Prescribing Gaussian curvature on compact surfaces and geodesic curvature on its boundary

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Joint work with R. López Soriano and A. Malchiodi

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Outline

The problem

Let Σ be a orientable compact surface with boundary. In this talk we consider the problem of prescribing the Gaussian curvature of Σ and the geodesic curvature of $\partial\Sigma$ via a conformal change of the metric.

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This question leads us to the boundary value problem:

$$\begin{cases} -\Delta u + 2\tilde{K}(x) = 2K(x)e^u, & x \in \Sigma, \\ \dfrac{\partial u}{\partial \nu} + 2\tilde{h}(x) = 2h(x)e^{u/2}, & x \in \partial\Sigma. \end{cases}$$

Here e^u is the conformal factor, ν is the normal exterior vector and

1. $\tilde{K}$, $\tilde{h}$ are the original Gaussian and geodesic curvatures, and
2. K, h are the Gaussian and geodesic curvatures to be prescribed.

Antecedents

- The higher order analogue: prescribing scalar curvature S on Σ and mean curvature H on $\partial\Sigma$.

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- The case $h = 0$: Chang-Yang.

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- The case of constants K, h:

  A parabolic flow converges to constant curvatures (Brendle).

  Classification of solutions in the annulus (Jiménez).

  Classification of solutions in the half-plane (Li-Zhu, Zhang, Gálvez-Mira).

Preliminaries

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By the Gauss-Bonnet Theorem,

$$\int_{\Sigma} Ke^{u} + \oint_{\partial\Sigma} he^{u/2} = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

It is easy to show that we can prescribe $h = 0$, $K = sgn(\chi(\Sigma))$. Then:

$$\begin{cases} -\Delta u + 2\tilde{K} = 2K(x)e^{u}, & x \in \Sigma, \\ \dfrac{\partial u}{\partial \nu} = 2h(x)e^{u/2}, & x \in \partial\Sigma, \end{cases}$$

where $\tilde{K} = sgn(\chi(\Sigma))$.

In this talk we are interested in the case of negative K and $\chi(\Sigma) \leq 0$.

The energy functional

The associated energy functional is given by $I : H^1(\Sigma) \to \mathbb{R}$,

$$I(u) = \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + 2\tilde{K}u + 2|K(x)|e^u \right) - 4 \oint_{\partial\Sigma} he^{u/2}.$$

For the statement of our results it will be convenient to define the function $\mathfrak{D} : \partial\Sigma \to \mathbb{R}$,

$$\mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}.$$

The function $\mathfrak{D}$ is scale invariant.

A trace inequality

Proposition

For any $\varepsilon > 0$ *there exists* $C > 0$ *such that:*

$$4\int_{\partial\Sigma} h(x)e^{u/2} \leq \left(\varepsilon + \max_{p\in\partial\Sigma} \mathfrak{D}^{+}(p)\right)\left[\int_{\Sigma} \frac{1}{2}|\nabla u|^2 + 2|K(x)|e^{u}\right] + C.$$

In particular, if $\mathfrak{D}(p) < 1\ \forall\, p \in \partial\Sigma$, then I is bounded from below.

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Assume that $h > 0$ is constant, and take N a vector field in Σ such that $N(x) = \nu(x)$ on the boundary, $|N(x)| \leq 1$. Then,

$$4 \int_{\partial\Sigma} h e^{u/2} = 4 \int_{\partial\Sigma} h e^{u/2} N(x) \cdot \nu(x)$$

$$= \ 4 \int_\Sigma h e^{u/2} \left[div \, N + \frac{1}{2} \nabla u \cdot N \right] \leq \ C \int_\Sigma e^{u/2} + 2 \int_\Sigma h e^{u/2} |\nabla u|$$

$$\leq C \int_\Sigma e^{u/2} + 2 \int_\Sigma h^2 e^u + \frac{1}{2} \int_\Sigma |\nabla u|^2.$$

The case $\chi(\Sigma) < 0$

Theorem

Assume that $\chi(\Sigma) < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then the functional I is coercive and attains its infimum.

