Optimal liquidation in target zone models and Neumann problem of Backward SPDE with singular terminal condition

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Outlines

1. Introduction of optimal liquidation in target zone model
2. Hamilton Jacobi Bellmann (HJB) equation
3. Neumann problem of semilinear BSPDEs
4. Verification theorem and feedback control

Outlines

Optimal liquidation in target zone model

We study the optimal buying and selling strategies when the price of an asset is in a target zone.

- Price is modelled by a diffusion process which is reflected at one barrier or two.
- Reflected stochastic differential equation with a pair of solutions $(y_\cdot, L_\cdot)$

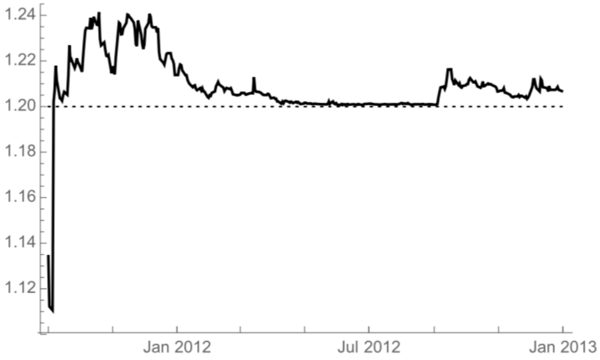
$$\begin{cases} y_r^{0,y} = y + \int_0^r \beta_s(y_s^{0,y})\, ds + \int_0^r \sigma_s(y_s^{0,y})\, dW_s + L_r, \quad r \in [0,T], \\ y_r \geq a, \text{a.s. for all } r \in [0,T]. \\ \int_0^T (y_s^{0,a} - a)\, dL_s = 0, \quad \text{(Skorohod condition)} \end{cases}$$

- The process $L_\cdot$ is endogenous. It is a nondecreasing process with $L_0 = 0$.

Refer to: Andrey Pilipenko, An Introduction to Stochastic Differential Equations with Reflection, Potsdam University Press, 2014

EUR/CHF

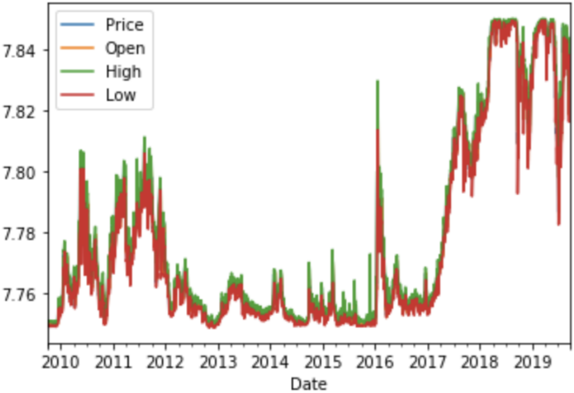
EUR/CHF exchange rate from Sep. 1, 2011 through Dec. 31, 2012



- On Sept. 6, 2011, the Swiss National Bank announced that it would enforce a minimum exchange rate of 1.20 EUR/CHF.

USD/HKD

USD/HKD exchange rate from Sep. 2009- Sep, 2019



- The prices are in $[7.75, 7.85]$

Optimal liquidation in target zone model (Continue)

- An agent try to close a position of x shares of asset before the terminal time T.

$$\begin{cases} x_r^{0,x} = x - \int_0^r \xi_s \, ds - \int_0^r \int_{\mathcal{Z}} \rho_s(z) \, \pi(dz, ds), \quad r \in [0, T], \\ x_T^{0,x} = 0 \end{cases}$$

  - $\{\xi_s, \ s \in [0, T]\}$: trading continuously with rate ξ, such as high frequency trading
  - $\{\rho_s, \ s \in [0, T]\}$: transact large blocks of shares at discrete time. e.g, trading in the dark pool
  - π Poisson random measure: Dark pool executions

Optimal Liquidation in target zone model (Continue)

- Assume that the transactions (HFT and dark pool) of agent has no effect on the price
- Then the overall liquidity costs entailed by the liquidation strategy can be written as

$$\begin{aligned} J(x,y;\xi,\rho) =& E\Big[\int_0^T \left(\eta_s(y_s^{0,y})|\xi_s|^q + \lambda_s(y_s^{0,y})|x_s^{0,x,\xi,\rho}|^q\right)\, ds \\ &+ \int_0^T \int_{\mathcal{Z}} \gamma_s(y_s^{0,y},z)|\rho_s(z)|^q\, \mu(dz)ds\Big] \end{aligned}$$

where $q \geq 1$.

