Non-convex super-level sets of Robin eigenfunctions

Daniel Hauer

THE UNIVERSITY OF SYDNEY

Physical, Geometrical and Analytical Aspects of Mean Field Systems of Liouville Type
4 April 2018 at Banff International Research Station (BIRS), Canada

Joint work with

Ben Andrews

Julie Clutterbuck

Preprint is available from:

www.maths.usyd.edu.au/u/dhauer

The Classical Eigenvalue Problem

For $0 \le \alpha \le +\infty$ & Ω a bounded domain in $\mathbb{R}^d$, $d \ge 1$, the classical eigenvalue problem for the Laplace operator $\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ on Ω is given by

$$(E_\alpha) \qquad \begin{cases} -\Delta \varphi_\alpha = \lambda_\alpha \varphi_\alpha & \text{in } \Omega \\ D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$

Here, ν is the outward pointing unit normal to Ω, D_ν the directional derivative in direction ν & the boundary $\partial\Omega$ is assumed to be Lipschitz.

In problem (E_α),

λ_α = first eigenvalue of $-\Delta$ equipped with the boundary conditions $D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0$ on $\partial\Omega$.

&

$$\boxed{\lambda_\alpha = \inf_{\substack{\varphi \in C(\overline{\Omega}) \\ \varphi \neq 0}} \frac{\int_\Omega |\nabla \varphi|^2 dx + \alpha \int_{\partial\Omega} |\varphi|^2 d\sigma}{\int_\Omega |\varphi|^2 dx} \quad (2)}$$

φ_α = corresponding "eigenfunction" – minimises (2) $\iff$ (E_α)

(there is only one satisfying $\varphi_\alpha > 0$ & $\int_\Omega |\varphi_\alpha|^2 dx = 1$)

Neumann, Robin & Dirichlet

The boundary condition $D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0$ on $\partial\Omega$ reduces to

Neumann	Robin	Dirichlet
$\alpha = 0$	$0 < \alpha < \infty$	"$\alpha = +\infty$"
$D_\nu \varphi_0 = 0$	$D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0$	$\varphi_\infty = 0$

The eigenvalue λ_α is monotone in $\alpha \nearrow$

$\Rightarrow 0 = \lambda_0 < \lambda_\alpha < \lambda_\infty < \infty$.

Why studying eigenvalue problems?

Laplace eigenfunctions $\varphi_{\alpha n} \sim \lambda_{\alpha n}$ appear

Why studying eigenvalue problems?

Laplace eigenfunctions $\varphi_{\alpha n} \sim \lambda_{\alpha n}$ appear

- as vibration modes in acoustic
  - of a thin membrane (drum) with a fixed boundary are given by Dirichlet Laplace eigenfunctions.

Why studying eigenvalue problems?

Laplace eigenfunctions $\varphi_{\alpha n} \sim \lambda_{\alpha n}$ appear

-) as vibration modes in acoustic

  ↳ of a thin membrane (drum) with a fixed boundary are given by Dirichlet Laplace eigenfunctions.

-) as electron wave functions in quantum waveguides.

Why studying eigenvalue problems?

Laplace eigenfunctions $\varphi_{\alpha n} \sim \lambda_{\alpha n}$ appear

-) as vibration modes in acoustic
  ↳ of a thin membrane (drum) with a fixed boundary are given by Dirichlet Laplace eigenfunctions.

-) as electron wave functions in quantum waveguides.

-) $\{\varphi_{\alpha n}\}_n$ as ONB in $L^2(\Omega)$

•) For every $u_0 \in L^2(\Omega)$, the unique solution $u_\alpha(x,t)$ of

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha = 0 & \text{in } \Omega \times (0,\infty) \\ D_\nu u_\alpha + \alpha \cdot u_\alpha = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u_\alpha(0,\cdot) = u_0 & \text{in } \Omega \end{cases}$$

is given by

$$\underline{\underline{u(x,t) = \sum_{n \geq 1} e^{\lambda_{\alpha n} t} (u_0, \varphi_{\alpha n})_{L^2} \cdot \varphi_{\alpha n}(x)}}$$

(if $\alpha = 0$ then $n \geq 2$) for $x \in \Omega$, $t \geq 0$

$\lambda_\alpha := \lambda_{\alpha,1} > 0 \Rightarrow$ expontial decay of $\|u(t)\|_{L^2} \to 0$ $t \to \infty$

$(0 < \alpha \leq \infty)$

Poincaré-inequality

$$\int_\Omega |\nabla \varphi|^2 dx + \alpha \int_{\partial\Omega} |\varphi|^2 d\sigma \geq \frac{1}{\lambda_\alpha} \int_\Omega |\varphi|^2 dx$$

for all $\varphi \in H^1(\Omega)$

Main interest: Geometric properties of φ_α

Neumann	Robin	Dirichlet
$\alpha = 0$	$0 < \alpha < \infty$	"$\alpha = +\infty$"
$D_\nu \varphi_0 \big\rvert_{\partial\Omega} = 0$	$(D_\nu \varphi_\alpha + \alpha \varphi_\alpha)\big\rvert_{\partial\Omega} = 0$	$\varphi_\infty \big\rvert_{\partial\Omega} = 0$

Main interest: Geometric properties of φ_α

Neumann	Robin	Dirichlet
$\alpha = 0$	$0 < \alpha < \infty$	"$\alpha = +\infty$"
$D_\nu \varphi_0 \big\vert_{\partial\Omega} = 0$	$(D_\nu \varphi_\alpha + \alpha \varphi_\alpha)\big\vert_{\partial\Omega} = 0$	$\varphi_\infty \big\vert_{\partial\Omega} = 0$

$\lambda_\infty > 0$

φ_∞ is log-concave

if Ω is convex

(Brascamp-Lieb '76)

Main interest: Geometric properties of φ_α

Neumann	Robin	Dirichlet
$\alpha = 0$	$0 < \alpha < \infty$	"$\alpha = +\infty$"
$D_\nu \varphi_0 \big\vert_{\partial\Omega} = 0$	$(D_\nu \varphi_\alpha + \alpha\varphi_\alpha)\big\vert_{\partial\Omega} = 0$	$\varphi_\infty \big\vert_{\partial\Omega} = 0$

$\lambda_0 = 0$

$\varphi_0(x) = \mathbb{1}_\Omega(x)$

$\lambda_\infty > 0$

φ_∞ is log-concave if Ω is convex (Brascamp–Lieb '76)

Daniel Hauer

Main interest: Geometric properties of φ_α

Neumann	Robin	Dirichlet
$\alpha = 0$	$0 < \alpha < \infty$	"$\alpha = +\infty$"
$D_\nu \varphi_0 \big\vert_{\partial\Omega} = 0$	$\left(D_\nu \varphi_\alpha + \alpha \varphi_\alpha\right)\big\vert_{\partial\Omega} = 0$	$\varphi_\infty \big\vert_{\partial\Omega} = 0$
$\lambda_0 = 0$	$\boxed{?}$	$\lambda_\infty > 0$
$\varphi_0(x) = \frac{\mathbb{1}(x)}{\Omega}$		φ_∞ is log-concave if Ω is convex (Brascamp-Lieb '76)

Daniel Hauer

THE UNIVERSITY OF SYDNEY

Reminder   Let Ω be convex.

- A function $\varphi : \Omega \to \mathbb{R}_+ := (0,\infty)$ is log-concave if $V(x) := \log \varphi(x)$ is concave

Reminder

Let Ω be convex.

- A function $\varphi : \Omega \to \mathbb{R}_+ = (0,\infty)$ is *log-concave* if $V(x) := \log \varphi(x)$ is *concave*

$\iff$ for all $x, \hat{x} \in \Omega$ & $\lambda \in (0,1)$, one has that

$$\varphi(\lambda x + (1-\lambda)\hat{x}) \geq \varphi^{\lambda}(x) \cdot \varphi^{1-\lambda}(\hat{x})$$

Reminder

Let Ω be convex.

