# Uniqueness of solutions to singular Liouville equations

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**Physical, Geometrical and Analytical Aspects of Mean Field Systems of Liouville Type**

BIRS, April 1-6, 2018

# Introduction

We consider first the following **singular Liouville** equation on a smooth bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \Delta u + \rho \dfrac{e^u}{\int_\Omega e^u \, dx} = 4\pi \sum_{j=1}^{N} \alpha_j \delta_{p_j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

- $\rho \in \mathbb{R}$ is a real parameter
- $\{p_1, \dots, p_N\} \subset \Omega$ and $\alpha_j > -1$ for $j = 1, \dots, N$.

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Motivations:

Mean field equation in statistical mechanics: turbulent Euler flows, self-gravitating systems.

# Introduction

There are by now many results concerning existence and multiplicity, blow-up phenomena and **uniqueness of solutions**.

We deduce here new uniqueness results (both on bounded domains and on spheres) as well as new self-contained proofs of previously known results.

# Introduction

The problem has an **equivalent formulation**: consider

$$-\Delta G_p(x) = \delta_p \text{ in } \Omega, \qquad G_p(x) = 0 \text{ on } \partial\Omega,$$

and the following substitution

$$u(x) \mapsto u(x) + 4\pi \sum_{j=1} \alpha_j G_{p_j}(x).$$

Then, we have

$$\Delta u + \rho \frac{h(x) e^u}{\int_\Omega h(x) e^u \, dx} = 0 \quad \text{in } \Omega,$$

where

$$h(x) = e^{-4\pi \sum_j \alpha_j G_{p_j}(x)},$$

$$h > 0 \text{ on } \Omega \setminus \{p_1, \dots, p_N\}, \qquad h(x) \simeq |x - p_j|^{2\alpha_j} \text{ near } p_j.$$

# Introduction

The latter problem has a **variational structure** and the solutions correspond to critical points of the functional

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \rho \log \int_\Omega h e^u \, dx, \qquad u \in H_0^1(\Omega).$$

The starting point in treating this kind of functionals is the following:

- **Regular case** $N = 0$**.**

**Moser-Trudinger inequality**

$$8\pi \log \int_\Omega e^u \, dx \leq \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + C, \qquad u \in H_0^1(\Omega).$$

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The starting point in treating this kind of functionals is the following:

- Singular case $N > 0$.

**Troyanov inequality**

$$8\pi(1+\alpha_-) \log \int_\Omega h e^u\,dx \le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + C, \qquad u \in H_0^1(\Omega).$$

$$\alpha_- = \min_j\{0, \alpha_j\}.$$

# Introduction

It follows that for:

- $\rho < 8\pi(1+\alpha_-)$ the functional $J_\rho$ is **bounded from below and coercive** and solutions can be found as global minima
- $\rho > 8\pi(1+\alpha_-)$ the functional $J_\rho$ is **unbounded from below** and one has to attack it for example by using either min-max or degree theory

  [D. Bartolucci, C.C. Chen, C.S. Lin, A. Malchiodi, G. Tarantello...]
- $\rho = 8\pi(1+\alpha_-)$ the problem is subtler since the functional $J_\rho$ is **bounded from below** but not coercive

  [D. Bartolucci, S.Y.A. Chang, C.C. Chen, C.S. Lin...]

# Introduction

Roughly speaking, the bigger is $\rho$ and the richer is the topology of $\Omega$, **the higher** is the number of solutions (by Morse theory).

[F. De Marchis]

On the other hand, for $\rho$ small and $\Omega$ simply-connected one expects to have **uniqueness of solutions**.

# Previous results for bounded domain case

Indeed, **uniqueness** holds for:

**Regular case $N = 0$.**

- $\rho < 8\pi$, $\Omega$ simply-conn. [T. Suzuki]
- $\rho \leq 8\pi$, $\Omega$ simply-conn. [S.Y.A. Chang - C.C. Chen - C.S. Lin]
- $\rho \leq 8\pi$, $\Omega$ multiply-conn. [D. Bartolucci - C.S. Lin]

**Singular case $N > 0$.**

- $\alpha_j > 0 \ \forall j$: $\rho \leq 8\pi$, $\Omega$ simply-conn. [D. Bartolucci - C.S. Lin]
- $\alpha_1 \in (-1, 0)$, $\alpha_j > 0 \ \forall j > 1$: $\rho \leq 8\pi(1 + \alpha_1)$, $\Omega$ simply-conn. [J. Wei - L. Zhang]
- multiple negative sources: **missing**.

