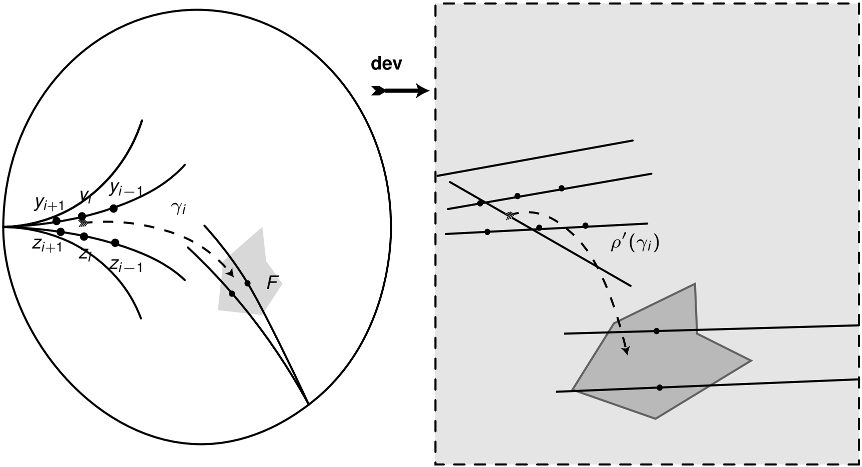

Figure: The proof of Theorem 1. Here γ_i is multiplied by an element to make the figure look better.

- We can choose two leaves l_y and l_z in $\mathcal{R}_p$ $y, z \in \mathbf{U}\tilde{M}$, so that $\tilde{\mathcal{I}_\infty}(l_y)$ and $\tilde{\mathcal{I}_\infty}(l_z)$ are in distinct subspaces $A^{0-}_{l_y}$ and $A^{0-}_{l_z}$ by Proposition 3.
- The following contradiction proves Theorem 1.

Proposition 5

There are no two leaves l_y and l_z in $\mathcal{R}_p$ for $y, z \in \mathbf{U}\tilde{M}$ so that so that $\tilde{\mathcal{I}_\infty}(l_y)$ and $\tilde{\mathcal{I}_\infty}(l_z)$ are in distinct subspaces $A^{0-}_{l_y}$ and $A^{0-}_{l_z}$

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- The following contradiction proves Theorem 1.

Proposition 5

There are no two leaves l_y and l_z in $\mathcal{R}_p$ for $y, z \in \mathbf{U}\tilde{M}$ so that so that $\tilde{\mathcal{I}_\infty}(l_y)$ and $\tilde{\mathcal{I}_\infty}(l_z)$ are in distinct subspaces $A^{0-}_{l_y}$ and $A^{0-}_{l_z}$

Proof begins

Suppose not. Also, under $\tilde{\pi}_M$, l_y and l_z respectively go to geodesics ending at p. We assume that an infinite order deck transformation γ acts on the axis containing l_y and fixes p.

$A^{0-}_{\phi_t(y)}$ is a fixed affine subspace independent of t, and $\rho'(\gamma)$ acts on $A^{0-}_{\phi_t(y)}$.

Pulling-back argument

- $A^0_{\phi_t(z)}$ contains I_z and $\mathbb{V}^-(\phi_t(z))$ is independent of t since they are parallel under the flat connection.

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- $A^0_{\phi_t(z)}$ contains l_z and $\mathbb{V}^-(\phi_t(z))$ is independent of t since they are parallel under the flat connection.
- Choose $y_i \in l_y$ so that $y_i = \phi_{t_i}(y)$, and $z_i \in l_z$ so that $z_i = \phi_{t_i}(z)$ where $t_i \to \infty$ as $i \to \infty$. Denote by

$$y_i' := \tilde{\mathcal{I}_\infty}(y_i) \text{ and } z_i' := \tilde{\mathcal{I}_\infty}(z_i) \text{ in } \mathbb{A}^n.$$

- Since $\langle\gamma\rangle$ acts on the axis containing l_y, $\gamma_i(y_i)$ is in a compact subset F of $\mathbf{U}\tilde{M}$ for a sequence $\gamma_i = \gamma^{-j_i}$ with j_i going to infinity. $\rho'(\gamma_i)(y_i')$ is in a compact subset of $\mathbb{A}^n$ for $y_i' = \tilde{\Pi}_M(y_i)$.

