LIOUVILLE PROBLEMS WITH SIGN CHANGING POTENTIALS

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Physical, Geometrical and Analytical Aspects of Mean Field Systems of Liouville Type

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1. Introduction: Motivation of the problem

Prescribing the Gaussian curvature on singular surfaces

(Σ, g) be a cpt Riemann surface;

$\rightsquigarrow$ K_g associated Gaussian curvature

$\tilde{g} = e^v g$ conformal in $\Sigma \setminus \{p_1, \dots, p_m\}$

$\rightsquigarrow$ $K_{\tilde{g}}$ associated Gaussian curvature in $\Sigma \setminus \{p_1, \dots, p_m\}$

$(\Sigma, \tilde{g})$ admits conical singularities at the points $p_1, \dots, p_m$ of orders $\alpha_1, \dots, \alpha_m$ respectively

$$-\triangle_g v + 2K_g = 2K_{\tilde{g}} e^v - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}. \qquad (**)$$

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Problem [Troyanov]

Given K defined on Σ, $p_1, \ldots, p_m \in \Sigma$, $\alpha_1, \ldots, \alpha_m > -1$

$$\exists\, \tilde{g} = e^v g \text{ in } \Sigma \setminus \{p_1, \ldots, p_m\}$$
s.t. $(\Sigma, \tilde{g})$ admits conical singularities at p_j's of orders α_j's and that $K_{\tilde{g}} = K$?

namely $\exists\, v$ solution of $(**)$?

Prescribing the Gaussian curvature on singular surfaces

Let us set

$$u(x) = v(x) + \underbrace{4\pi \sum_{j=1}^{m} \alpha_j G(x, p_j)}_{=:h_m(x)} \qquad \text{where} \qquad \begin{cases} -\triangle_g G(x,y) = \delta_y - \frac{1}{|\Sigma|} \\ \int_\Sigma G(x,y) dV_g = 0 \end{cases}$$

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Then the equation can be rewritten as follows

$$-\triangle_g u + 2K_g = 2\tilde{K} e^u - \frac{4\pi}{|\Sigma|} \sum_{j=1}^{m} \alpha_j,$$

where

$$\tilde{K}(x) = e^{-h_m(x)} K(x), \quad \tilde{K}(x) \simeq d(x, p_j)^{2\alpha_j} K(x) \quad \text{near each } p_j.$$

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Integrating (**) and applying the Gauss-Bonnet Theorem

$$\lambda := 2 \int_\Sigma \tilde{K} e^u dV_g = 2 \int_\Sigma K_g dV_g + 4\pi \sum_{j=1}^{m} \alpha_j \overset{\text{GB}}{=} 4\pi(\chi(\Sigma) + \sum_{j=1}^{m} \alpha_j).$$

Prescribing the Gaussian curvature on singular surfaces

Assuming that K_g is constant by the Uniformization Theorem, we can rewrite the equation as

$$-\triangle_g u = \lambda \left(\frac{\tilde{K} e^u}{\int_\Sigma \tilde{K} e^u dV_g} - \frac{1}{|\Sigma|} \right).$$

This problem is usually called the *mean field* equation of Liouville type.

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The mean field problem not only appears in geometrical contexts, but also in Physics.

Physical motivations

Periodic vortices in Electroweak theory of Glashow-Salam-Weinberg
[Lai, 1981], [Yang, 2001], [Bartolucci-Tarantello, 2002]

Periodic vortices in Chern-Simons-Higgs theory
[Dunne, 1994], [Tarantello, 2007]

Stationary turbulence for Euler flow with vortices
[Caglioti-Lions-Marchioro-Pulvirenti, 1992], [Tur-Yanovsky, 2004]

Variational formulation

Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \frac{\lambda}{|\Sigma|}\int_\Sigma u\, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g$$

defined in $\quad X = \{u \in H^1(\Sigma) \,:\, \int_\Sigma \tilde{K} e^u dV_g > 0\}.$

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Moser-Trudinger type inequality [Troyanov, 1991]

There exists a constant $C > 0$ such that

$$\log \int_\Sigma \tilde{K} e^u dV_g \leq \frac{1}{16\pi \min_{j=1,\ldots,m}\{1, 1+\alpha_j\}} \int_\Sigma |\nabla u|^2 dV_g + C, \qquad \text{(MT)}$$

for every $u \in H^1(\Sigma)$ with $\int_\Sigma u\, dV_g = 0$.

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for every $u \in H^1(\Sigma)$ with $\int_\Sigma u\, dV_g = 0$.