Literature (selected)

- P. R. Krugman, Target zones and exchange rate dynamics, Q. J. Econ., 106 (1991)
- G. Bertola and R. J. Caballero, Target zones and realignments, Am. Econ. Rev., (1992)
- E. Neuman and A. Schied, Optimal portfolio liquidation in target zone models and catalytic superprocesses, Finance Stoch. (2016)
  - trading only happen on the barrier
  - no terminal constraint
  - price process is Markovian
  - ······
- P. Graewe, U. Horst, and J. Qiu, A non-markovian liquidation problem and backward SPDEs with singular terminal conditions, SIAM J. Control Optim.(2015)
- E. Bayraktar and J. Qiu, Controlled reflected sdes and neumann problem for backward spdes, Ann. Appl. Probab., (2018).

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Optimal Liquidation in target zone

- $(\Omega, \bar{\mathscr{F}}, (\bar{\mathscr{F}}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space
- two independent Wiener processes W and B.
- an point process $\tilde{J}$ on a non-empty Borel set $\mathcal{Z} \subset \mathbb{R}^l$ with finite characteristic measure $\mu(dz)$

$$\begin{cases} x_r^{0,x,\xi,\rho} = x - \int_0^r \xi_s \, ds - \int_0^r \int_{\mathcal{Z}} \rho_s(z) \, \pi(dz, ds), \quad r \in [0,T], \\ x_T^{0,x,\xi,\rho} = 0, \\ \quad y_r^{0,y} = y + \int_0^r \beta_s(y_s^{0,y}) \, ds + \int_0^r \sigma_s(y_s^{0,y}) \, dW_s + \int_0^r \bar{\sigma}_s(y_s^{0,y}) \, dB_s + L_r, \\ \quad\quad y_r \geq a, \text{a.s. for all } r \in [0,T]. \\ \quad \int_0^T (y_s^{0,a} - a) \, dL_s = 0, \quad \text{(Skorohod condition)} \end{cases}$$

Dynamic System

To use the dynamic program principle, we consider the dynamic system, for $t \in [0, T]$,

$$
\begin{cases}
x_r^{t,x;\xi,\rho} = x - \int_t^r \xi_s \, ds - \int_t^r \int_{\mathcal{Z}} \rho_s(z) \, \pi(dz, ds), \quad r \in [t, T], \\
x_T^{t,x;\xi,\rho} = 0, \\
\quad y_r^{t,y} = y + \int_t^r \beta_s(y_s^{t,y}) \, ds + \int_t^r \sigma_s(y_s^{t,y}) \, dW_s + \int_t^r \bar{\sigma}_s(y_s^{t,y}) \, dB_s + L_r, \\
\quad y_r \geq a, \text{a.s. for all } r \in [t, T]. \\
\quad \int_t^T (y_s^{0,a} - a) \, dL_s = 0, \quad \text{(Skorohod condition)}
\end{cases}
$$

- Denote by $(\mathscr{F}_t)_{t \in [0,T]}$ the augmented filtration generated by W.
- $\beta, \sigma, \bar{\sigma}$ are all adapted to $(\mathscr{F}_t)_{t \in [0,T]}$.

Value function

Define the dynamical cost function

$$J_t(x,y;\xi,\rho) = E\left[\int_t^T \left(\eta_s(y_s^{t,y})|\xi_s|^q + \lambda_s(y_s^{t,y})|x_s^{t,x;\xi,\rho}|^q\right)\,ds + \int_t^T \int_{\mathcal{Z}} \gamma_s(y_s^{t,y},z)|\rho_s(z)|^q\,\mu(dz)ds \;\middle|\bar{\mathscr{F}}_t\right], \quad t\in[0,T],$$

where the coefficients $\eta_s(y)$, $\lambda_s(y)$ and $\gamma_s(y,z)$ are $\mathscr{F}$-adapted.