- A function $\varphi : \Omega \to \mathbb{R}_+ := (0,\infty)$ is *log-concave* if $v(x) := \log \varphi(x)$ is *concave*

$\iff$ for all $x, \hat{x} \in \Omega$ & $\lambda \in (0,1)$, one has that

$$\varphi(\lambda x + (1-\lambda)\hat{x}) \geq \varphi^{\lambda}(x) \cdot \varphi^{1-\lambda}(\hat{x})$$

Consequence:

φ *log-concave* $\underset{\not\Leftarrow}{\Rightarrow}$ for every $c \in \mathbb{R}_+$ the super-level set $E_c := \{x \mid \varphi(x) \geq c\}$ is *convex*.

Why is log-concavity so important?

Why is log-concavity so important?

In 1983, Michael van der Berg [5] stated

"for a bounded convex domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$,
& for a convex potential $V : \Omega \to \mathbb{R}$,
the difference between the 1^{st} & 2^{nd} Dirichlet-eigenvalues $\lambda^V_{\infty 1}$, $\lambda^V_{\infty 2}$ of the Schrödinger operator $-\Delta + V$ satisfy

$$\lambda^V_{\infty 2} - \lambda^V_{\infty 1} \geq \frac{3\pi^2}{D}$$

where $D := \mathrm{diam}(\Omega)$."

Fundamental Gap Conjecture

- The fundamental gap conjecture was also independently suggested by

  - Ashbaugh & Benguria [2] in 1989
  - Yau [10] in 1986.

Fundamental Gap Conjecture

- The fundamental gap conjecture was also independently suggested by
  - Ashbaugh & Benguria [2] in 1989
  - Yau [10] in 1986.

The difference $\lambda_{\infty 2} - \lambda_{\infty 1}$ is called *fundamental gap* or *spectral gap*.

Fundamental Gap Conjecture

The case $d=1$ and $V\equiv 0$

$$-\varphi''_{\infty n} = \lambda_{\infty n}\,\varphi_{\infty n} \quad \text{on } \Omega=\left(-\frac{D}{2},\frac{D}{2}\right)$$

$$\Rightarrow \lambda_{\infty n} = \frac{n^2\pi^2}{D^2} \quad \Rightarrow \lambda_{\infty 2}-\lambda_{\infty 1} = \frac{3\pi^2}{D^2}$$

showing that the fundamental gap conjecture holds.

Fundamental Gap Conjecture

"for a bounded convex domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$,
& for a convex potential $V : \Omega \to \mathbb{R}$,
the difference between the 1^{st} & 2^{nd} Dirichlet-eigenvalues $\lambda^V_{\infty 1}$, $\lambda^V_{\infty 2}$ of the Schrödinger operator $-\Delta + V$
satisfy

$$\lambda^V_{\infty 2} - \lambda^V_{\infty 1} \geq \frac{3\pi^2}{D} = \lambda^{(-\frac{D}{2},\frac{D}{2})}_{\infty 2} - \lambda^{(-\frac{D}{2},\frac{D}{2})}_{\infty 1}$$

where $D := \operatorname{diam}(\Omega)$."

Fundamental Gap Conjecture

- In quantum mechanics, the spectral gap determines the energy needed to jump from the ground state $\varphi^V_{\infty,1}$ to the next excited eigen state.

Fundamental Gap Conjecture

- In quantum mechanics, the spectral gap determines the energy needed to jump from the ground state $\varphi_{\infty 1}^{V}$ to the next excited eigen state.
- The fundamental gap governs the rate

$$\left\| e^{\lambda_{\infty 1}^{V} t} u(\cdot,t) - a_1 \varphi_{\infty 1}^{V}(\cdot) \right\|_{\infty} \lesssim e^{-(\lambda_{\infty 2}^{V} - \lambda_{\infty 1}^{V}) t}$$

Fundamental Gap Conjecture

- Ashbaugh & Benguria [2] proved the fundamental gap conjecture in dimension $d=1$ & for symmetric single-well potential V & V is nondecreasing on $[0,\frac{D}{2}]$.

Fundamental Gap Conjecture

- Ashbaugh & Benguria [2] proved the fundamental gap conjecture in dimension $d=1$ & for symmetric single-well potential V & V is nondecreasing on $[0,\frac{D}{2}]$.

- Horváth [7] partly removed the above symmetry assumption by allowing single-well potentials V with minimum at $x=0$.

Fundamental Gap Conjecture

- Ashbaugh & Benguria [2] proved the fundamental gap conjecture in dimension $d=1$ & for symmetric single-well potential V & V is nondecreasing on $[0,\frac{D}{2}]$.

- Horváth [7] partly removed the above symmetry assumption by allowing single-well potentials V with minimum at $x=0$.

- In 1994, Lavine [8] proved the fundamental gap conjecture in dimension $d=1$.

Fundamental Gap Conjecture

In higher dimensions

- Bañuelos & Méndes-Hernándes[4] proved the fundamental gap conjecture for $V \equiv 0$.

Fundamental Gap Conjecture

In higher dimensions

- Bañuelos & Méndes-Hernándes [4] proved the fundamental gap conjecture for $V \equiv 0$.
- Davis [6] and Bañuelos & Kröger [3] proved it for $\Omega \subseteq \mathbb{R}^2$ symmetric

Fundamental Gap Conjecture

In higher dimensions

- Bañuelos & Méndes-Hernándes [4] proved the fundamental gap conjecture for $V \equiv 0$.
- Davis [6] and Bañuelos & Kröger [3] proved it for $\Omega \subseteq \mathbb{R}^2$ symmetric

Nonoptimal bounds

- In 1985, Singer, Wong, Yau & Yau [9] proved $\lambda^V_{\infty 2} - \lambda^V_{\infty 1} \geq \frac{\pi^2}{4D^2}$

Fundamental Gap Conjecture

In higher dimensions

- Bañuelos & Méndes-Hernándes[4] proved the fundamental gap conjecture for $V \equiv 0$.
- Davis [6] and Bañuelos & Kröger [3] proved it for $\Omega \subseteq \mathbb{R}^2$ symmetric

Nonoptimal bounds

- In 1985, Singer, Wong, Yau & Yau [9] proved $\lambda^V_{\infty 2} - \lambda^V_{\infty 1} \geq \frac{\pi^2}{4D^2}$
- This result was improved by Yu & Zhong [11] to

$$\lambda^V_{\infty 2} - \lambda^V_{\infty 1} \geq \frac{\pi^2}{D^2}.$$

Fundamental Gap Conjecture

In 2011, the fundamental gap conjecture was proved completely by Andrews & Clutterbuck [1].

THE UNIVERSITY OF SYDNEY

Daniel Hauer

First Main Result

1. Theorem [Andrews, Clutterbuck, H. '18]

Let $\lambda^V_{\alpha 1}$ & $\lambda^V_{\alpha 2}$ be the Robin-eigenvalues on a bounded convex domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, with diameter D & $V \in L^1_{loc}(\Omega)$.

If $\varphi^V_{\alpha 1}$ is log-concave, then $\lambda^V_{\alpha 2} - \lambda^V_{\alpha 1} \geq \frac{\pi^2}{D^2}$.

Proof.