The argument is based on a non-trivial **eigenvalue analysis** for Liouville-type linearized problems showing uniqueness of the branch of solutions.

# First main result

Take $\alpha_j > -1, \ j = 1, \dots, N$ (positive or negative) and let

$$\alpha = \sum_{j \in \mathcal{N}} \alpha_j, \quad \mathcal{N} = \Big\{ j \in \{1, \dots, N\} \, : \, \alpha_j \in (-1, 0) \Big\}.$$

**Theorem** [Bartolucci-Gui-J.-Moradifam]

Let $\Omega \subset \mathbb{R}^2$ be bounded, smooth and $\rho \leq 8\pi(1+\alpha)$. Then, there exists **at most one solution**.

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Let $\Omega \subset \mathbb{R}^2$ be bounded, smooth and $\rho \leq 8\pi(1+\alpha)$. Then, there exists **at most one solution**.

**Remarks.**

1. In particular observe that the result holds for $\Omega$ multiply-conn.
2. It covers all the previously known results.
3. Whenever the coercivity condition $\rho < 8\pi(1+\alpha_-)$ is also satisfied, we have **existence** and uniqueness.

**Open problem.**
Does uniqueness still hold for $\rho \in \Big(8\pi(1+\alpha), 8\pi(1+\alpha_-)\Big)$?

# First main result

The result holds for more general problems. Recall,

$$\begin{cases} \Delta u + \rho \dfrac{e^u}{\int_\Omega e^u \, dx} = 4\pi \sum_{j=1}^{N} \alpha_j \delta_{p_j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \rho \le 8\pi(1+\alpha), \ \alpha = \sum_{\mathcal{N}} \alpha_j .$$

We can consider

$$\begin{cases} \Delta u + \rho \dfrac{e^u}{\int_\Omega e^u \, dx} = \boxed{4\pi \mu} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \rho \le 8\pi(1+\alpha), \ \boxed{\alpha = -\mu_-(\Omega)}.$$

# Sphere case

Let us now consider the problem **on the sphere**:

$$\Delta_g v + \rho \left( \frac{e^v}{\int_{\mathbb{S}^2} e^v \, dV_g} - \frac{1}{4\pi} \right) = 4\pi \sum_{j=1}^{N} \alpha_j \left( \delta_{p_j} - \frac{1}{4\pi} \right) \quad \text{on } \mathbb{S}^2, \quad |\mathbb{S}^2| = 4\pi.$$

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Motivations:

- gauge fields
- cosmic strings
- prescribed Gaussian curvature problem

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**geometric case**: $\rho = 4\pi \left( 2 + \sum_j \alpha_j \right)$

# Sphere case: previous results

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**Uniqueness** holds for:

- $N \leq 2$: $\rho < 4\pi \left(2 + \sum_j \alpha_j\right)$ [C.S. Lin, J. Prajapat - G. Tarantello]

  Project on $\mathbb{R}^2$ and apply moving plane argument to show that solutions are **radial**. Next prove uniqueness of radial solutions.

- $N \geq 3$, $\alpha_j \in (-1, 0)$ $\forall j$, **geometric case**: $\rho = 4\pi \left(2 + \sum_j \alpha_j\right)$

  [F. Luo - G. Tian]. By using algebraic geometric approach.

- $N > 2$, **not geometric case**: **missing**.

# Second main result

**Theorem** [Bartolucci-Gui-J.-Moradifam]

Let $\alpha_j \in (-1,0),\ j = 1,\dots,N$. Then we have:

(i) $N \geq 0,\ \rho < 4\pi\big(2 + \sum_j \alpha_j\big)$: there exists **at most one solution**;

(ii) $N \geq 3,\ \rho = 4\pi\big(2 + \sum_j \alpha_j\big)$: there exists **at most one solution**.

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$(ii)$ $N \geq 3,\ \rho = 4\pi\big(2 + \sum_j \alpha_j\big)$: there exists **at most one solution**.

