Physical, Geometrical and Analytical aspects of Mean Field Systems of Liouville type

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1 Overview of mean field systems of Liouville type

The study of mean field systems of Liouville type, in particular of the Chern–Simons vortices, has received a great interest in recent years, especially in the critical coupling of the self-dual regime, where the theory can be embedded into a supersymmetric field theory. In this way, a new and unexpected role of non-abelian vortices has emerged also in connection with the delicate issue of quark confinement, see [18, 21, 63].

In this workshop we have discussed some rigorous mathematical results concerning the existence of such vortex configurations and their asymptotic behaviour, as relevant physical parameters approach their limiting values.

To motivate the nature of the mathematical questions we need to address in the study of gauge field vortices, we start to discuss the "pure" abelian Chern–Simons model introduced by Jackiw–Weinberg [31] and Hong–Kim–Pac [29]. It provides a simple yet non-trivial theory, which captures the most interesting features about self-dual Chern–Simons (CS) vortices.

The model in [31] and [29] proposes a 2D theory governed only by the CS-electromagnetism which supports charged vortices (both electrically and magnetically). For this reason, such model is particularly appropriate to describe phenomena such as Superconductivity and the Quantum Hall effect [19].

In comparison, we recall that the more familiar Ginzburg–Landau theory can support only magnetically charged but electrically neutral vortices.

More precisely, the electromagnetic theory in [29, 31] is formulated within an abelian $U(1)$-gauge field theory in terms of the complex valued Higgs field ϕ and the electromagnetic potential: $\mathbf{A} = (A_0, A_1, A_2)$, which are weakly coupled through the covariant derivative:

$$D_\alpha \phi = \partial_\alpha \phi - \mathrm{i} A_\alpha \phi, \quad \alpha = 0, 1, 2.$$

As usual the value of the parameter $\alpha = 0$ refers to the time-variable, while $\alpha = 1, 2$ are used for the space variables.

The self-dual equations (formulated in the temporal gauge) for the Abelian CS- model with a 6th order potential [29, 31] are given as follows:

$$\begin{cases} D_1\phi \pm \mathrm{i}D_2\phi = 0 \\ F_{12} = \pm\dfrac{2}{\kappa^2}|\phi|^2(1 - |\phi|^2) \end{cases} \tag{1.1}$$

supplemented by the following Gauss-law constraint governing the system:

$$\kappa F_{12} = \mathrm{i}(\overline{\phi} D_0\phi - \phi\overline{D_0\phi}), \tag{1.2}$$

with $\rho = \mathrm{i}(\overline{\phi} D_0\phi - \phi\overline{D_0\phi})$ the charge density, $\mathbf{F} = (F_{12}, -F_{02}, F_{01}) = \mathrm{curl}\mathbf{A}$ the electromagnetic field, $\kappa > 0$ the Chern–Simons coupling parameter.

In particular, $F_{12} = \partial_1 A_2 - \partial_2 A_1$ defines the (time-independent) magnetic field (a scalar in 2D), while ϕ is the (time independent) complex wave function.

To obtain a self-dual CS vortex, we need to provide a complex function $\phi : \mathbb{C} \to \mathbb{C}$ and real functions $A_\alpha : \mathbb{R}^2 \to \mathbb{R}, \quad \alpha = 0, 1, 2$; such that $(\phi; A_1, A_2)$ solves (1.1), while A_0 is determined by (1.1) and (1.2), so that (away from zero set of of ϕ) we have :

$$A_0 = \pm\frac{2}{\kappa^2}(1 - |\phi|^2). \tag{1.3}$$

We can focus only to the vortex equation with "plus" sign, since the opposite sign is obtained simply by complex conjugation. In the self-dual regime, the following relation holds between the total energy E, the magnetic flux Φ and the electric charge Q :

$$E = \int F_{12}\mathrm{d}x = \Phi \quad \text{and} \quad Q = \kappa\Phi, \tag{1.4}$$

(see [29, 31] for details) so to ensure finite total energy and fluxes we need to supplement (1.1) with the condition:

$$\int F_{12}\mathrm{d}x < +\infty, \tag{1.5}$$

Although (1.1) provide a first-order reduction of the more involved second-order field equations, still they are analytically delicate to handle for their invariance with respect to the following gauge transformations:

$$\phi \to \mathrm{e}^{\mathrm{i}\omega\phi}, \quad A_\alpha \to A_\alpha + \partial_\alpha\omega, \quad \alpha = 0, 1, 2$$

$\omega : \mathbb{R}^2 \to \mathbb{R}$ a smooth (gauge) function.