The value function is given by

$$V_t(x,y) = \operatorname*{ess\,inf}_{(\xi,\rho)\in\mathscr{A}} J_t(x,y;\xi,\rho) \quad t\in[0,T),$$

where $\mathscr{A}$ is the set of admissible controls.

Admissible control

An *admissible controls* is $(\xi,\rho)\in\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T;\mathbb{R})\times\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T;L^q(\mathcal{Z}))$ $(q\in(1,\infty))$ that satisfy almost surely the *terminal state constraint*

$$x_T^{t,x;\xi,\rho}=0$$

Lemma

Given any admissible control pair $(\xi,\rho)\in\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T)\times\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T;L^q(\mathcal{Z}))$, we may find a corresponding admissible control pair $(\hat{\xi},\hat{\rho})\in\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T)\times\mathcal{L}^q_{\bar{\mathscr{F}}}(0,T;L^q(\mathcal{Z}))$ satisfying:

(i) the cost associated to $(\hat{\xi},\hat{\rho})$ is no more than that of (ξ,ρ);

(ii) the corresponding state process $x^{0,x;\hat{\xi},\hat{\rho}}$ is a.s. monotone;

(iii) it holds that for each $t\in[0,T]$,

$$E\left[\sup_{s\in[t,T]}|x_s^{0,x;\hat{\xi},\hat{\rho}}|^q\middle|\bar{\mathscr{F}}_t\right]=|x_t^{0,x;\hat{\xi},\hat{\rho}}|^q\leq C(T-t)^{q-1}E\left[\int_t^T|\hat{\xi}_s|^q\,ds\middle|\bar{\mathscr{F}}_t\right].$$

where the constant $C>0$ is independent of $(x,\hat{\xi},\hat{\rho})$.

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HJB equation for the Value function

The qth-power structure of the cost functional suggests a multiplicative decomposition of the value function of the form

$$V_t(x,y) = u_t(y)|x|^q.$$

Here the processes u, together with another adapted process ψ, satisfies the following backward SPDE of Neumann type with singular terminal term:

$$\begin{cases} -du_t(y) = \Big[\alpha D^2 u + \sigma^T D\psi + \beta D u + \lambda - \dfrac{|u|^{q^*}}{(q^*-1)|\eta|^{q^*-1}} - \mu(\mathcal{Z})u \\ \qquad\qquad + \displaystyle\int_{\mathcal{Z}} \frac{\gamma_\cdot(\cdot,z)u}{(|\gamma_\cdot(\cdot,z)|^{q^*-1} + |u|^{q^*-1})^{q-1}}\,\mu(dz)\Big](t,y)\,dt - \psi_t(y)dW_t, \\ Du_t(a) = 0, \quad t \in [0,T). \\ u_T(y) = \infty, \quad y \in \mathcal{D}, \end{cases} \tag{1}$$

where $q^* = \frac{q}{q-1}$ is the Hölder conjugate of q and

$$\alpha_t(y) := \frac{1}{2}\big[\sigma_t^T(y)\sigma_t(y) + \bar{\sigma}_t^T(y)\bar{\sigma}_t(y)\big].$$

Notations

we denote by $\mathscr{S}^p_{\mathscr{F}}(s,t;\mathbb{H})$ the set of all the $\mathbb{H}$-valued and $\mathscr{F}_r$-adapted continuous processes $(X_r)_{r\in[s,t]}$ such that

$$\|X\|_{\mathscr{S}^p_{\mathscr{F}}(s,t;\mathbb{H})} := \left\| \sup_{r\in[s,t]} \|X_r\|_{\mathbb{H}} \right\|_{L^p(\Omega)} < \infty.$$

By $\mathscr{L}^p_{\mathscr{F}}(s,t;\mathbb{H})$ we denote the class of $\mathbb{H}$-valued $\mathscr{F}_r$-adapted processes $(u_r)_{r\in[s,t]}$ such that

$$\|u\|_{\mathscr{L}^p_{\mathscr{F}}(s,t;\mathbb{H})} := \|\|u(\cdot)\|_{\mathbb{H}}\|_{L^p(\Omega\times[s,t])} < \infty.$$

$$\mathcal{H}^k([s,t]\times\mathcal{O}) := \left(\mathscr{S}^2_{\mathscr{F}}(s,t;H^k(\mathcal{O}))\cap\mathscr{L}^2_{\mathscr{F}}(s,t;H^{k+1}(\mathcal{O}))\right)\times\mathscr{L}^2_{\mathscr{F}}(s,t;H^k(\mathcal{O})),$$

where $\mathcal{O}\subset\mathcal{D}$, and $H^k(\mathcal{O})$ being Sobolev space.