By the trace inequality,

$$I(u) \geq \int_\Sigma \varepsilon|\nabla u|^2 + 2\varepsilon|K(x)|e^u + 2\tilde{K}u - C.$$

Since $\tilde{K} < 0$, $\lim_{u\to\pm\infty} 2\varepsilon|K(x)|e^u + 2\tilde{K}u = +\infty$, so I is coercive.

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If $\chi(\Sigma) = \tilde{K} = 0$, the above reasoning gives you boundedness from below but not coercivity! The reason is that $\int_\Sigma u_n$ could go to $-\infty$ for a minimizing sequence u_n.

Minimizers for $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$ and:

1. $\mathfrak{D}(p) < 1$ *for all* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h > 0$.

Then I attains its infimum.

Observe that if $u_n = -n$, then: $I(u_n) = \int_\Sigma 2|K(x)|e^{-n} - 4\oint_{\partial\Sigma} he^{-n/2} \nearrow 0$.

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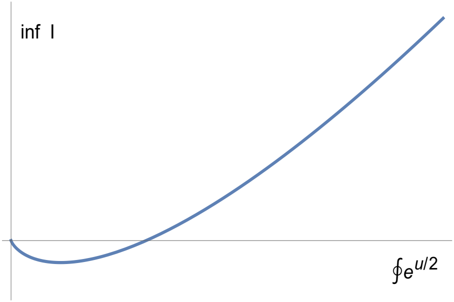


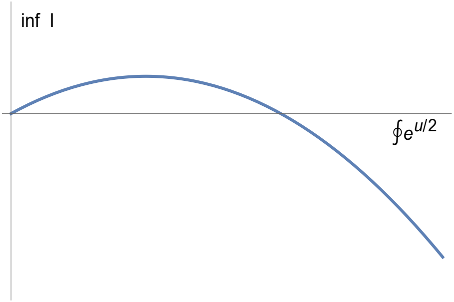
Min-max for $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$ and:

1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h < 0$.

Then I has a mountain-pass geometry.



Obstructions to existence

Proposition (Jiménez 2012)

If Σ is an cylinder and $K = -1$, h_1 and h_2 are constants, then our problem is solvable iff

1. $h_1 + h_2 > 0$ *and both* $h_i < 1$ *(minima).*
2. $h_1 + h_2 < 0$ *and some* $h_i > 1$ *(mountain-pass).*
3. $h_1 = 1$, $h_2 = -1$ *or viceversa.*

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Proposition (Rosenberg 2006, Sun 2009)

Let Σ be a compact surface with boundary, and assume that $h(p) > \sqrt{|K^-(q)|}$ for all $p \in \partial\Sigma$, $q \in \Sigma$. Then $\partial\Sigma$ is connected.

Blow-up analysis

Here the (PS) condition is not known to hold. But we can bypass this problem by using the monotonicity trick of Struwe, if we can guarantee compactness of solutions.

Let u_n be a blowing-up sequence (namely, $\sup\{u_n(x)\} \to +\infty$) of solutions to the problem:

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \dfrac{\partial u_n}{\partial \nu} = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma. \end{cases} \tag{1}$$

Here $\tilde{K}_n \leq 0$, $\tilde{K}_n \to \tilde{K}$, $K_n \to K$, $h_n \to h$ in C^1 sense, with $K < 0$. By integrating:

$$\int_\Sigma K_n e^{u_n} + \oint_{\partial\Sigma} h_n e^{u_n/2} \to 2\pi\chi(\Sigma),$$

Hence there could be compensation of diverging masses!!