Proof. The function

$$V(x,t) := \frac{e^{-\lambda^{V}_{\alpha_2} t}\, \varphi^{V}_{\alpha_2}(x)}{e^{-\lambda^{V}_{\alpha_1} t}\, \varphi^{V}_{\alpha_1}(x)} \quad \text{for } x \in \Omega \;\&\; t \geq 0$$

Proof. The function

$$V(x,t) := \frac{e^{-\lambda^V_{\alpha 2} t}\, \varphi^V_{\alpha 2}(x)}{e^{-\lambda^V_{\alpha 1} t}\, \varphi^V_{\alpha 1}(x)} \quad \text{for } x \in \Omega \;\&\; t \geq 0$$

solves $\partial_t V = \Delta V + 2\, \nabla \log \varphi^V_{\alpha 1} \cdot \nabla V$ on $\Omega \times (0,\infty)$

Proof. The function

$$V(x,t) := \frac{e^{-\lambda^V_{\alpha_2} t}\, \varphi^V_{\alpha_2}(x)}{e^{-\lambda^V_{\alpha_1} t}\, \varphi^V_{\alpha_1}(x)} \quad \text{for } x \in \Omega \;\&\; t \geq 0$$

solves $\partial_t V = \Delta V + 2\, \nabla \log \varphi^V_{\alpha_1} \cdot \nabla V$ on $\Omega \times (0,\infty)$

and on $\partial\Omega \times (0,\infty)$:

$$D_\nu V = \frac{e^{-\lambda_{\alpha_2} t}}{e^{-\lambda_{\alpha_1} t}} \cdot \frac{D_\nu \varphi^V_{\alpha_2} \cdot \varphi^V_{\alpha_1} - \varphi^V_{\alpha_2} \cdot D_\nu \varphi^V_{\alpha_1}}{(\varphi^V_{\alpha_1})^2}$$

Proof. The function

$$V(x,t) := \frac{e^{-\lambda^V_{\alpha 2} t}\,\varphi^V_{\alpha 2}(x)}{e^{-\lambda^V_{\alpha 1} t}\,\varphi^V_{\alpha 1}(x)} \qquad \text{for } x \in \Omega \;\&\; t \geq 0$$

solves $\partial_t V = \Delta V + 2\,\nabla \log \varphi^V_{\alpha 1} \cdot \nabla V$ on $\Omega \times (0,\infty)$

and on $\partial\Omega \times (0,\infty)$:

$$D_\nu V = \frac{e^{-\lambda_{\alpha 2} t}}{e^{-\lambda_{\alpha 1} t}} \cdot \frac{D_\nu \varphi^V_{\alpha 2} \cdot \varphi^V_{\alpha 1} - \varphi^V_{\alpha 2} \cdot D_\nu \varphi^V_{\alpha 1}}{(\varphi^V_{\alpha 1})^2}$$

$$= \frac{e^{-\lambda_{\alpha 2} t}}{e^{-\lambda_{\alpha 1} t}} \cdot \frac{-\alpha\, \varphi^V_{\alpha 2} \cdot \varphi^V_{\alpha 1} + \alpha\, \varphi^V_{\alpha 2} \cdot \varphi^V_{\alpha 1}}{(\varphi^V_{\alpha 1})^2} = 0$$

Proof.

By hypothesis $X := 2\nabla \log \varphi_{\alpha_1}^{V}$ has modulus of contraction $\omega \equiv 0$: $\left(X(y) - X(x)\right) \cdot \dfrac{y - x}{|y - x|} \leq 2\omega\left(\dfrac{|y-x|}{2}\right)$

$(\omega : [0, D/2] \to \mathbb{R})$

Proof.

By hypothesis $X := 2\nabla \log \varphi_{\alpha 1}^{V}$ has modulus of contraction $\omega \equiv 0$: $\left(X(y) - X(x)\right) \cdot \frac{y-x}{|y-x|} \le 2\omega\left(\frac{|y-x|}{2}\right)$

$(\omega : [0, D/2] \to \mathbb{R})$

Thm 2.1 in [1] $\Longrightarrow$ $\varphi(s,t) := e^{-t\frac{\pi^2}{D^2}} \cdot \sin\left(\frac{\pi}{D} s\right)$

for $s \in [0, D/2]$, $t \ge 0$

is a modulus of continuity of V:

$$V(y,t) - V(x,t) \le 2\,\varphi\left(\frac{y-x}{|y-x|}, t\right) \text{ for all}$$

$x, y \in \overline{\Omega}$ & $t \ge 0$.

Proof.

$$\Longrightarrow \quad e^{-(\lambda^{V}_{\alpha 2}-\lambda^{V}_{\alpha 1})t} \operatorname{osc} \frac{\varphi_{\alpha 2}}{\varphi_{\alpha 1}} \leqslant C_1 \cdot e^{-t\frac{\pi^2}{D^2}} \quad \text{for all } t \geq 0$$

Proof.

$$\Rightarrow \quad e^{-(\lambda^V_{\alpha_2}-\lambda^V_{\alpha_1})t} \operatorname{osc} \frac{\varphi_{\alpha_2}}{\varphi_{\alpha_1}} \leq C \cdot e^{-t\frac{\pi^2}{D^2}} \text{ for all } t \geq 0$$

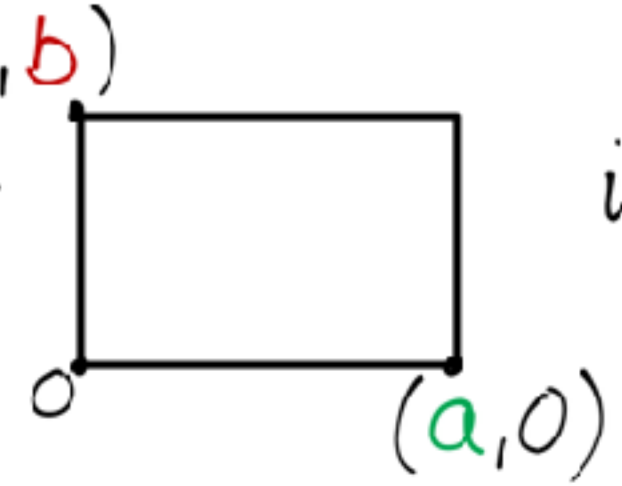
$$\Rightarrow \quad \lambda^V_{\alpha_2}-\lambda^V_{\alpha_1} \geq \frac{\pi^2}{D^2} \qquad \square$$

Is the 1^{st} Robin eigenfunction log-concave?

Is the 1^{st} Robin eigenfunction log-concave?

1^{st} Example

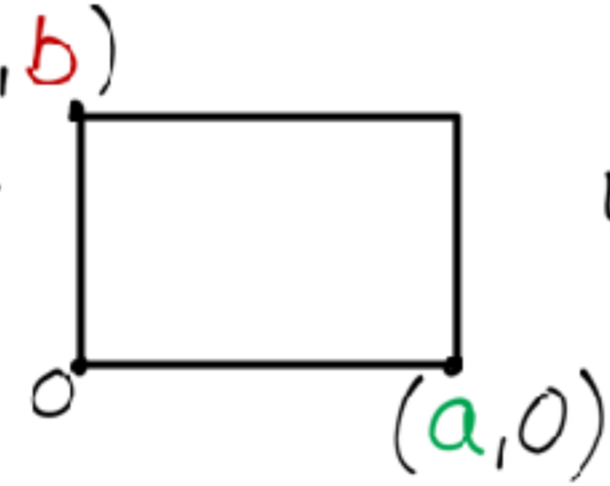
Let $\Omega =$



in $\mathbb{R}^2$

Is the 1^{st} Robin eigenfunction log-concave?

1^{st} Example

Let $\Omega =$ 

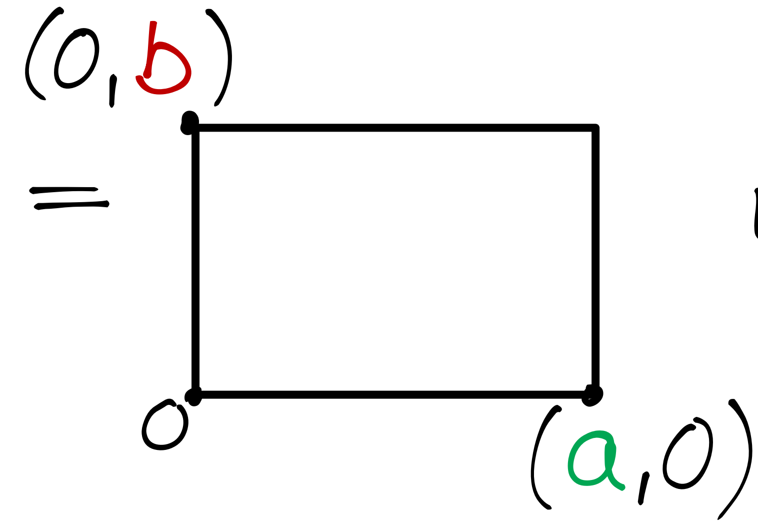
in $\mathbb{R}^2$

If φ_α is the 1^{st} Robin eigenfunction on $I := (0,a) \subseteq \mathbb{R}$

& $\hat{\varphi}_\alpha$ is the 1^{st} Robin eigenfunction on $I := (0,b) \subseteq \mathbb{R}$

Is the 1st Robin eigenfunction log-concave?