Thus analytically, we are able to express explicitly the wave function ϕ only in terms of a (suitably) fixed gauge. An appropriate choice of gauge function was introduced by Taubes for the Maxwell–Higgs model (see [32]) based on the observation that the equation:

$$D_1\phi + \mathrm{i}D_2\phi = 0 \tag{1.6}$$

provides a gauge invariant version of the Cauchy–Riemann equation. Therefore, by the Poincaré $\overline{\partial}$ lemma, for a suitable choice of the gauge function η, we have that $\mathrm{e}^{\mathrm{i}\eta}\phi$ is holomorphic (see [32] for details). Hence, ϕ admits isolated zeros (vortex points), say at $\{p_1, \cdots, p_N\}$, with integral multiplicity ("quantized " local flux). The vortex points and their multiplicities are "observable" quantities, (i.e. gauge independent) together with F_{12} (magnetic field) and $|\phi|^2$ (density).

Hence, to obtain an analytical expression for solutions, we need to formulate problem (1.1) only in terms of such "observable" quantities. To this purpose, we express ϕ in polar (complex) coordinates, according to the following ansatz:

$$\phi(x) = \mathrm{e}^{\frac{u(x)}{2} + \mathrm{i}\left(\sum_{j=1}^{N} \mathrm{Arg}(x - p_j)\right)}. \tag{1.7}$$

So that,

$$|\phi|^2 = \mathrm{e}^u \quad \text{with} \quad u(x) = \ln|x - p_j|^{2n_j} + \mathrm{O}(1), \quad \text{as} \quad x \to p_j \tag{1.8}$$

(with n_j the multiplicity of p_j, $j \in \{1, \ldots, N\}$), and by (1.1), we find that u must satisfy:

$$-\Delta u = \frac{4}{\kappa^2}\mathrm{e}^u(1 - \mathrm{e}^u) - 4\pi \sum_{j=1}^{N} \delta_{p_j}. \tag{1.9}$$

Conversely, from any solution u of (1.9) we can reconstruct a full vortex solution $(\phi, (A_\alpha)_{\alpha=0,1,2})$, with ϕ vanishing exactly at $p_1, \ldots, p_N$, by means of (1.7) and (1.3).

Therefore, if we search for periodic CS vortices in the style of Abrikosov's mixed states [1] then it suffices to solve (1.9) over the torus, while to obtain planar vortices we need to to solve (1.9) over $\mathbb{R}^2$ supplemented by the appropriate boundary conditions that ensure the finite energy condition:

$$\int_{\mathbb{R}^2} \mathrm{e}^u(1 - \mathrm{e}^u)\mathrm{d}x < +\infty. \tag{1.10}$$

Based on physically consistent ansatz, a similar procedure can be adopted in the search of non-abelian CS vortices. In this way, we reduce their construction to the search of solutions of elliptic problems involving exponential nonlinearities in system form. More precisely, the corresponding system of PDE's governing self-dual non-abelian CS vortices takes the following form:

$$-\Delta u_i = \lambda_i \left(\sum_{j=1}^{n} K_{ji}\mathrm{e}^{u_j} - \sum_{j=1}^{n}\sum_{k=1}^{n} K_{kj}K_{ji}\mathrm{e}^{u_j}\mathrm{e}^{u_k} \right) - 4\pi \sum_{s=1}^{N_i} \delta_{p_{i,s}}(x), \quad i = 1, \ldots, n, \tag{1.11}$$