**Remarks.**

1. It covers (most of) the previously known results.
2. Part $(ii)$ is somehow **sharp**:
   - for $N = 0$ and $N = 2$ solutions are classified and uniqueness does not hold;
   - for $N = 1$ solutions do not exist since the 'tear drop' does not admit constant curvature;
   - uniqueness fails if some $\alpha_j > 0$.
3. Whenever the coercivity condition $\rho < 8\pi(1 + \alpha_-)$ is also satisfied, we have **existence** and uniqueness.

# Second main result

Observe that we may have

$$\text{unique. threshold} \quad 4\pi\Big(2+\sum_j \alpha_j\Big) > 8\pi(1+\alpha_-) \quad \text{subcrit. threshold.}$$

Since by [C.C. Chen - C.S. Lin] the degree

$$d_\rho = 0 \quad \text{for} \quad \rho \in \Big(8\pi(1+\alpha_-), 4\pi\Big(2+\sum_j \alpha_j\Big)\Big), \quad N \geq 3, \; \alpha_j \in (-1,0)\, \forall j,$$

if we knew that any such a solution is **non degenerate**, then we would get a **non existence** result in this supercritical region.

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if we knew that any such a solution is **non degenerate**, then we would get a **non existence** result in this supercritical region.

**Open problem.**
Is it true that the problem has **no solutions** for

$$\rho \in \big(8\pi(1+\alpha_-), 4\pi\big(2+\sum_j \alpha_j\big)\big), \quad N \geq 3, \ \alpha_j \in (-1,0)\, \forall j?$$

**Remark.**
Up to now one can treat only the (**radial**) case $N \leq 2$ via Pohozaev identities [D. Bartolucci - A. Malchiodi, G. Mancini, G. Tarantello] showing **non existence** for

$$\rho \in \big(8\pi(1+\alpha_1), 8\pi(1+\alpha_2)\big).$$

# The argument

## Sphere Covering Inequality [Gui-Moradifam]

Let $\Omega \subset \mathbb{R}^2$ be simply-connected and consider two solutions

$$\Delta u_i + e^{2u_i} = f \geq 0 \quad \text{in } \Omega, \quad i = 1, 2.$$

Suppose that,

$$\begin{cases} u_2 \not\equiv u_1 & \text{in } \Omega, \\ u_2 = u_1 & \text{on } \partial\Omega. \end{cases}$$

Then it holds,

$$\int_\Omega \left( e^{2u_1} + e^{2u_2} \right) dx \geq 4\pi = |\mathbb{S}^2|.$$

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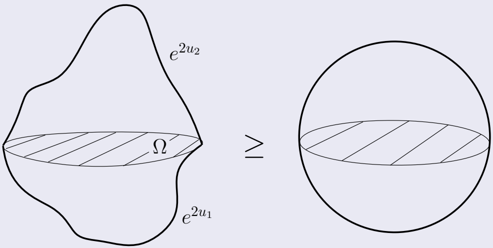


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# The argument

**Singular Sphere Covering Inequality** [Bartolucci-Gui-J-Moradifam]

Let $\Omega \subset \mathbb{R}^2$ be simply-connected and consider two solutions

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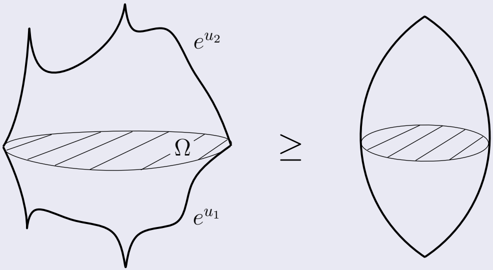


equality holds $\Leftrightarrow$ 'for the American football case'

# The argument

**The equality case.**

$$U_{\lambda,\alpha}(x) = \ln\left(\frac{\lambda(1+\alpha)}{1+\frac{\lambda^2}{8}|x|^{2(1+\alpha)}}\right)^2, \quad \alpha \in (-1,0],\ \lambda > 0.$$

$$\Delta U_{\lambda,\alpha} + |x|^{2\alpha} e^{U_{\lambda,\alpha}} = 0 \quad \text{a.e. in } B_R.$$

Take $\lambda_2 > \lambda_1$ such that,

$$\begin{cases} U_{\lambda_2,\alpha} > U_{\lambda_1,\alpha} & \text{in } B_R, \\ U_{\lambda_2,\alpha} = U_{\lambda_1,\alpha} & \text{on } \partial B_R. \end{cases}$$