A classification result in the half-plane

Theorem (Zhang 2003, Gálvez-Mira 2009)

Let u be a solution of:

$$\begin{cases} -\Delta u = 2K_0 e^u, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2h_0 e^{u/2}. & \text{in } \partial\mathbb{R}^2_+, \end{cases} \implies \begin{cases} -\Delta u = -2e^u, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2\mathfrak{D}_0 e^{u/2} & \text{in } \partial\mathbb{R}^2_+. \end{cases}$$

with $\mathfrak{D}_0 = \dfrac{h_0}{\sqrt{|K_0|}}$. *Then the following holds:*

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with $\mathfrak{D}_0 = \dfrac{h_0}{\sqrt{|K_0|}}$. Then the following holds:

- *If $\mathfrak{D}_0 < 1$ there is no solution.*
- *If $\mathfrak{D}_0 = 1$ the only solutions are:*

$$u(s,t) = 2\log\left(\frac{\lambda}{1+\lambda t}\right), \ \lambda > 0, \ s \in \mathbb{R}, \ t \geq 0.$$

A classification result in the half-plane

- *If $\mathfrak{D}_0 > 1$, then:*

$$u(z) = 2\log\left(\frac{2|g'(z)|}{1-|g(z)|^2},\right),$$

where g is locally univalent holomorphic map from $\mathbb{R}^2_+$ to a disk of geodesic curvature $\mathfrak{D}_0$ in the Poincaré disk $\mathbb{H}^2$. For instance, to $B(0,R)$ with $\mathfrak{D}_0 = \frac{1+R^2}{2R}$.

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Moreover, g is a Möbius transform if and only if

$$\text{either } \int_{\mathbb{R}^2_+} e^u < +\infty \text{ and / or } \oint_{\partial\mathbb{R}^2_+} e^{u/2} < +\infty.$$

In such case u can be written as:

$$u(s,t) = 2\log\left(\frac{2\lambda}{(s-s_0)^2+(t+t_0)^2-\lambda^2}\right), t \geq 0,$$

where $\lambda > 0$, $s_0 \in \mathbb{R}$, $t_0 = \mathfrak{D}_0\lambda$. We call these solutions "bubbles".

Passing to a limit problem in the half-plane

Let us define the singular set, as Brezis-Merle:

$$S = \{p \in \Sigma : \ \exists y_n \in \Sigma, \ y_n \to p, \ u_n(y_n) \to +\infty\}.$$

Proposition

Let $p \in S$. Then there exists $x_n \in \Sigma$, $x_n \to p$ such that, after a suitable rescaling, we obtain a solution of the problem in the half-plane in the limit.

In particular $S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1\}$.

In [Brezis-Merle, 1991] this is done via a key integral estimate which, together with the finite mass assumption, implies that S is finite.

Here the idea is to choose a **good sequence** x_n, even if they are not local maxima!

Choosing good sequences

Let $p \in S$, $y_n \in \Sigma$ with $y_n \to p$ and $u_n(y_n) \to +\infty$, and define:

$$\varphi_n = e^{-\frac{u_n}{2}}, \quad \varepsilon_n = e^{-\frac{u_n(y_n)}{2}} \to 0.$$

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By Ekeland variational principle there exists a sequence $x_n \subset \Sigma$ such that

- $e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(y_n)}{2}}$,
- $d(x_n, y_n) \leq \sqrt{\varepsilon_n}$,
- $e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(z)}{2}} + \sqrt{\varepsilon_n}\, d(x_n, z)$ for every $z \in \Sigma$.

The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.

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The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.

- Since $K(p) < 0$, there is no entire solution of $-\Delta u = 2K(p)e^u$ in $\mathbb{R}^2$.
- Hence p in $\partial\Sigma$, the limit problem is posed in a half-plane and $\mathfrak{D}(p) \geq 1$.