- if $\lambda < 8\pi \min_{j=1,\dots,m}\{1, 1+\alpha_j\}$: I_λ is coercive and w.l.s.c. $\Rightarrow \exists$ a minimizer.
- if $\lambda = 8\pi \min_{j=1,\dots,m}\{1, 1+\alpha_j\}$: I_λ is bounded below but is no longer coercive;
- if $\lambda > 8\pi \min_{j=1,\dots,m}\{1, 1+\alpha_j\}$: the functional I_λ is not even bounded from below.

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The case $\alpha_j < 0$ has been treated (for $K > 0$) in [Carlotto-Malchiodi, 2012], [Carlotto, 2014].

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Compactness ($K > 0$)

Blow–up alternative [Bartolucci-Tarantello, 2002] ([Brezis-Merle, 1991], [Li-Shafrir, 1994])

Let $K \in C^{0,1}(\Sigma)$ and $K > 0$, and let u_n a sequence of solutions of (1) such that $\int_\Sigma \tilde{K} e^{u_n} dV_g \leq C$, then as $\lambda_n \to \lambda$ the following alternative holds (up to a subsequence)

a) u_n is bounded from above in Σ;

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a) u_n is bounded from above in Σ;

b) $\max_\Sigma \left(u_n - \log \int_\Sigma \tilde{K} e^{u_n} dV_g \right) \to +\infty$, and there exist a finite (blow up) set $S = \{x_1, \ldots, x_r\} \subset \Sigma$ such that $u_n(x_{j,n}) \to +\infty$ with $x_{j,n} \to x_j \in S$ and $u_n \to -\infty$ uniformly on compact sets of $\Sigma \setminus S$.

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$$\lambda \frac{\tilde{K} e^{u_n}}{\int_\Sigma \tilde{K} e^{u_n} dV_g} \rightharpoonup \sum_{j=1}^{r} \beta_j \delta_{x_j} \qquad \text{weakly in the sense of measure,}$$

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In particular,
$\lambda = \sum_{j=1}^{r} \beta_j \in \left\{ 8\pi r + \sum_{j=1}^{m} 8\pi(1 + \alpha_j) n_j \,|\, r \in \mathbb{N} \cup \{0\}, n_j \in \{0, 1\} \right\} \setminus \{0\}$.

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Theorem [Bartolucci-Tarantello, 2002], ([Brezis-Merle, 1991], [Li-Shafrir, 1994])

If $K \in C^0(\Sigma)$ and $K > 0$, the set of solutions of the problem is compact if

$$\lambda \notin \Lambda_m = \left\{ 8\pi r + \sum_{j=1}^{m} 8\pi(1+\alpha_j)n_j \,|\, r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}.$$

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These values are related with the integral of the entire solutions

$$-\Delta u = |x|^{2\alpha} e^u \text{ in } \mathbb{R}^2 \qquad \text{such that } \int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx < C,$$

which satisfies that

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx = 8\pi(1+\alpha),$$

[Prajapat-Tarantello, 2001], [Chen-Li, 1991].

Some previous results

Theorem [Bartolucci-De Marchis-Malchiodi, 2011]

If $\chi(\Sigma) \leq 0$, then for any positive $K \in C^0(\Sigma)$, the mean field problem admits a solution for any $\lambda \in (8\pi, +\infty) \setminus \Lambda_m$.

Theorem [Malchiodi-Ruiz, 2011]

If $\Sigma = \mathbb{S}^2$, $\Theta_\lambda = \{p_j \in \Sigma : \lambda < 8\pi(1+\alpha_j)\}$ and $K > 0$. Let $\lambda \in (8\pi, 16\pi) \setminus \Lambda_m$, $\alpha_j \in (0, 1]$ and $|\Theta_\lambda| \neq 1$, the mean field problem admits a solution.

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Non existence result [Bartolucci-Lin-Tarantello, 2011]

If $\Sigma = \mathbb{S}^2$, $K \equiv 1$, $m = 1$, $\alpha_1 > 0$, $\lambda \in (8\pi, 8\pi(1+\alpha_1))$ then $(*)$ does not admit a solution.

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Other results with $K > 0$

Leray-Schauder degree: [Chen-Lin, 2003, 2015] Blowing–up solutions: [Esposito-Grossi-Pistoia, 2005], [Chen-Lin, 2015] Generic Multiplicity: [De Marchis, 2010], [Bartolucci-De Marchis-Malchiodi, 2011]

2. Our contribution:

The singular mean field problem with sign changing potentials.

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We study the existence of solutions for the mean field type problem

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As far as we know, this case has not much been considered in the literature. For that reason we analyze some of the most fundamental questions in the analysis of PDEs:

Existence and compactness of solutions.