Definition of the strong solution

A pair of processes (u, ψ) is a strong solution to equation (1) if for all $\tau \in (0, T)$ and $b \in \mathbb{R}$ with $b > a$, it holds that $(u, \psi)1_{[0,\tau]\times[a,b]} \in \mathcal{H}^1([0,\tau]\times[a,b])$, and with probability 1, for all $t \in [0,\tau]$,

$$u_t(y) = u_\tau(y) + \int_t^\tau \left[\frac{1}{2}\alpha D^2 u + \sigma^T D\psi + \beta Du + \lambda - \frac{|u|^{q^*}}{(q^*-1)|\eta|^{q^*-1}} - \mu(\mathcal{Z})u + \int_{\mathcal{Z}} \frac{\gamma_\cdot(\cdot,z)u}{(|\gamma_\cdot(\cdot,z)|^{q^*-1} + |u|^{q^*-1})^{q-1}}\,\mu(dz)\right](s,y)\,ds - \int_t^\tau \psi_s(y)dW_s, \quad \text{dy-a}$$

with

$$Du_t(a) = 0, \text{ for } t \in [0,\tau], \quad \text{and } \lim_{\tau\to T} u_\tau(y) = \infty, \quad \text{for all } y \in \mathcal{D}, \text{ a.s.}$$

We would note that the zero Neumann boundary condition is holding in the sense that with probability 1, for each $t \in [0, T)$,

$$\lim_{\delta\to 0^+} \frac{1}{\delta}\int_a^{a+\delta} Du_t(x)\,dx = 0.$$

Assumptions

- (Measurability and boundedness) The function $\gamma : \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{Z} \longrightarrow [0,+\infty]$ is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d) \times \mathscr{Z}$-measurable, and the functions

$$\beta, \sigma, \bar{\sigma}, \eta, \lambda : \Omega \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+$$

  are $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d)$-measurable and essentially bounded by $\Lambda > 0$.
- (Lipschitz-continuity) For $h = \lambda, \eta, \beta, \sigma^i, \bar{\sigma}^j$, $i = 1, \ldots, d$, $j = 1 \ldots, m$, it holds that for all $y_1, y_2 \in \mathbb{R}$ and $(\omega, t) \in \Omega \times [0,T]$,

$$|h_t(y_1) - h_t(y_2)| + \operatorname*{ess\,sup}_{z \in \mathcal{Z}} |\gamma_t(y_1, z) - \gamma_t(y_2, z)| \leq \Lambda \, |y_1 - y_2| \, ,$$

  where Λ is the constant in $(\mathcal{A}1)$.
- There exist constants $\kappa > 0$ and $\kappa_0 > 0$ such that $\eta_s(y) \geq \kappa_0$ and

$$\text{(Superparabolicity)} \qquad \sum_{i=1}^{m} \left| \bar{\sigma}_s^i(y) \right|^2 \geq \kappa, \quad \text{a.s.,} \quad \forall \, (s, y) \in [0,T] \times \mathbb{R}.$$

Outlines

Newmann problems

For any $y \in [a, +\infty)$, $u_t(y)$ is bounded. To consider it in Sobolev space $H^k([a, +\infty))$, we introduce a weight function

$$\theta : \mathbb{R} \to \mathbb{R}, \quad y \mapsto \left(1 + |y - a|^2\right)^{-1},$$

and we may analyze θu instead of u.
(u, ψ) is a solution to (1) if and only if $(v, \zeta) := (\theta u, \theta\psi)$ solves