Pulling-back argument

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- Choose a subsequence so that

$$\rho'(\gamma_i)(y_i') \to y'_\infty \text{ for a point } y'_\infty \in \mathbb{A}^n. \tag{7}$$

- Since s_∞ is continuous by Proposition 2, we obtain

$$d_{\mathbb{A}^n/\Gamma}(\tilde{\mathcal{I}_\infty}(y_i), \tilde{\mathcal{I}_\infty}(z_i)) \to 0. \tag{8}$$

- Since γ_i is an isometry of $d_{\mathbb{A}^n}$,

$$d_{\mathbb{A}^n}(\rho'(\gamma_i)(y_i'), \rho'(\gamma_i)(z_i')) \to 0 \tag{9}$$

as $i \to \infty$.

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  as $i \to \infty$.
- We claim that $A^{0-}_{l_z}$ is affinely parallel to $A^{0-}_{l_y}$: Otherwise, we can show $\rho(\gamma_i)(A^{0-}_{l_z}) = A^{0-}_{\gamma_i(z_i)}$ does not converge to $A^0_{l_y}$. But $d_M(\gamma_i(z_i), \gamma_i(y_i)) \to 0$.
- Also the sequence of the Hausdorff distance between

$$A^{0-}_{\gamma_i(z_i)} = \rho'(\gamma_i)(A^{0-}_{l_z}) \text{ and } A^{0-}_{\gamma_i(y_i)} = \rho'(\gamma_i)(A^{0-}_{l_y})$$

  is going to 0.

- Let $\vec{v}$ denote the vector in the direction of $\mathbb{V}_+(y_i)$ going from y_i to $A^{0-}_{l_z}$, independent of y_i. Then for the linear part A_{γ_i} of the affine transformation γ_i,

$$\|v_i' := A_{\gamma_i}(\vec{v})\|_n^E \to \infty.$$

- Hence affine subspaces

$$A^{0-}_{\gamma_i(z_i)} = \rho'(\gamma_i)(A^{0-}_{l_z}) \text{ and } A^{0-}_{\gamma_i(y_i)} = \rho'(\gamma_i)(A^{0-}_{l_y})$$

are not getting close to each other. This is a contradiction to the third paragraph above.

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are not getting close to each other. This is a contradiction to the third paragraph above.

- See following diagram as a proof.

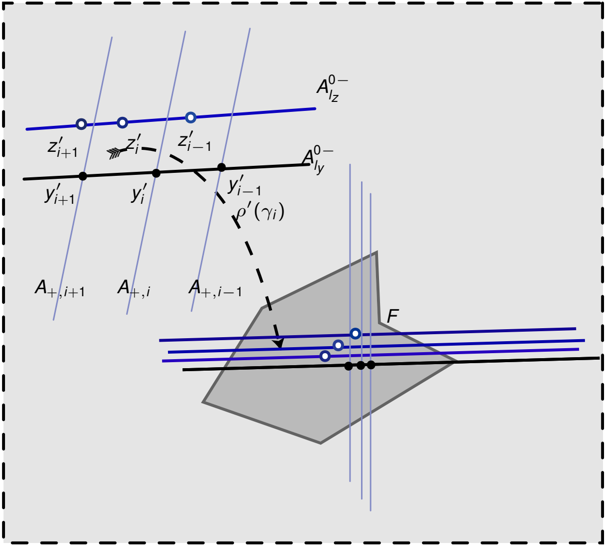

Figure: The proof of Theorem 1

II: P-Anosov corollaries

P-Anosov property

We can characterize the P-Anosov property of the linear holonomy group in $SL(n, \mathbb{R})$ in a few different way:

- Guichard-Weinhard: Every point of the limit set Λ is attached a flag associated with a parabolic subgroup P. There is a flow action where the tangent spaces of the flag in the flag space is exponentially decreasing or increasing.