1st Example

Let Ω = 

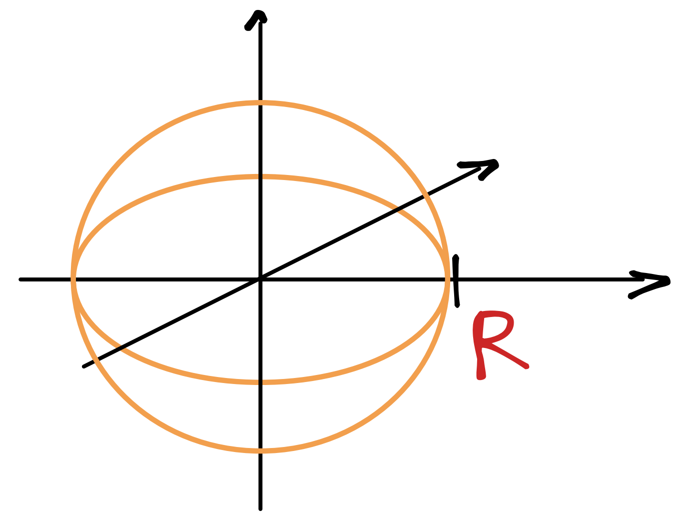
in $\mathbb{R}^2$

If φ_α is the 1st Robin eigenfunction on $I := (0,a) \subseteq \mathbb{R}$

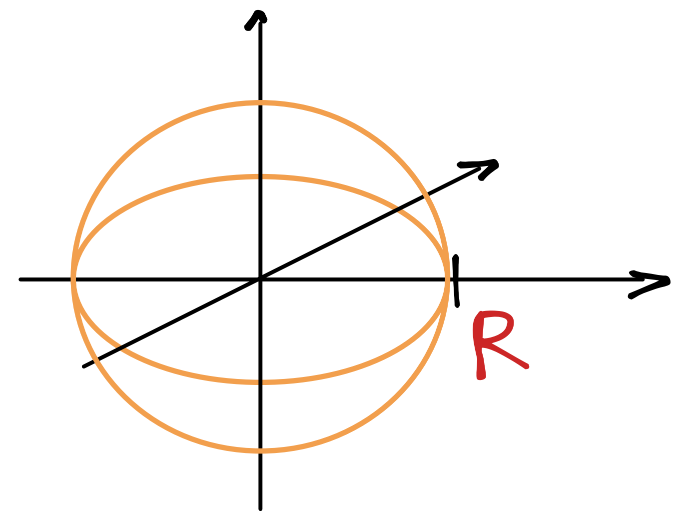
& $\hat{\varphi}_\alpha$ is the 1st Robin eigenfunction on $I := (0,b) \subseteq \mathbb{R}$

then $\varphi(x,y) := \varphi_\alpha(x) \cdot \hat{\varphi}_\alpha(y)$ for $(x,y) \in \Omega$ is the 1st Robin eigenfunction on Ω & φ is log-concave ✓

2^{nd} Example

Let $\Omega =$  $B(0,R) \subseteq \mathbb{R}^d$

2^{nd} Example

Let $\Omega =$ 

$B(0,R) \subseteq \mathbb{R}^d$

Since Ω is radial symmetric

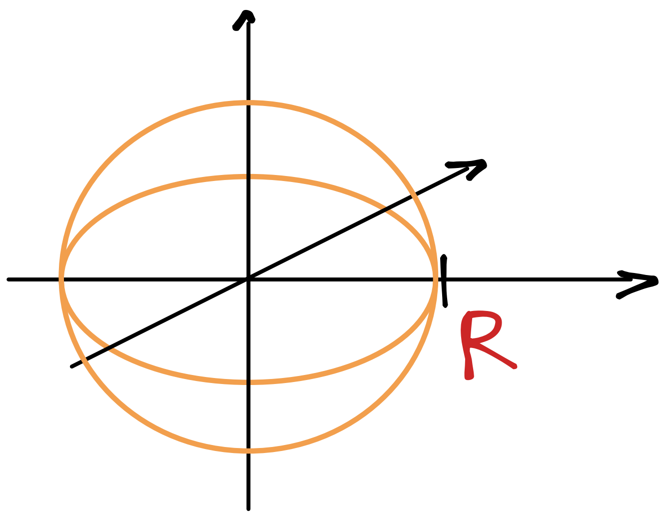
$\Longrightarrow \quad \varphi_\alpha(x) = \varphi_\alpha(r) \quad$ for $r = |x|$ and

$$(E_\alpha) \quad \begin{cases} -\Delta \varphi_\alpha = \lambda_\alpha \varphi_\alpha & \text{in } \Omega \\ D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$

reduces to

$$(E_\alpha^r) \quad \begin{cases} -\varphi_\alpha'' - \frac{d-1}{r} \varphi_\alpha' = \lambda_\alpha \varphi_\alpha \text{ on } (0,R); \\ \varphi_\alpha'(R) + \alpha \varphi_\alpha(R) = 0. \end{cases}$$

2^{nd} Example

Let $\Omega =$  $B(0,R) \subseteq \mathbb{R}^d$

Since Ω is radial symmetric

$\Longrightarrow$ $\varphi_\alpha(x) = \varphi_\alpha(r)$ for $r = |x|$ and

$$(E_\alpha) \quad \begin{cases} -\Delta \varphi_\alpha = \lambda_\alpha \varphi_\alpha & \text{in } \Omega \\ D_\nu \varphi_\alpha + \alpha \varphi_\alpha = 0 & \text{on } \partial\Omega \end{cases}$$

reduces to

$$(E_\alpha^r) \quad \begin{cases} -\varphi_\alpha'' - \frac{d-1}{r} \varphi_\alpha' = \lambda_\alpha \varphi_\alpha \text{ on } (0,R), \\ \varphi_\alpha'(R) + \alpha \varphi_\alpha(R) = 0. \end{cases}$$

$\Longrightarrow \varphi_\alpha$ is concave.

Natural conclusion

One might believe that the 1^{st} Robin eigenfunction φ_α is log-concave for any bounded convex domain & all α.

Natural conclusion

One might believe that the 1^{st} Robin eigenfunction φ_α is log-concave for any bounded convex domain & all α.

But this is false!

2. Theorem [Andrews, Clutterbuck, H. '18]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is NOT a product of circumsolids. Then for sufficiently small $\alpha > 0$, the 1^{st} Robin eigenfunction φ_α is not log-concave.

2. Theorem [Andrews, Clutterbuck, H. '18]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is NOT a product of circumsolids. Then for sufficiently small $\alpha > 0$, the 1^{st} Robin eigenfunction φ_α is <u>not</u> log-concave.

Non-concavity of 1st-Robin eigenfuncts

Some Definitions

Non-concavity of 1st-Robin eigenfuncts

Some Definitions

- $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, is a convex polyhedron if Ω is open, bounded & the intersection of finitely many open half-spaces:

$$\Omega = \bigcap_{i=1}^{m} \{ x \in \mathbb{R}^d \mid x \cdot \nu_i < b_i \}$$

where $\nu_1, \dots, \nu_m$ are unit vectors & $b_1, \dots, b_m$ are constants.