with respect to the unknown functions $(u_1, \ldots, u_n)$; where $\lambda_i > 0$, $\{p_{i,s}, s = 1, \ldots, N_i\}$ are the vortex points of the ith-component with total multiplicity $N_i \in \mathbb{N}$, $i = 1, \ldots, n$ and the coupling matrix:

$$K = (K_{ij})_{i,j=1,\ldots,n}$$

is fixed by the gauge group G specified by the physical model under exam, since a unified theory including all fundamental particle interaction is not available yet. To give some examples, we recall that the abelian group $U(1)$ describe eletro-magnetic interactions, the gauge group $G = SU(2)$ allows for weak particle interactions, $G = SU(3)$ for strong particle interactions, and $G = U(1) \times SU(2)$ describe electroweak particle interactions. While we refer to [11,24–28,71] for more details, we mention that for the "pure" non-abelian CS model introduced by Dunne [17], the matrix K corresponds to the Cartan matrix of the group G. So for example, if $G = SU(n+1)$ then K is given by the following $n \times n$ matrix:

$$K = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \tag{1.12}$$

related to the integrable Toda system, see [26, 54, 72]

Furthermore, if G is a semi-simple Lie group with rank 2, then besides $G = SU(3)$, we have the choices of a group G of the type $B_2(= C_2)$ with corresponding 2×2 corresponding Cartan matrix:

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{1.13}$$

or $G = G_2$ with Cartan matrix:

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{1.14}$$

More recently, self-duality for Chern–Simons gauge field theories has been attained by exploiting the general framework of Supersymmetry. Under this point of view, several classes of new genuinely non-abelian vortices (or supersymmtric monopoles) have been identified, see [22, 23, 46] and references therein. For example the model in [46] unifies some of the previous models by proposing a theory over the gauge group $G = U(1) \times SU(N)$, $N \geq 2$, whose vortex configurations can be obtained by solving a 2×2 system of the type (1.11) with coupling matrix given as follows:

$$K = \frac{1}{N} \begin{pmatrix} N-1+\kappa & 1-\kappa \\ (N-1)(1-\kappa) & 1+(N-1)\kappa \end{pmatrix}, \tag{1.15}$$

where $\kappa = \frac{\kappa_1}{\kappa_2} > 0$, and κ_1 and κ_2 are the coupling parameters respectively of the $U(1)$-fields and $SU(N)$-fields, see [22, 23, 46] and [11, 24] for details.

Notice that, for $N = 2$ and $\kappa = 3$, then K in (1.15) coincides with the 2×2-Toda matrix in (1.12).

Another interesting example can be found in [52, 53]. There the authors propose an abelian truncation of the Aharony–Bergman–Jafferis–Maldacena (ABJM) model [2], for which it is possible to introduce a suitable vortex ansatz, and obtain a system of the type (1.11) with an off-diagonal 2×2 coupling matrix $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, (in normalized units). This model has been analyzed in [27].

Again, for the physical applications, it is relevant to consider solutions of the system (1.11) over a compact surface (e.g. the flat 2-torus) or over $\mathbb{R}^2$ supplemented by suitable integrability conditions of type (1.10), which guarantee finite energy and fluxes.

As discussed during the workshop the analysis of the governing CS-vortex equations (1.9) and (1.11) (also referred to as the Master equations), touch upon many analytical issues concerning singular Mean Field equations of Liouville-type, see e.g. [66, 67], surfaces with conical singularities, see e.g. [69], sharp Moser–Trudinger or log(HLS)-inequalities, see e.g. [61, 62, 70], bubbling phenomena, see e.g. [5, 13, 14] etc.

In this respect, we have learnt that, while for the single equation (1.9) the understanding of the (abelian) vortex-equation is rather satisfactory, in the system case the results available are only partial, as pointed out by the talks of L. Battaglia, Z. Nie and A. Poliakovsky.