$$\begin{cases} -dv_t(y) = \Big[\alpha D^2 v + \sigma^T D\zeta + \lambda\theta - \dfrac{|v|^{q^*}}{(q^* - 1)|\theta\eta|^{q^*-1}} - \mu(\mathcal{Z})v \\ \qquad\qquad + \displaystyle\int_{\mathcal{Z}} \frac{\theta\gamma(\cdot, z)v}{(|\theta\gamma(\cdot, z)|^{q^*-1} + |v|^{q^*-1})^{q-1}}\,\mu(dz) + f(t, y, Dv, v, \zeta)\Big](t, y)\,dt \\ \qquad\qquad - \zeta_t(y)\,dW_t, \quad (t, y) \in (0, T) \times \mathcal{D}, \\ Dv_t(a) = 0, \quad \text{for } t \in [0, T], \\ v_T(y) = \infty, \end{cases} \tag{2}$$

with f being linear.

Solution of (2)

Two difficulties:

- q^*th-power growth in v in the drift term
- the terminal term is ∞

To deal with them

- Step 1: Lipschitz continuous equation with finite terminal condition
- Step 2: truncations on the quadratic growth and the infinite terminal value

$$\begin{cases} -dv_t(y) = \left[\alpha D^2 v + \sigma^T D\psi + F(t,y,Dv,v,\psi)\right] dt - \psi_t(y)\, dW_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,y) \in (0,T) \times \mathcal{D}, \\ Dv_t(a) = 0, \quad t \in [0,T], \\ v_T(y) = G(y), \quad \forall y \in \bar{\mathcal{D}}. \end{cases} \tag{3}$$

Existence and Uniqueness theorem

The BSPDE (3) admits a unique strong solution (v, ζ) lying in $\mathcal{M}^1$, and

$$\|(v,\zeta)\|_{\mathcal{H}^1} \le C\Big(\|G\|_{L^\infty(\Omega;H^1(\mathcal{D}))} + \|F_0\|_{\mathscr{M}^2_{\mathscr{F}}(0,T;L^2(\mathcal{D}))}\Big),$$

with the constant C depending on κ, Λ, T and K.

Comparison theorem

Suppose that (v^1, ψ^1) and (v^2, ψ^2) are strong solutions of (3) with (G^1, F^1) and (G^2, F^2), If $G^1 \le G^2$ and $F^1(t,y,Dv^1,v^1,\psi^1) \le F^2(t,y,Dv^1,v^1,\psi^1)$, then

$$v_t^1(y) \le v_t^2(y), \quad \text{a.s.}$$

Step 2: Truncation of the q^*-power term in the drift term

$$-\frac{\left|\left(\theta^{-1}|v|\right)\wedge N\right|^{q^*-1}}{(q^*-1)|\eta|^{q^*-1}}|v| \quad \text{instead of} \frac{\theta^{-1}|v|^{q^*}}{(q^*-1)|\eta|^{q^*}}.$$

Let $N\to+\infty$.

Step 3: Truncation of the singular terminal term

$$v_T(y)=M\theta(y) \quad \text{instead of} \quad V_T(y)=+.$$

Let $M\to+\infty$.

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Verification theorem and optimal feedback control

Theorem

Suppose that (u,ψ) *is a strong solution to BSPDE* (1) *that*

$$(\theta u, \theta\psi)1_{[0,t]} \in \mathcal{H}^1([0,t]\times\mathcal{D}), \quad t\in(0,T), \tag{4}$$

and

$$\frac{c_0}{(T-t)^{q-1}} \le u_t(y) \le \frac{c_1}{(T-t)^{q-1}}, \quad \text{a.s. } \forall (t,y)\in[0,T)\times\mathcal{D}, \tag{5}$$

with c_0 *and* c_1 *being two positive constants. Then,*

$$V(t,y,x) := u_t(y)|x|^q, \quad (t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R},$$

coincides with the value function. Moreover, the optimal (feedback) control is given by

$$(\xi^*_t,\, \rho^*_t(z)) = \left(\frac{|u_t(y_t)|^{q^*-1}\, x_t}{|\eta_t(y_t)|^{q^*-1}},\, \frac{|u_t(y_t)|^{q^*-1}\, x_{t-}}{|\gamma_t(y_t,z)|^{q^*-1} + |u_t(y_t)|^{q^*-1}} \right), \quad \text{for } t\in[0,T). \tag{6}$$

Thank you for your attention!