Our hypotheses

Let Σ be a compact surface without boundary, consider the problem

$$-\triangle_g u = \lambda \left(\frac{\tilde{K} e^u}{\int_\Sigma \tilde{K} e^u \, dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma, \tag{1}$$

where $\lambda > 0$, $\tilde{K} = K e^{-h_m}$ and K and the singular points p_i's verify

(H1) K is a sign changing $C^{2,\alpha}$ function with $\nabla K(x) \neq 0$ for any $x \in \Sigma$ with $K(x) = 0$.

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(H1) K is a sign changing $C^{2,\alpha}$ function with $\nabla K(x) \neq 0$ for any $x \in \Sigma$ with $K(x) = 0$.

Let us define

$$\Sigma^+ = \{x \in \Sigma : K(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma : K(x) < 0\}, \quad \Gamma = \{x \in \Sigma : K(x) = 0\}.$$

By (H1), Γ is a union of regular curves.

Our hypotheses

Let Σ be a compact surface without boundary, consider the problem

$$-\triangle_g u = \lambda \left(\frac{\tilde{K} e^u}{\int_\Sigma \tilde{K} e^u \, dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma, \tag{1}$$

where $\lambda > 0$, $\tilde{K} = K e^{-h_m}$ and K and the singular points p_i's verify

(H1) K is a sign changing $C^{2,\alpha}$ function with $\nabla K(x) \neq 0$ for any $x \in \Sigma$ with $K(x) = 0$.

Let us define

$$\Sigma^+ = \{x \in \Sigma : K(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma : K(x) < 0\}, \quad \Gamma = \{x \in \Sigma : K(x) = 0\}.$$

By (H1), Γ is a union of regular curves.

(H2) $\{p_1, \dots, p_\ell\} \subset \Sigma^+$ and $\{p_{\ell+1}, \dots, p_m\} \subset \Sigma^-$.

Therefore $p_j \notin \Gamma$ for all $j \in \{1, \dots, m\}$.

A strategy to find critical points of I_λ of saddle type

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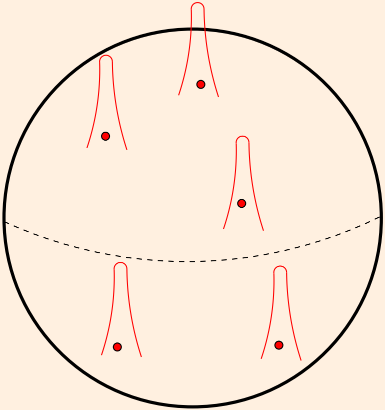

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- Find $\mathcal{Z}$ compact and non contractible and construct Φ and Ψ s.t. for some large L enough s.t.

$$\mathcal{Z} \xrightarrow{\Phi} \{I_\lambda \leq -L\} \xrightarrow{\Psi} \mathcal{Z} \qquad \text{s.t.} \quad \Psi \circ \Phi \simeq \mathrm{Id}_{|\mathcal{Z}}$$

  then $\Phi(\mathcal{Z})$ is not contractible in $\{I_\lambda \leq -L\}$. As a consequence, the functional has a min-max geometry.

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This approach has been used in: [Djadli, 2008], [Carlotto-Malchiodi, 2012], [Malchiodi-Ruiz, 2012], [Battaglia-Jevnikar-Malchiodi-Ruiz, 2015], [LS-Ruiz, 2016], [Jevnikar-Yang, 2017]...

Compactness

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$$\int_\Sigma |\tilde{K}| e^u \, dV_g \tag{2}$$

is uniformly bounded or not. By standard regularity results, this would give a priori $W^{1,p}$ estimates ($p \in (1,2)$) on u. Instead, integrating the mean field problem, we know that $\int_\Sigma \tilde{K} e^u$ is bounded. Observe that if $K > 0$, then (2) holds directly.

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- Our strategy is obtain uniform integral estimates, which allow one to derive a priori estimates in the region $\{\, K(x) < 0\}$. Then we obtain a priori estimates in a neighborhood of the region $\{\, K(x) = 0\}$. Finally, we can apply the classical compactness-quantization results in the region $\{\, K(x) > 0\}$

Compactness

- Recall that, given a positive function K, the set of solutions is compact if λ does not belong to the critical set Λ_m. For the sign changing case, consider the set

$$\Lambda_\ell = \left\{ 8\pi r + \sum_{j=1}^{\ell} 8\pi(1+\alpha_j)n_j : \ r \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\} \right\} \setminus \{0\}. \tag{3}$$

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Theorem, [De Marchis-LS-Ruiz, 2016]

Assume that $\alpha_1, \ldots, \alpha_m > -1$ and let K s.t. (H1) and (H2) are satisfied, then the set of solutions of the problem (1) is compact if $\lambda \notin \Lambda_\ell$.