A convex polyhedral domain $\Omega \subseteq \mathbb{R}^d$ is a circumsolid if there are $x_0 \in \Omega$ and $R > 0$ s.th. the open ball $B_R(x_0) \subseteq \overline{\Omega}$ touching every face

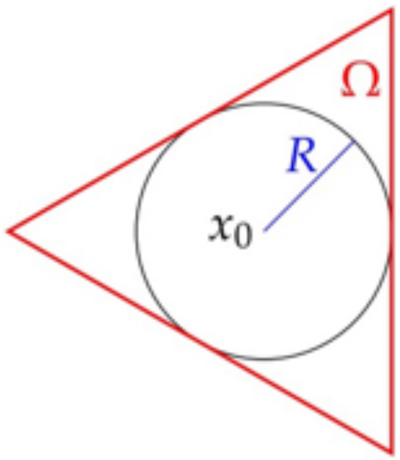
$$\Sigma_i := \{ x \in \overline{\Omega} \mid x \cdot \nu_i = b_i \}$$ of Ω.

(that is, $\partial B_R(x_0) \cap \Sigma_i$ contains exactly one point)

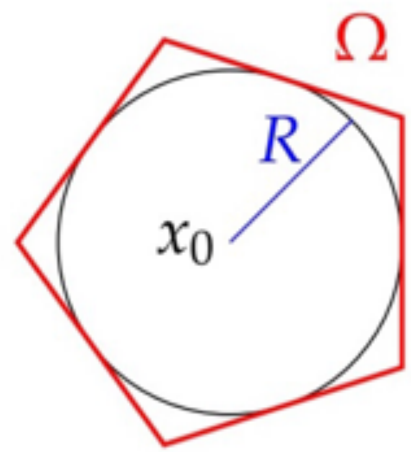
THE UNIVERSITY OF SYDNEY

Daniel Hauer

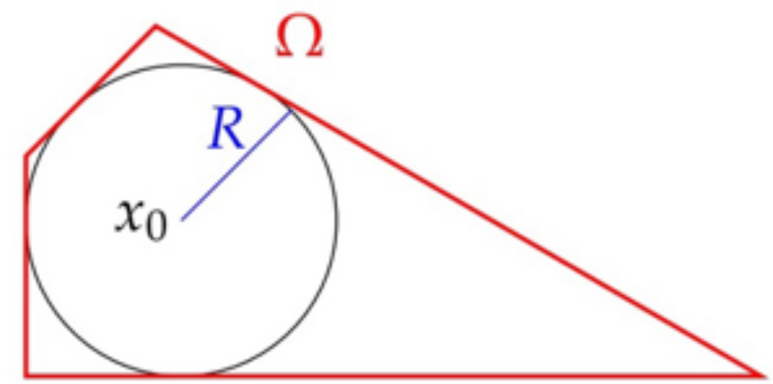
Examples of planar circumsolids:



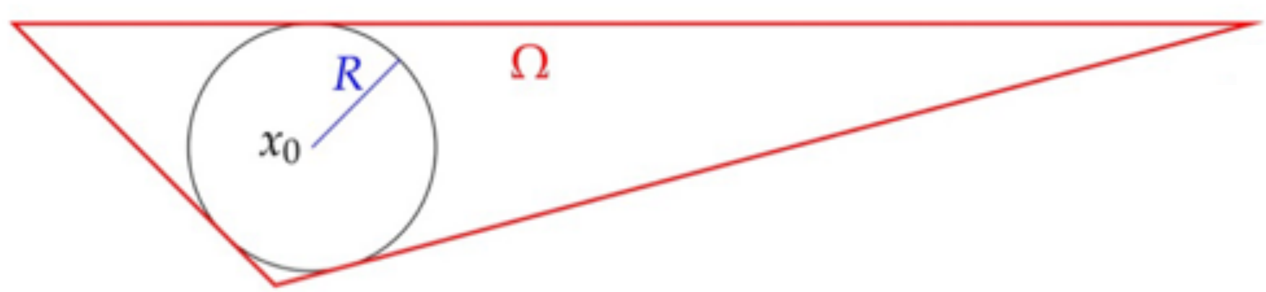
Regular triangle



Regular pentagon



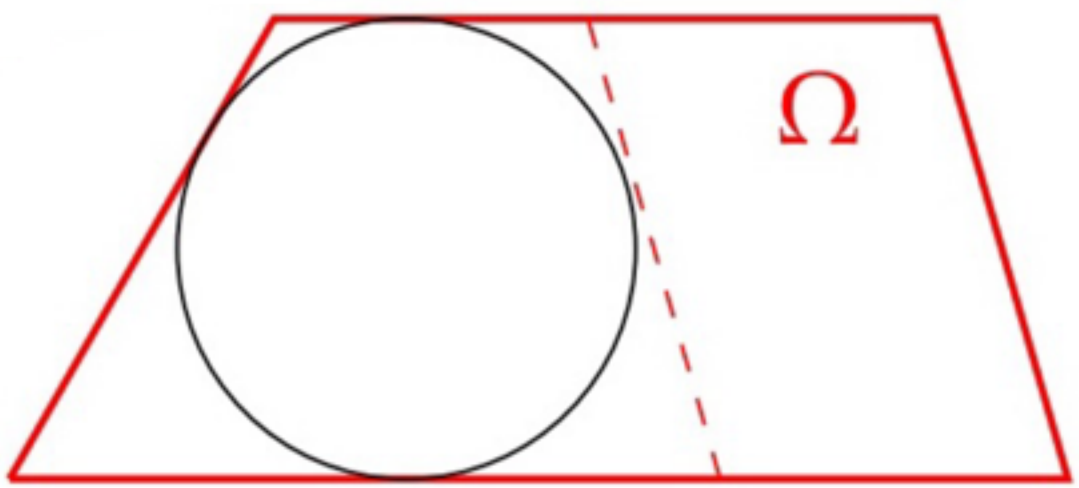
Skew quadrilateral



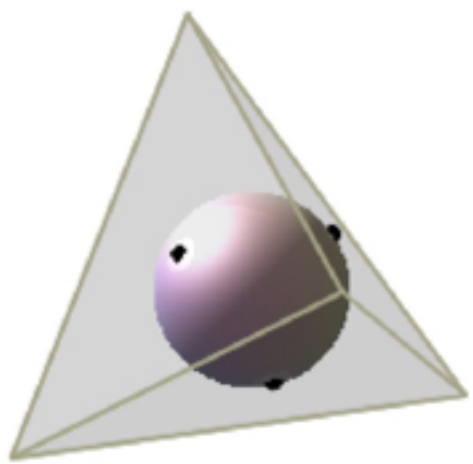
Skew triangle

Every triangle in the plane is a circumsolid.

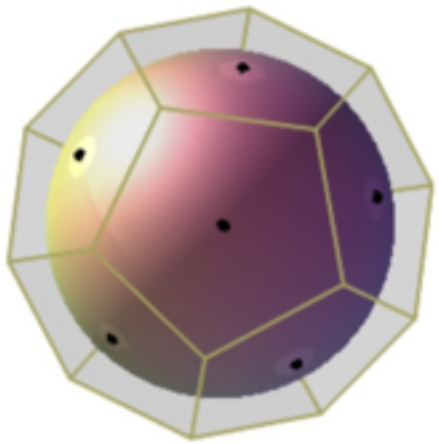
But this is not true for quadrilaterals:



In $\mathbb{R}^3$:



Tetrahedron

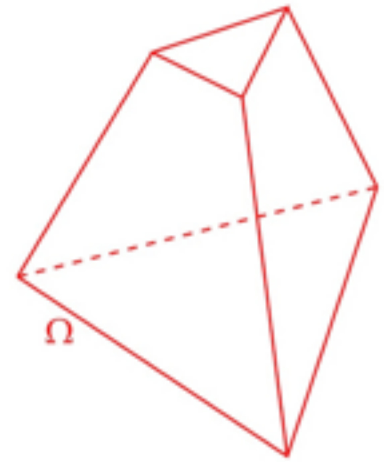


Regular dodecahedron

or $\left\{ \sum_{i=0}^{d} \lambda_i x_i \,\middle|\, \lambda_i \geq 0, \sum_{i=0}^{d} \lambda_i = 1 \right\}$

are all circumsolids.