Compensation

Proposition

There exists a positive unit measure σ on $\partial\Sigma$ such that:

$$\frac{|K_n|e^{u_n}}{\int_\Sigma |K_n|e^{u_n}} \rightharpoonup \sigma, \ \frac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} |h|e^{u_n/2}} \rightharpoonup \sigma.$$

Multiplying (4) by $\phi \in C^2(\Sigma)$ and integrating:

$$2\oint_{\partial\Sigma} h_n e^{u_n/2}\phi - 2\int_\Sigma |K_n|e^{u_n}\phi = \underbrace{2\int_\Sigma \tilde{K}_n\phi}_{O(1)} + \int_\Sigma u_n\Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu}u_n,$$

and we can show that $\int_\Sigma u_n\Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu}u_n << \int_\Sigma e^{u_n} + \oint_{\partial\Sigma} e^{u_n/2}$.

On the support of σ

Clearly $supp\ \sigma \subset S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}$. Moreover, we have:

Proposition

The support of σ is contained in the set $\{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}$.

The proof consists in multiplying the equation by $\nabla u_n \cdot F$, where F is a tangential vector field.

However the proof is a bit tricky since we do not have information on the asymptotic behavior of the term $\int_\Sigma |\nabla u_n|^2$.

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This is all the information that we have without further hypotheses on u_n.

Morse index

From now on we assume that the sequence of solutions u_n has bounded Morse index.

This is the case, for instance, of solutions coming from usual variational methods.

Morse index bounds is a typical assumption in the context of minimal surfaces, but not in this type of PDE's.

If u_n has bounded Morse index, the solutions of the limit problem obtained by rescaling have finite Morse index.

Morse index of the limit problem

Theorem

Let u be a solution of the problem:

$$\begin{cases} -\Delta u = -2e^{u} & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2\mathfrak{D}_0 e^{u/2} & \text{in } \partial\mathbb{R}^2_+. \end{cases} \tag{2}$$

Define:

$$Q(\psi) = \int_{\mathbb{R}^2_+} |\nabla \psi|^2 + 2\int_{\mathbb{R}^2_+} e^{u}\psi^2 - \mathfrak{D}_0 \int_{\partial\mathbb{R}^2_+} e^{u/2}\psi^2, \text{ and}$$

$$ind(v) = \sup\{dim(E) : \ E \subset C_0^{\infty}(\mathbb{R}^2_+) \text{ vector space}, \ Q(\psi) < 0 \ \forall \, \psi \in E\}.$$

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1. *If $\mathfrak{D}_0 = 1$, then $ind(u) = 0$, that is, u is stable.*
2. *If $\mathfrak{D}_0 > 1$ and u is a bubble, then $ind(u) = 1$. Otherwise, $ind(u) = +\infty$.*

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If $\mathfrak{D}_0 > 1$, then we pass to the problem posed in $B(0,R) \subset \mathbb{H}^2$:

$$\begin{cases} -\Delta\gamma + 2\gamma = 0, & \text{in } B(0,R), \\ \dfrac{\partial \gamma}{\partial \nu} = \lambda\gamma, & \text{in } \partial B(0,R). \end{cases} \tag{3}$$

The Morse index is the number of eigenvalues λ smaller than $\mathfrak{D}_0$.

- The functions $\gamma_i(z) = \frac{z_i}{1-|z|^2}$ solve (3) with $\lambda = \mathfrak{D}_0$.

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- For a convenient cut-off ϕ, $\psi = \phi(g \circ \gamma)$ satisfies $Q(\psi) < 0$.

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- The function $\gamma(z) = \frac{1+|z|^2}{1-|z|^2}$ solves (3) with $\lambda = \frac{1}{\mathfrak{D}_0}$.
- For a convenient cut-off ϕ, $\psi = \phi(g \circ \gamma)$ satisfies $Q(\psi) < 0$.
- If moreover $\oint_{\partial\mathbb{R}^2_+} e^{u/2} = +\infty$ we can choose ϕ to be 0 outside any arbitrary compact set.