To be more specific, for simplicity we shall focus on 2×2 systems and assume (as in the physical applications) that in (1.11) the 2×2 coupling matrix $K = (K_{ij})_{i,j=1,2}$ satisfies:

$$\det K \neq 0 \quad \text{and} \quad K_{ii} > 0, \quad i = 1, 2. \tag{1.16}$$

Furthermore, via the transformation: $u_i \to u_i + \ln(K_{ii})$, and by setting:

$$\tau_i = \frac{K_{ji}}{K_{jj}} \quad i \neq j \in \{1, 2\}, \tag{1.17}$$

we arrive at the following normalized expression of (1.11):

$$\begin{cases} -\Delta u_1 = \lambda_1 \left(\mathrm{e}^{u_1} + \tau_1 \mathrm{e}^{u_2} - \mathrm{e}^{2u_1} - \tau_1 \mathrm{e}^{2u_2} - \tau_1 (1 + \tau_2) \mathrm{e}^{u_1 + u_2} \right) - 4\pi \sum_{j=1}^{N_1} \delta_{p_{1,j}} \\ -\Delta u_2 = \lambda_2 \left(\mathrm{e}^{u_2} + \tau_2 \mathrm{e}^{u_1} - \mathrm{e}^{2u_2} - \tau_2 \mathrm{e}^{2u_1} - \tau_2 (1 + \tau_1) \mathrm{e}^{u_1 + u_2} \right) - 4\pi \sum_{j=1}^{N_2} \delta_{p_{2,j}} \end{cases} \tag{1.18}$$

with $\lambda_i > 0$, $i = 1, 2$.

As in the abelian case, in the search of planar non-abelian vortices, we need to solve (1.18) over $\mathbb{R}^2$, under appropriate boundary conditions which guarantee the integrability of the right hand side of the equations in (1.18), in order to ensure finite energy for the corresponding vortex.

Therefore, we recognize the <u>topological</u> boundary condition to be given as follows:

$$u_i(x) \to \ln \left(\frac{1 - \tau_i}{1 - \tau_1 \tau_2} \right) \quad \text{as} \quad |x| \to +\infty, \quad i = 1, 2; \tag{1.19}$$

whenever the right hand side of (1.19) is well defined. Indeed, we easily check that (1.19) is always well defined for the physical examples (1.12)–(1.15).

The existence of such class of solutions has been established under rather general assumptions for the coupling matrix K, which cover all the interesting physical models. We refer to [11, 24–26, 71] for more details about non-abelian topological vortices and their "quantization" property.

On the contrary, much less is understood about the presence of "non-topological vortices" satisfying:

$$u_i(x) \to -\infty \quad \text{as} \quad |x| \to +\infty, \quad i = 1, 2, \tag{1.20}$$

Also it is of interest the existence of the so-called "mixed" vortices where the topological and non-topological boundary conditions are each satisfied by either one of the two components available. Unfortunately, mixed vortices have been constructed only in some special situation, see [20, 54], and are difficult to grasp also in the radial case, see [30]; so that much of their study remains open to future investigation.

Concerning the construction of non-topological vortices, we mention a "perturbation" approach where we introduce the following scaled version of the solutions:

$$u_{i,\varepsilon}(x) = u_i\left(\frac{\sqrt{\lambda}x}{\varepsilon}\right) + 2\ln\frac{1}{\varepsilon}, \quad \varepsilon > 0 \tag{1.21}$$

so that planar solutions of (1.18) satisfying:

$$\int_{\mathbb{R}^2} \mathrm{e}^{u_i}\mathrm{d}x < +\infty, \quad i = 1, 2 \tag{1.22}$$

can be sought (for $\varepsilon \to 0$) as "bifurcating" from solutions of the following system of Liouville type:

$$\begin{cases} -\Delta u_1 = \mathrm{e}^{u_1} + \tau_1\mathrm{e}^{u_2} - 4\pi N_1\delta_0 & \text{in} \quad \mathbb{R}^2 \\ -\Delta u_2 = \mathrm{e}^{u_2} + \tau_2\mathrm{e}^{u_1} - 4\pi N_2\delta_0 & \text{in} \quad \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} \mathrm{e}^{u_i}\mathrm{d}x < +\infty, \quad i = 1, 2. \end{cases} \tag{1.23}$$