Remark

The assumption (H1) is necessary. Otherwise, there are some examples of blowing-up solutions. [Borer-Galimberti-Struwe, 2015], [Del Pino-Román, 2015], [Struwe, 2017]

A sketch of the proof of the compactness theorem

Let u_n be a solution of (1) with $\lambda = \lambda_n$:

0. By the Kato's inequality, we obtain that

$$\|u_n^- - \fint_\Sigma u_n^-\|_{L^p} \leq C \quad \text{for any } p \in [1, +\infty) \quad \text{and} \quad u_n^- - \fint_\Sigma u_n^- \geq -C.$$

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3. As a consequence of the estimates obtained by the Kato's inequality,

$$u_n(x_0) - u_n(x_1) < C \qquad \text{where} \quad K(x_0) < 0 \text{ and } x_1 \in \Sigma. \tag{6}$$

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$$u_n(x) + C_1 \geq u_n(x_0), \text{ for } x_0 \in \{K \leq \varepsilon\} \text{ and } x \in \Delta_{x_0} \subset \overline{\Sigma}^+, \tag{7}$$

   where $\varepsilon > 0$ and Δ_{x_0} is a cone with vertex at x_0.
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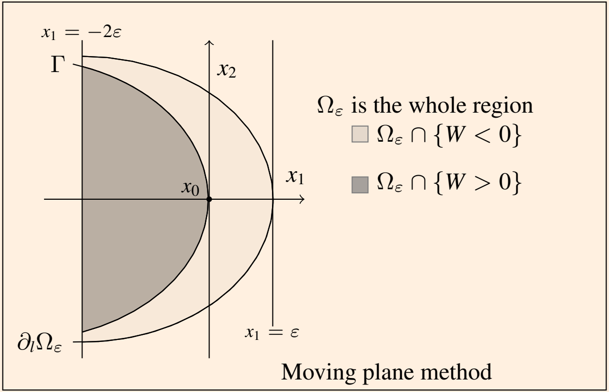

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Open Question

Extend the compactness theorem to surfaces with boundary (Dirichlet or Neumann condition)

Two existence results

Theorem 1 [De Marchis-LS, 2016] [De Marchis-LS-Ruiz, 2016]

Let $\alpha_1, \ldots, \alpha_\ell > 0$, and $\lambda \in (8k\pi, 8(k+1)\pi) \setminus \Lambda_\ell$. Assume (H1), (H2) and

(H3) $N^+ := \#\{\text{connected components of } \Sigma^+\} > k$ or Σ^+ has a connected component which is not simply connected,

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For the special case $k = 1$, we obtain

Theorem 2 [De Marchis-LS, 2016] [De Marchis-LS-Ruiz, 2016]

Let $\alpha_1, \ldots, \alpha_\ell \geq 0$ and $\lambda \in (8\pi, 16\pi) \setminus \Lambda_\ell$. Assume (H1), (H2) and

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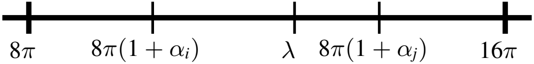

In this situation $p_i \notin \Theta_\lambda$ and $p_j \in \Theta_\lambda$.

Description of the low sublevels, $\lambda \in (8k\pi, 8(k+1)\pi)$

The problem (1) is the Euler-Lagrange equation of the energy functional

$$I_\lambda(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g + \frac{\lambda}{|\Sigma|}\int_\Sigma u\, dV_g - \lambda \log \int_\Sigma \tilde{K} e^u dV_g,$$

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Assume (H1), (H2) and $\lambda \in (8k\pi, 8(k+1)\pi)$. Let $u_n \in X$, if $I_\lambda(u_n) \to -\infty$, then

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We can retract $\overline{\Sigma^+}$ to a compact set $Z \subset \Sigma^+ \setminus \{p_1, \cdots, p_m\}$.

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Applying Proposition 1 and using the retraction from $\overline{\Sigma^+}$ onto Z, we prove that.

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Assume (H1), (H2). Then for $L > 0$ sufficiently large there exists a continuous projection

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For $\mu > 0$ and $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in Bar_k(Z)$, we define

$$\Phi_\mu : Bar_k(Z) \to I_\lambda^{-L}, \quad \Phi_\mu(\sigma) = \varphi_{\mu,\sigma}(x) = \log \sum t_i \left(\frac{\mu}{1 + (\mu d(x, x_i))^2} \right)^2,$$

Lemma

Given $L > 0$ there exists $\mu(L) > 0$ such that for $\mu \geq \mu(L)$, $I_\lambda(\varphi_{\mu,\sigma}) < -L$;

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- Following the ideas of [Chen-Li, 1995] for the Nirenberg problem, we are able to show some cases of non existence if $k \geq N^+$ and $\Theta_\lambda = \emptyset$.