But



Tetrahedron with a flat top
(cut off the tip!)

is NOT a circumsolid

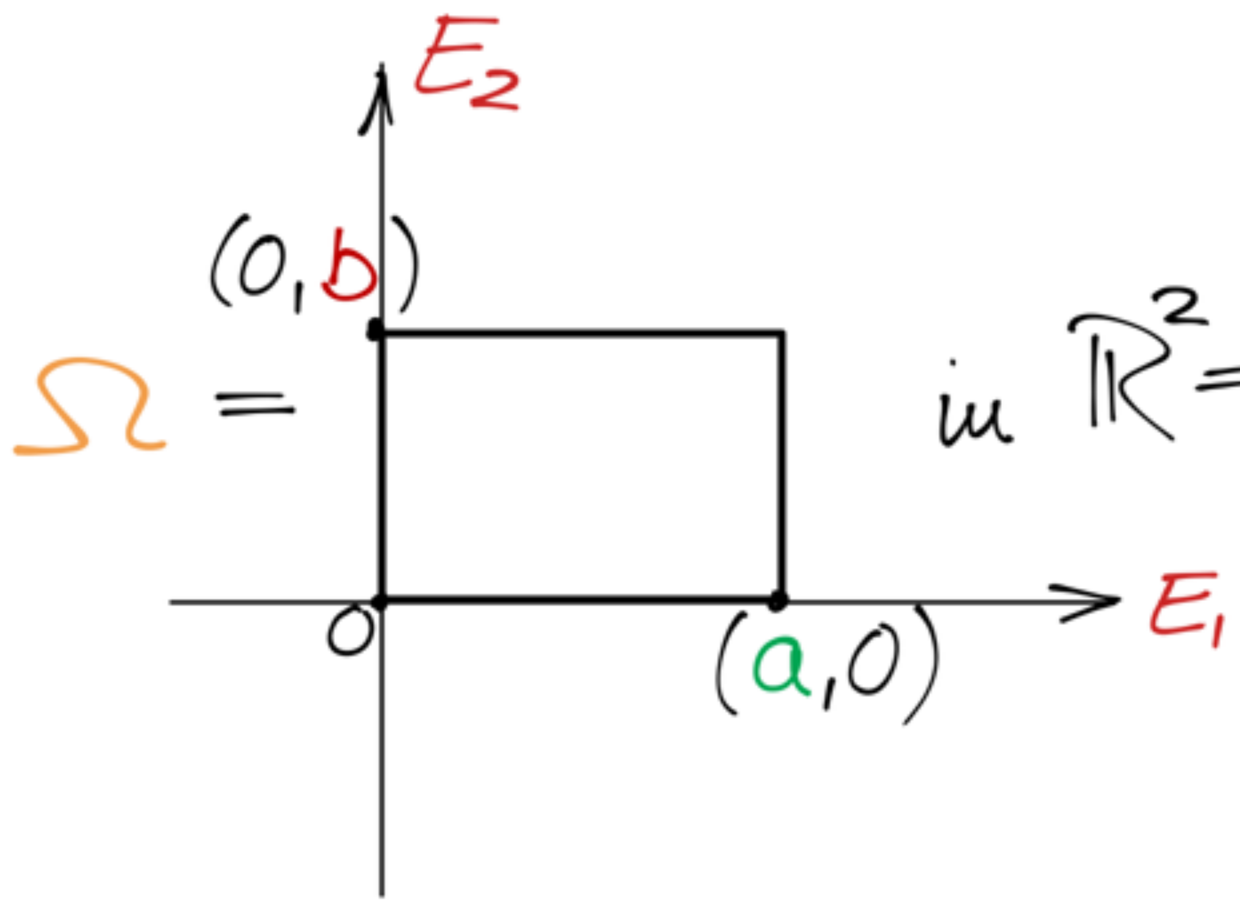
unless the plane of the flat top matches
the inscribed sphere $\partial B_R(x_0)$.

A convex polyhedral domain $\Omega \subseteq \mathbb{R}^d$ is a product of circumsolid if there is a decomposition of $\mathbb{R}^d = \bigoplus_{j=1}^{k} E_j$ into orthogonal subspaces $E_1, \dots, E_k$ of $\mathbb{R}^d$ and there circumsolids $\Omega_1, \dots, \Omega_k$ with $\Omega \subseteq E_i$ such that

$$\Omega = \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{proj}_{E_i}(x) \in \Omega_i \text{ for } i = 1, \dots, k \right\}$$

Examples of products of circumsolids

- **In $\mathbb{R}^2$:** Rectangles

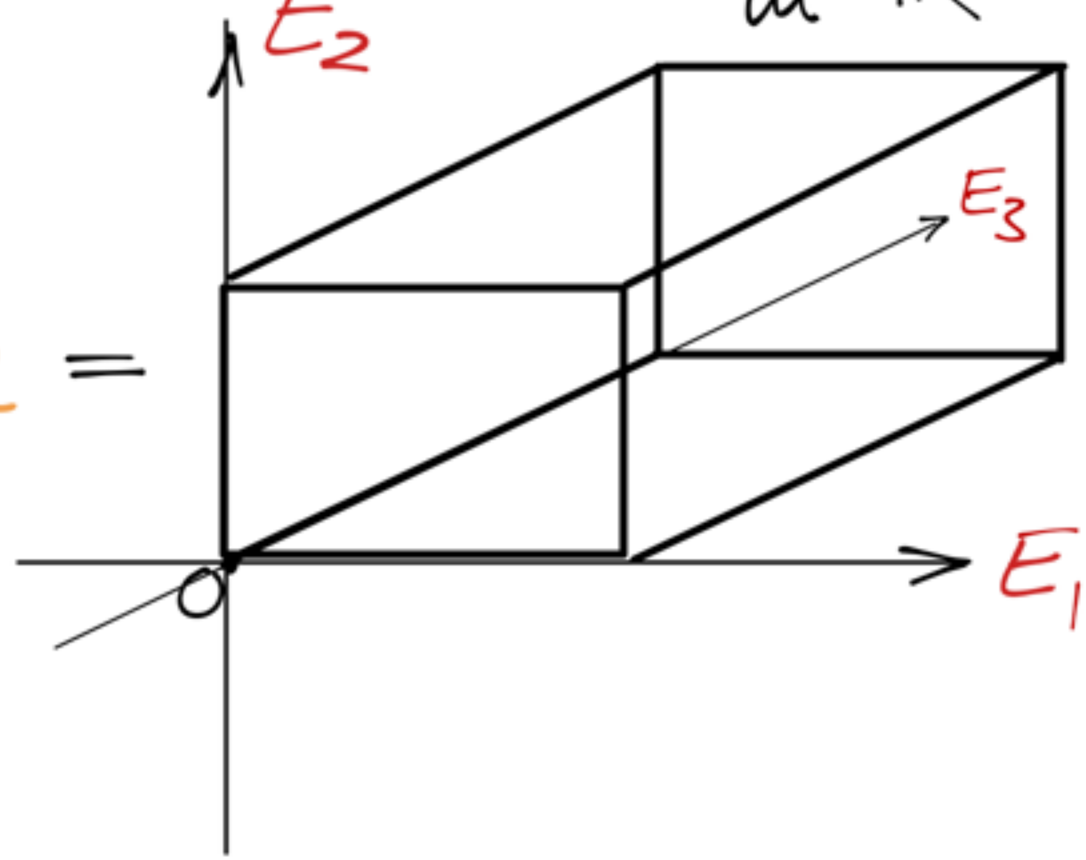


$$\text{in } \mathbb{R}^2 = \underset{\substack{\rotatebox{90}{$\cong$} \\ E_1}}{\mathbb{R}} \oplus \underset{\substack{\rotatebox{90}{$\cong$} \\ E_2}}{\mathbb{R}}$$

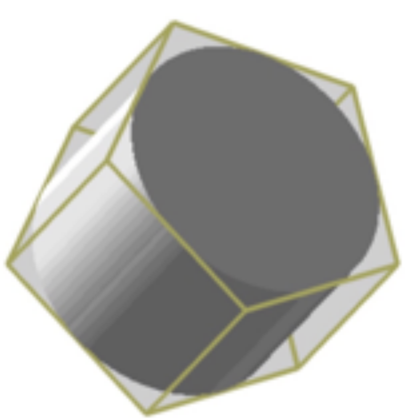
In $\mathbb{R}^3$:

- in $\mathbb{R}^3 = \underset{\substack{\cong \\ E_1}}{\mathbb{R}} \oplus \underset{\substack{\cong \\ E_2}}{\mathbb{R}} \oplus \underset{\substack{\cong \\ E_3}}{\mathbb{R}}$

$\Omega =$



- in $\mathbb{R}^3 = \underset{\substack{\cong \\ E_1}}{\mathbb{R}^2} \oplus \underset{\substack{\cong \\ E_2}}{\mathbb{R}}$



Prism over a regular pentagon.