Unlike the single-(singular) Liouville equation, (corresponding to the decoupled case: $\tau_1 = \tau_2 = 0$), in general the solvability of (1.23) is far from understood, except for the integrable and fully conformal case of the Toda-system (1.12), which according to our normalization corresponds to the case where, $\tau_1 = \tau_2 = -\frac{1}{2}$. In fact, for (1.23), a full characterization of solutions (analogous to single Liouville equation [60]) has been obtained by Lin–Wei–Ye in [40], which extends and completes the previous characterization result of Jost–Wang [36] concerning the "regular" case, where $N_1 = N_2 = 0$.

In particular from [40] and [31] we know that all solutions of (1.23) with $\tau_1 = \tau_2 = -\frac{1}{2}$ satisfy:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{u_1}\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{u_2}\mathrm{d}x = 4(N_1 + N_2 + 2). \tag{1.24}$$

Furthermore, the explicit expression of the solutions of (1.23) as provided in [31,40] together with their non-degeneracy properties (see [40]) have been exploited by Ao–Lin–Wei in [3] to carry out a "perturbation" approach in the same spirit of Chae–Imanuvilov [9] . In this way the authors in [3] obtained a class of non-topological solutions for the $SU(3)$-model [17]. Such construction was extended by the same authors in [4] for the gauge group G of the type B_2 and G_2. Moreover, we mention the recent work of Lin–Nie–Wei [38] which uses more algebraic tools to fully characterize solutions and their total integrals, for any $n \times n$ Liouville system of the type (1.23) with coupling matrix corresponding to a Cartan matrix.

However, in view of the physical applications, we need to treat with the same accuracy, also the case where the coupling matrix admits a structure other than a Cartan matrix, as for example (1.15). In particular, for the normalized system (1.23), we need to identify the sharp range of admissible pairs (β_1, β_2) such that problem (1.23) with $(\tau_1, \tau_2) \neq (-\frac{1}{2}, -\frac{1}{2})$ admits a solution satisfying:

$$\beta_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^{u_i} \mathrm{d}x, \quad i = 1, 2. \tag{1.25}$$

This task was carried out by Poliakovsky-Tarantello [59] with respect to the radial solvability of (1.23)–(1.25), see also [10,15,57,58,61,62] for related results. Notice that, as in the analysis of the single Liouville equations, [12,60], we expect the radial problem to furnish a full description about the pairs (β_1, β_2) admissible for the solvability of (1.23), (1.25), in the sense that, non-radial solutions can only occur as "bifurcation" from radial ones while keeping the same integral pair (β_1, β_2).

In addition the role of (1.23) and (1.25) is relevant also in the analysis of the "bubbling" phenomenon in the context of systems, which is a very delicate task (see [34,41,43–45,55]) but crucial to obtain solutions via topological, variational and perturbative methods, as e.g. [7,8,14,33,49–51].

Even more involved is the blow-up analysis when the different components blow-up at the same point but with a different rate. Here a new phenomenon occurs, namely the possibility of blow-up without an associated "concentration" phenomenon and a "quantization" property, as recently shown in [39]. Similar blow-up phenomena with "residual mass" (in the terminology of [56]) have been shown to occur when different vortex points collapse into each other (see to [37,39] for details), a situation of great interest from the physical point of view.

All such new, interesting and delicate aspects have been discussed during the meeting and we expect to see some further developments and progress in the near future as an outcome of the workshop.

2 Scientific activities of the workshop

The workshop has brought together a group of expert mathematicians as well as junior researchers studying different aspects of Mean Field Equations of Liouville type, as they arise from physical and geometrical problems. The topics discussed during the workshop have concerned the analysis of self-dual gauge field vortices, including the Abrikosov's configurations for the celebrated Electroweak theory of Glashow-Salam-Weinberg, and the "uniformization" problem in differential geometry for Riemann surfaces with conical singularities and various other generalizations concerning the assigned (also sign-changing) Gauss curvature problem.