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Remark

We can say that both existence theorems are somehow sharp.

3. The problem with negative orders

Existence results for $\alpha_j < 0$ & $K > 0$

Theorem [Carlotto-Malchiodi, 2012] (using [Bartolucci-Montefusco, 2007])

Let $\lambda \in (8\pi(1 + \min_j \alpha_j), +\infty) \setminus \Lambda_m$, then $(*)_\lambda$ admits a solution if $Bar_{\lambda,\underline{\alpha}}(\Sigma)$ is not contractible, where

$$Bar_{\lambda,\underline{\alpha}}(\Sigma) = \left\{ \sum_{q_j \in J} t_j \delta_{q_j} : \sum_{q_j \in J} t_j = 1,\ t_j \geq 0,\ q_j \in \Sigma,\ 8\pi \sum_{q_j \in J} \xi(q_j) < \lambda \right\},\quad \xi(q_j) = \begin{cases} 1 + \alpha_i & \text{if } q_j = p_i \\ 1 & \textit{otherwise} \end{cases}$$

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Algebraic conditions for the solvability [Carlotto, 2014]

$Bar_{\lambda,\underline{\alpha}}(\Sigma)$ is not contractible if and only if there exist a number $k \in \mathbb{N}$ and a set $I \subset \{1, 2, \ldots, m\}$, possibly empty, such that $k + \mathrm{card}(I) > 0$ and

$$\lambda > 8\pi \left[k + \sum_{i \in I} (1+\alpha_i) \right] \ \wedge\ \lambda < 8\pi \left[k + \sum_{i \in \{1\} \cup I} (1+\alpha_i) \right].$$

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Algebraic conditions for the solvability [Carlotto, 2014]

$Bar_{\lambda,\underline{\alpha}}(\Sigma)$ is not contractible if and only if there exist a number $k \in \mathbb{N}$ and a set $I \subset \{1, 2, \ldots, m\}$, possibly empty, such that $k + \mathrm{card}(I) > 0$ and

$$\lambda > 8\pi \left[k + \sum_{i \in I} (1+\alpha_i) \right] \wedge \lambda < 8\pi \left[k + \sum_{i \in \{1\} \cup I} (1+\alpha_i) \right].$$

Remark

$$\lambda_{geom} = 4\pi(\chi(\Sigma) + \textstyle\sum_j \alpha_j) < 8\pi \qquad \text{and} \qquad \lambda_{crit} = 8\pi(1+\min_j \alpha_j) \in (0, 8\pi).$$

Therefore $\lambda_{geom} > \lambda_{crit}$ only if $\chi(\Sigma) = 2$ and $\frac{1}{2}\sum_j \alpha_j > \min_j \alpha_j$.

Work in progress for $\alpha_j < 0$ & K sign-changing

[De Marchis-Kallel-LS, w.i.p.] using ([De Marchis-LS-Ruiz, 2016])

Let K sign-changing, $K \in C^{2,\alpha}(\Sigma)$, $\nabla K \neq 0$ in $\{K = 0\}$ and $p_j \notin \{K = 0\}$.
Let $\lambda \in (\pi(1 + \min_j \alpha_j), +\infty) \setminus \Lambda_\ell$, then $(*)_\lambda$ admits a solution if $Bar_{\lambda,\underline{\alpha}}(\Sigma^+)$ is not contractible, where

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Algebraic conditions for the solvability [De Marchis-Kallel-LS, w.i.p.]

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Algebraic conditions for the solvability [De Marchis-Kallel-LS, w.i.p.]

? ? ?

Remark

Since $\alpha_j > -1$, then $\lambda_{geom} = 4\pi(\chi(\Sigma) + \sum_j \alpha_j)$ can be arbitrarily large.

4. Remarks and open problems

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- What happens if $K \geq 0$? Could the solution blow–up at minimum?
- Is it possible to obtain an analogous result for the Toda system or other Liouville type systems?

$$\begin{cases} -\Delta u_1 = 2\tilde{K}_1 e^{u_1} - \tilde{K}_2 e^{u_2}, & \text{in } \Sigma, \\ -\Delta u_2 = 2\tilde{K}_2 e^{u_2} - \tilde{K}_1 e^{u_1}, & \text{in } \Sigma, \end{cases}$$

with K_1, K_2 sign changing.

Than you for your attention!