Then it holds,

$$\int_{B_R} \left(|x|^{2\alpha} e^{U_{\lambda_1,\alpha}} + |x|^{2\alpha} e^{U_{\lambda_2,\alpha}}\right) dx = 8\pi(1+\alpha).$$

# Idea of the proof

By using the **Alexandrov-Bol isoperimetric inequality**

[A.D. Alexandrov, C. Bandle, D. Bartolucci - D. Castorina, Y.G. Reshetnyak]

one can show that a **radial subsolution**

$$\begin{cases} \int_{\partial B_r} |\nabla \psi| \, d\sigma \leq \int_{B_r} |x|^{2\alpha} e^{\psi} \, dx & \text{for a.e. } r \in (0, R), \\ \psi = U_{\lambda_1,\alpha} = U_{\lambda_2,\alpha} & \text{on } \partial B_R \end{cases}$$

satisfies either

$$\int_{B_R} |x|^{2\alpha} e^{\psi} \, dx \leq \int_{B_R} |x|^{2\alpha} e^{U_{\lambda_1,\alpha}} \, dx$$

OR

$$\int_{B_R} |x|^{2\alpha} e^{\psi} \, dx \geq \int_{B_R} |x|^{2\alpha} e^{U_{\lambda_2,\alpha}} \, dx.$$

# Idea of the proof

Consider

$$\Delta u_i + e^{u_i} = 4\pi \sum_{j=1}^{N} \alpha_j \delta_{p_j}, \quad \text{w.l.o.g. } u_2 > u_1 \text{ in } \Omega, \quad u_2 = u_1 \text{ on } \partial\Omega.$$

Take $\lambda_2 > \lambda_1$ such that $U_{\lambda_2,\alpha} = U_{\lambda_1,\alpha}$ on $\partial B_R$ and

$$\int_\Omega e^{u_1}\, dx = \int_{B_R} |x|^{2\alpha} e^{U_{\lambda_1,\alpha}}\, dx.$$

Consider a

**rad. rearrang.** $\phi^*$ of $u_2 - u_1$ w.r.t. $e^{u_1}\, dx$ and $|x|^{2\alpha} e^{U_{\lambda_1,\alpha}}\, dx$

so that,

$$\int_\Omega (e^{u_1} + e^{u_2})\, dx = \int_\Omega \left(e^{u_1} + e^{u_1 + (u_2 - u_1)}\right) dx = \int_{B_R} \left(|x|^{2\alpha} e^{U_{\lambda_1,\alpha}} + |x|^{2\alpha} e^{U_{\lambda_1,\alpha} + \phi^*}\right)$$

# Idea of the proof

We have

$$\begin{cases} -\Delta(u_2 - u_1) = e^{u_2} - e^{u_1} & \Rightarrow \int_{\partial B_r} |\nabla U_{\lambda_1,\alpha} + \phi^*| d\sigma \leq \int_{B_r} |x|^{2\alpha} e^{U_{\lambda_1,\alpha} + \phi^*} dx \\ u_2 = u_1 \text{ on } \partial\Omega & \Rightarrow U_{\lambda_1,\alpha} + \phi^* = U_{\lambda_1,\alpha} = U_{\lambda_2,\alpha} \text{ on } \partial B_R. \end{cases}$$

Therefore, by the **previous alternative**,

$$\begin{aligned} &\text{since} \quad u_2 > u_1 \text{ in } \Omega \quad \Rightarrow \quad U_{\lambda_1,\alpha} + \phi^* > U_{\lambda_1,\alpha} \text{ in } B_R, \\ &\text{then} \quad \int_{B_R} |x|^{2\alpha} e^{U_{\lambda_1,\alpha} + \phi^*} \, dx \geq \int_{B_R} |x|^{2\alpha} e^{U_{\lambda_2,\alpha}} \, dx \end{aligned}$$

and thus

$$\int_\Omega \left(e^{u_1} + e^{u_2}\right) dx \geq \int_{B_R} \left(|x|^{2\alpha} e^{U_{\lambda_1,\alpha}} + |x|^{2\alpha} e^{U_{\lambda_2,\alpha}}\right) dx = 8\pi(1+\alpha).$$

Thank you for your attention!