2. Theorem [Andrews, Clutterbuck, H. '18]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$, $d\geq 2$, which is NOT a product of circumsolids. Then for sufficiently small $\alpha>0$, the 1^{st} Robin eigenfunction φ_α is not log-concave.

Note: φ_α is not log-concave on Ω

$\Longleftrightarrow$ there x & $\hat{x} \in \overline{\Omega}$ & $t_0 \in (0,1)$ s.th.

$$\varphi_\alpha^{t_0}(x) \cdot \varphi_\alpha^{1-t_0}(\hat{x}) > \varphi_\alpha(t_0 x + (1-t_0)\hat{x})$$

Note: φ_α is not log-concave on Ω

$\Longleftrightarrow$ there x & $\hat{x} \in \overline{\Omega}$ & $t_0 \in (0,1)$ s.th.

$$\varphi_\alpha^{t_0}(x) \cdot \varphi_\alpha^{1-t_0}(\hat{x}) > \varphi_\alpha(t_0 x + (1-t_0)\hat{x})$$

If $\Omega_n \to \Omega$ in Hausdorff & $\Omega_n, \Omega \subseteq D \subseteq \mathbb{R}^d$ open & bounded.

$\Longrightarrow$ $\varphi_{\alpha_n} \mathbb{1}_{\Omega_n} \to \varphi_\alpha \mathbb{1}_\Omega$ in $H^1(D)$ & $\varphi_{\alpha_n} \to \varphi_\alpha$ in $C^{0,\delta}(B)$ for all $B \subseteq \bigcap_{n\geq 1} \Omega_n$

1st Corollary [Andrews, Clutterbuck, H. '18]

Let Ω_0 be a convex polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, which is NOT a product of circumsolids. Then for sufficiently small $\alpha > 0$, for any convex domain Ω which is sufficient close to Ω_0 in Hausdorff distance, the 1st Robin eigenfunction φ_α is not log-concave.

Idea of the proof of Theorem 2.

Idea of the proof of Theorem 2.

1) The map $\alpha \mapsto \varphi_\alpha$ of 1^{st} Robin-eigenfunction is $C^1(\mathbb{R}, H^1(\Omega)) \cap C^1(\mathbb{R}, C^{0,\gamma}(\Omega))$ for some $\gamma \in (0,1)$.

Idea of the proof of Theorem 2.

1) The map $\alpha \mapsto \varphi_\alpha$ of 1^{st} Robin-eigenfunction is $\mathcal{C}^1(\mathbb{R}, H^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}, \mathcal{C}^{0,\gamma}(\Omega))$ for some $\gamma \in (0,1)$.

More precisely, if for $\alpha \in \mathbb{R}$, $\boxed{V_\alpha := \log \varphi_\alpha}$ $(\rightsquigarrow V_0 \equiv 0)$

$$\Rightarrow \quad V_\alpha = \alpha V + o(\alpha) \quad \text{in } \mathcal{C}^{0,\beta}(\Omega) \text{ for } \alpha \to 0,\ \beta \in (0,1)$$

where

$$(TP) \begin{cases} \Delta V + \mu = 0 & \text{in } \Omega \\ D_\nu V + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with $\gamma_i := 1$ & $\mu := \frac{d}{d\alpha}\lambda_\alpha\Big|_{\alpha=0} = \frac{|\partial\Omega|}{|\Omega|} \geq 0$.

1^{st}. Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\gamma_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

1st. Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

1.st Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^d(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

Proof. If v is semi-concave, then there is $C \in \mathbb{R}$ such that

$$D^2 v \leq C\, E_{d\times d}.$$

Proof. If v is semi-concave, then there is $C \in \mathbb{R}$ such that $D^2 v \leq C\, E_{d\times d}$.

Let $e \in \mathbb{R}^d$ be a unit vector & choose an ONB $\{e_1, \dots, e_d\}$ with $e_d = e$.

Proof If v is semi-concave, then there is $C \in \mathbb{R}$ such that $D^2 v \leq C\, E_{d\times d}$.

Let $e \in \mathbb{R}^d$ be a unit vector & choose an ONB $\{e_1, \dots, e_d\}$ with $e_d = e$.

$$\Rightarrow e^t D^2 v\, e = \underbrace{\Delta v}_{=\mu} - \sum_{i=1}^{d-1} \underbrace{e_i^t D^2 v\, e_i}_{\leq C \underbrace{|e_i|^2}_{=1}} \geq \mu - C(d-1).$$

Proof If v is semi-concave, then there is $C \in \mathbb{R}$ such that $$D^2 v \leq C\, E_{d\times d}.$$

Let $e \in \mathbb{R}^d$ be a unit vector & choose an ONB $\{e_1, \dots, e_d\}$ with $e_d = e$.

$$\Longrightarrow e^t D^2 v e = \underbrace{\Delta v}_{=\mu} - \sum_{i=1}^{d-1} \underbrace{e_i^t D^2 v e_i}_{\leq C |e_i|^2,\ |e_i| = 1} \geq \mu - C(d-1).$$

$\Longrightarrow$ $D^2 v$ is bounded from below & above

Proof If v is semi-concave, then there is $C \in \mathbb{R}$ such that $D^2 v \leq C\, E_{d\times d}$.

Let $e \in \mathbb{R}^d$ be a unit vector & choose an ONB $\{e_1, \dots, e_d\}$ with $e_d = e$.

$$\Rightarrow \quad e^t D^2 v\, e = \underbrace{\Delta v}_{=\mu} - \sum_{i=1}^{d-1} \underbrace{e_i^t D^2 v\, e_i}_{\leq C\,\underbrace{|e_i|^2}_{=1}} \geq \mu - C(d-1).$$

$\Rightarrow$ $D^2 v$ is bounded from below & above

$\Rightarrow$ Dv is Lipschitz on $\overline{\Omega}$. $\square$

1st. Lemma

Let V be a weak solution of

$$(TP)\quad \begin{cases} \Delta V + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} V + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

(1) V is semi-concave;

(2) V is $\mathcal{C}^{1,1}(\overline{\Omega})$;

1^{st}. Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

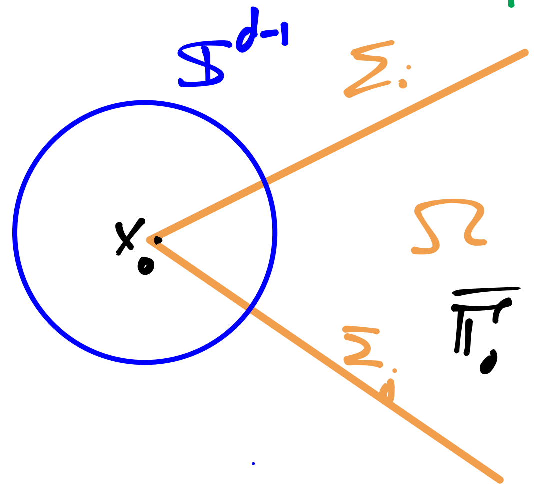
(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