A first successful outcome of the workshop was stimulated by the new "sphere covering inequality " of C. Gui and A. Moradifam and its application towards uniqueness results. Already, other applications of Gui- Moradifam's sharp inequality were presented during the workshop by D. Bartolucci and A. Jevnikar in the setting of the Riemann sphere with conical singularities. But during the discussions section many of the participants have remarked about the strength of such inequality and have proposed several other applications, in the form of future projects and collaborations. For example, the "Sphere covering inequality" seems flexible enough to yield new non-existence results in the supercritical and in the non-radial setting, in the presence of more than three conical singularities.

Another important problem discussed during the workshop has concerned traveling waves of the Gross-Pitaevskii equation, for which ground state solutions have been already established. The results presented by J. Wei show the asymptotic behaviour of a new class of solutions for small values of the velocity. This however leaves open the existence of excited state for fixed values of the velocity. During the workshop D. Ruiz and J. Wei discussed about a possible approach to handle such problem, and on this basis formulated a future project in collaboration.

The recent developments about the spectral gap conjecture, as reported in D. Hauer's talk got also a lot of attention. For example, it was observed by D. Bartolucci that in order to describe the global bifurcation diagram for solutions of mean field equations, so far it has been possible under a spectral gap assumption about entropy maximizers. It was suggested that (by further investigation) Hauers result may be helpful in the understanding of the above mentioned bifurcation diagram. Furthermore, J. Dolbeault and D. Hauer have been discussing an issue related with stability in functional inequalities, based on hyper-contractivity. For the fast diffusion equation in the subcritical range, one can get explicit rates of regularization. A more or less explicit control of the relative uniform norm in terms of the initial data has recently been obtained by M. Bonforte and N. Simonov (arXiv:1804.03537v1). Then using the improved spectral gap and an improved entropy (i.e. entropy production inequality), one should be in position to get an explicit stability result for the Gagliardo-Nirenberg inequality. In the subcritical case, this is a project of M. Bonforte, J. Dolbeault, B. Nazaret and N. Simonov. The major issue is the critical case (Sobolev inequality). Techniques based on hyper-contractivity (using the framework of the book of T. Coulhon and D. Hauer, in preparation) could provide at least a partial answer.

Finally, another important topic widely discussed during the meeting concerned systems of Liouville type, which arise in the construction of non-abelian vortices. Indeed, the most natural extension of the Mean Field Equations into a system form is given by the Toda-system, whose solutions describe the asymptotic profile of non-abelian vortices for the $SU(n+$

1)-Chern-Simons model. More generally in this context mean field equations of Liouville type are coupled together via the Cartan matrix associated to the gauge group. After the talks of L. Battaglia, Z. Nie and A. Poliakovsky we have seen that many interesting questions still need to be answered. Especially concerning the blow-up analysis for systems we still need a complete classification of solutions for the "limiting" problem in the plane (describing the blow-up profile) away from the conformal case.

According to D. Hauer, who recently started a new collaboration on Mean Field Systems with C. Gui, this workshop provided many backgrounds, geometrical and mathematical motivations and new knowledge about this topic, and helped to push forward his current research project. All the participants have remarked that: the workshop was very central to their research interest about a field in great development in recent years, with exciting novelties in unexpected directions. There were people from different backgrounds so that fruitful exchanges of ideas were encouraged. The intellectual environment during the workshop was notably vibrant.

The discussions during question sessions were remarkably interesting, and served as further motivation to encourage and enhance the participants interactions. Also they took a great advantage of the breaks to actively interact clarifying doubts and intellectual curiosities raised by the talks.The BIRS center offered a very warm and cozy atmosphere, (despite the unusual winter temperatures of -10 to -15 centi-grades). To have a scientific workshop in an environment of arts, dance and drama certainly provided an inspiring atmosphere and contributed to the success of the meeting.

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