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- Kapovich-Leeb-Porti: the singular-value direction vectors $(\!(a_1(g), a_2(g), \dots, a_n(g))\!)$ are bounded away from the union of faces not containing a side τ of the Weyl chamber Δ.
- Bochi-Sambarino: There exists k such that $\log \left| \frac{a_k(g)}{a_{k+1}(g)} \right| \to \infty$ uniformly for $g \in \Gamma$.
- Linear bundle dominance condition: The domination part of the partial hyperbolicity. (Bochi-Gourmelon and Kapovich-Leeb-Porti)

Hirsch-Kostant-Sullivan condition

Theorem 6 (HKS)

Let M be a complete affine manifold. Let ρ be the linear part of the affine holonomy group ρ'. Then $\rho(g)$ has an eigenvalue equal to 1 for each $g \in \Gamma$.

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Note

The following is incorrect as pointed out by Danciger after the talk

Theorem 7

Suppose ρ is semi-simple. Then there is an index i for $1 \leq i < n/2$ so that the following holds for singular values:

$$a_i(g) = a_{i+1}(g) = \cdots = a_{n-i+1}(g) = 1 \text{ for every } g \in \rho(\pi_1(M)).$$

- From the equivalences, we obtain the linear bundle dominance condition.
- Since the neutral bundle $\mathbb{V}_0$ contains the subspaces corresponding to singular values 1, we obtain partially hyperbolic decomposition.
- This proves Corollary 1 by Theorem 1.
- For Proof of Corollary 2: When ρ has images in the specified groups in the premises, the singular values satisfies the same conditions.

III: Generalization without negative curvature conditions

- Assume that $\tilde{M}$ is Gromov hyperbolic.
- A *complete isometric geodesic* in $\tilde{M}$ is a geodesic that is an isometry of $\mathbb{R}$ into $\tilde{M}$ equipped with a Riemannian metric. A *complete isometric geodesic* in M is a geodesic that lifts to a complete isometric geodesic in $\tilde{M}$.

III: Generalization without negative curvature conditions

- Assume that $\tilde{M}$ is Gromov hyperbolic.
- A *complete isometric geodesic* in $\tilde{M}$ is a geodesic that is an isometry of $\mathbb{R}$ into $\tilde{M}$ equipped with a Riemannian metric. A *complete isometric geodesic* in M is a geodesic that lifts to a complete isometric geodesic in $\tilde{M}$.
- We consider the subset of $\mathbf{U}M$ where complete isometric geodesics pass. We denote this set by $\mathbf{UC}M$, and call it the *complete-isometric-geodesic unit-tangent bundle*.
- The inverse image in $\mathbf{U}\tilde{M}$ is denoted by $\mathbf{UC}\tilde{M}$. Clearly, $\mathbf{UC}M$ is compact and $\mathbf{UC}\tilde{M}$ is locally compact. However, $\tilde{\pi}_M(\mathbf{UC}\tilde{M})$ may be a proper subset of $\tilde{M}$.
- Now we define partial hyperbolicity over $\mathbf{UC}M$ only.

Generalization of Theorem 1

Theorem 8

Let M be a closed complete special affine n-manifold. Then the linear part of a holonomy homomorphism ρ is not a partially hyperbolic representation in a bundle sense.

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Let M be a closed complete special affine n-manifold. Then the linear part of a holonomy homomorphism ρ is not a partially hyperbolic representation in a bundle sense.

- Partial hyperbolicity $\longrightarrow$ P-Anosov for $k = \dim \mathbb{V}_+$.
- Now, by Kapovich-Lee-Porti, $\pi_1(M)$ is hyperbolic.
- Hence, $\tilde{M}$ is Gromov hyperbolic by Svarc-Milnor.