A blow-up analysis

Theorem

Let u_n be a blowing-up sequence of solutions to the problem:

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \dfrac{\partial u_n}{\partial \nu} = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma. \end{cases} \tag{4}$$

Here $\tilde{K}_n \leq 0$, $\tilde{K}_n \to \tilde{K}$, $K_n \to K$, $h_n \to h$ in C^1 sense, with $K < 0$. Then

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1. $S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1\}$.
2. $\int_\Sigma |K_n| e^{u_n} \to +\infty$, $\oint_{\partial\Sigma} h_n e^{u_n/2} \to +\infty$, *and*

   a) $\dfrac{|K_n| e^{u_n}}{\int_\Sigma |K_n| e^{u_n}} \rightharpoonup \sigma$, $\dfrac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} h_n e^{u_n/2}} \rightharpoonup \sigma$.

   b) $supp\ \sigma \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1, \ \mathfrak{D}_\tau(p) = 0\}$.

A blow-up analysis

Theorem

Let u_n be a blowing-up sequence of solutions to the problem:

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \dfrac{\partial u_n}{\partial \nu} = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma. \end{cases} \tag{4}$$

Here $\tilde{K}_n \leq 0$, $\tilde{K}_n \to \tilde{K}$, $K_n \to K$, $h_n \to h$ in C^1 sense, with $K < 0$. Then

1. $S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1\}$.
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   a) $\dfrac{|K_n|e^{u_n}}{\int_\Sigma |K_n|e^{u_n}} \rightharpoonup \sigma$, $\dfrac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} h_n e^{u_n/2}} \rightharpoonup \sigma$.
   b) $supp\ \sigma \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1, \ \mathfrak{D}_\tau(p) = 0\}$.
3. *Moreover, if $ind(u_n)$ is bounded,*

$$S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) = 1, \ \mathfrak{D}_\tau(p) = 0\}.$$

Min-max for $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$ and:

1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h < 0$.

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1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h < 0$.
3. $\mathfrak{D}_\tau(p) \neq 0$ *for any* $p \in \partial\Sigma$ *with* $\mathfrak{D}(p) = 1$.

Then I has a mountain-pass critical point.

In the proof we combine the monotonicity trick of Struwe ([Struwe 1985]), with boundedness of the Morse index for variational solutions ([Fang-Ghoussoub 1992, 94]).

Explicit examples of blow-up

Let us consider the problem:

$$\begin{cases} -\Delta u = -2e^u, & \text{in } A(0; r, 1), \\ \frac{\partial u}{\partial \nu} + 2 = 2h_1 e^{u/2}, & \text{on } |x| = 1, \\ \frac{\partial u}{\partial \nu} - \frac{2}{r} = 2h_2 e^{u/2}, & \text{on } |x| = r. \end{cases}$$

The classification of all solutions to this problem has been done in [Jiménez, 2012].

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For example, the function:

$$u(x) = \log\left(\frac{4}{|x|^2(\lambda + 2\log|x|)^2}\right), \quad \text{for any } \lambda < 0,$$

is a solution with $h_1 = 1$ and $h_2 = -1$. Observe that if λ tends to 0 then u blows up at a whole component of the boundary.

The singular set $S = \{|x| = 1\}$ is not finite.

A second example

Given any $h_1 > 1$, $\gamma \in \mathbb{N}$, there exists a explicit solution:

$$u_\gamma(z) = 2\log\left(\frac{\gamma|z|^{\gamma-1}}{h_1 + Re(z^\gamma)}\right),$$

where $h_2 = -h_1 r^{-\gamma}$.

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The asymptotic profile is:

$$u(s,t) = 2\log\left(\frac{e^{-t}}{h_1 + e^{-t}\cos s}\right),$$

defined in the half-plane $\{t \geq 0\}$. This is indeed a solution to the limit problem in the half-space with $K = -1$ and $h_1 > 1$, with infinite Morse index.

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5. Higher-order analogue.

Thank you for your attention!