Let p be a point of the Gromv boundary $\partial\tilde{M}$. Let $\mathcal{R}_p$ denote the union of complete isometric geodesics in $\mathbf{UC}\tilde{M}$ mapping to complete isometric geodesic in $\tilde{M}$ ending at p.

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Proposition 9

Let M be a closed manifold with a Riemannian metric. Suppose that $\pi_1(M)$ is hyperbolic. Let $p \in \partial\tilde{M}$. Then $\pi_{\tilde{M}}(\mathcal{R}_p)$ is C-dense in $\tilde{M}$.

Proposition 10 (Modification)

There is a section s_∞ homotopic to the developing section s in the C^0-topology with the following conditions:

- $\nabla_\phi s_\infty$ *is in* $\mathbb{V}_0(x)$ *for each* $x \in \mathbf{UC}M$.
- $d_{\mathbb{A}^n_{\rho'}}(s(x), s_\infty(x))$ *is uniformly bounded for every* $x \in \mathbf{UC}M$.
- $d_{\mathbb{A}^n}(\tilde{\mathcal{I}}(x), \tilde{\mathcal{I}_\infty}(x))$ *is uniformly bounded for* $x \in \mathbf{UC}\tilde{M}$.
- $\tilde{\mathcal{I}_\infty} : \mathbf{UC}\tilde{M} \to \mathbb{A}^n$ *is properly homotopic to* $\tilde{\mathcal{I}}$ *and is coarsely Lipschitz.*

Now, the proof of Theorem 8 proceeds similar to that of Theorem 1. However, we need some rough geometry ideas.

Theorem 11 (Choi-Kapovich)

Suppose that M is a closed complete affine manifold covered by an affine space $\tilde{M} = \mathbb{A}^n$ with the Riemannain metric d_M induced from that of M. Let L be an affine subspace of lower-dimension of $\tilde{M}$. Then $\tilde{M}$ is not a C-neighborhood $N_C(L)$ of L.

Proof.

Follows from the following two theorems. □

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Proof.

Follows from the following two theorems. □

Proposition 12 (Choi-Kapovich)

Let M and L be as above. Then L with induced path-metric d_L is uniformly properly embedded in $\tilde{M} = \mathbb{A}^n$.

Proof.

Just need to show if two points are of bounded distance under d_M, the path-distance in L cannot go to infinity. Here, we may asume one point is in a fundamental domain using deck transformations. □

Theorem 13

Let M and L be as above. Then L is uniformly contractible with respect to the path metric on L induced from d_M.

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Proof.

Any sphere map $f : S^i \to L$ with a d_M-diameter C may be moved by a deck transformation γ to a one passing a fundamental domain F of $\mathbb{A}^n$. Hence, a Euclidean ball B_R of some radius contains the image of $\gamma \circ f$. Here R depends only on C. Now, $B_R \subset B^M_{R'}$ for a d_M-ball $B^M_{R'}$ for a radius R' depending only on R. Hence, f is homotopic to a point inside $\gamma^{-1}(B^M_{R'})$ for R' depending only on C. □

Recall $H^n_C(X) := \varinjlim H^n(X, X-K)$ for K a compact subset of X. For $X = R^n$, $H^n_C(X) = \mathbb{Z}$.

Theorem 14 (Kapovich)

Let X be an open n-manifold that is a contractible δ-hyperbolic complete Riemannian metric space with the path metric d_X. Let U be a uniformly properly embedded open cell with the induced path-metric so that U is uniformly contractible and coarsely equivalent to X. Then U must have the topological dimension n.

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Proof.

There is an inclusion map $f : U \to X$ and its rough inverse map $g : X \to U$. We may assume that both are continuous. Then $f \circ g$ is homotopic to identity by a bounded continuous homotopy. Then $g_* \circ f_* : H^n_C(X) \to H^n_C(X)$ is an isomorphism. Since $H^n_C(U)$ has to be nonzero, $\dim U = \dim X$. □

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