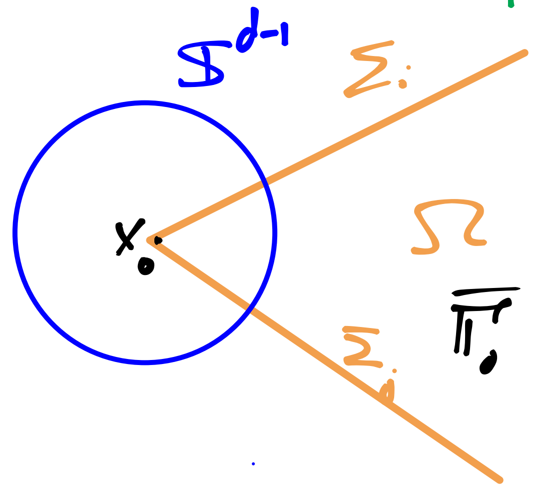
(3) v is $C^{2}(\overline{\Omega})$;

- Let $x_0 \in \partial\Omega$ & choose $r>0$ sufficiently small

  such that $B_r(x_0) \cap \Omega = x_0 + r\left(B_1(0) \cap \overline{\Pi}_{x_0}\right)$



- Let $x_0 \in \partial\Omega$ & choose $r > 0$ sufficiently small such that $B_r(x_0) \cap \Omega = x_0 + r\left(B_1(0) \cap \Gamma_{x_0}\right)$



Note $v \in C^{1,1}(\overline{\Omega})$

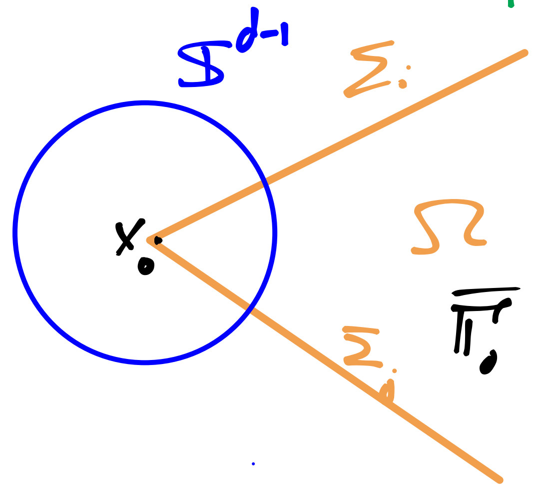
Daniel Hauer

- Let $x_0 \in \partial\Omega$ & choose $r>0$ sufficiently small

such that $B_r(x_0)\cap\Omega = x_0 + r\,(B_1(0)\cap \overline{\Pi}_{x_0})$

Note

$V \in C^{1,1}(\overline{\Omega})$



& set $W(rz) := V(x_0 + rz) - \nabla v(x_0)(rz) + \frac{\mu}{d2}|rz|^2$

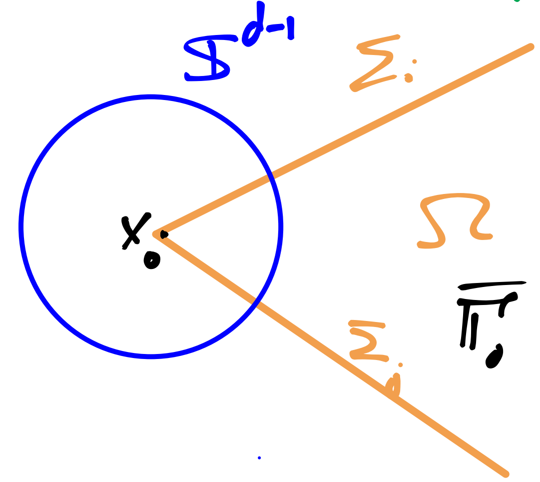
for every $rz \in B_1(0)\cap\overline{\Pi}_{x_0}$

- Let $x_0 \in \partial\Omega$ & choose $r>0$ sufficiently small

such that $B_r(x_0)\cap\Omega = x_0 + r\left(B_1(0)\cap \Pi_{x_0}\right)$

Note

$V \in C^{1,1}(\overline{\Omega})$



& set $W(rz) := V(x_0 + rz) - \nabla v(x_0)(rz) + \frac{\mu}{d2}|rz|^2$

for every $rz \in B_1(0)\cap \Pi_{x_0}$

where if $\Omega = \bigcap_{i=1}^{m}\{x\in\mathbb{R}^d \mid x\cdot\nu_i < b_i\}$

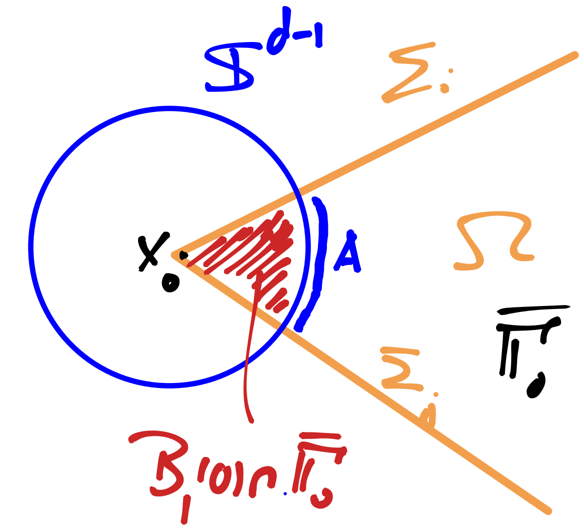
then the tangent cone Π_{x_0} to Ω at $x_0\in\Omega$ is

$\Pi_{x_0} = \bigcap_{i\in J(x_0)}\{y\in\mathbb{R}^d \mid y\cdot\nu_i < 0\}$ with index set $J(x_0) := \{i\in\{1,\dots,m\} \mid x_0\cdot\nu_i = b_i\}$

$\Longrightarrow \quad \Delta W = 0 \quad$ on $B_1(0) \cap \overline{\Pi_{x_0}}$

$$\Longrightarrow \qquad \Delta W = 0 \qquad \text{on } B_1(0) \cap \overline{\Pi}_{x_0}$$

$$\& \quad D_{\gamma_i} W(x) = \underbrace{r D_{\gamma_i} V\big|_{x_0 + r x}}_{= -\gamma_i} - \underbrace{r D_{\gamma_i} V(x_0)}_{= -\gamma_i} + \frac{\mu}{d} r^2 \underbrace{x \cdot \gamma_i}_{= 0} = 0 \quad \text{on } \Sigma_i$$

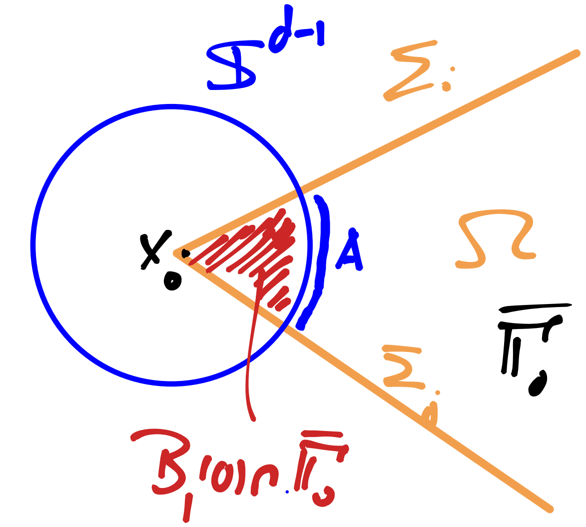


$$\Longrightarrow \qquad \Delta W = 0 \quad \text{on } B_1(0)\cap\overline{\Pi_{x_0}}$$

$$\& \quad D_{\gamma_i} W(x) = \underbrace{r D_{\gamma_i} v\big|_{x_0+rx}}_{=-\gamma_i} - \underbrace{r D_{\gamma_i} v(x_0)}_{=-\gamma_i} + \underbrace{\frac{\mu}{d} r^2 \, x\cdot\gamma_i}_{=0} = 0 \quad \text{on } \Sigma_i$$

$$\Longrightarrow \quad W(rz) = \sum_{i=0}^{\infty} f_i \, r^{\beta_i} \varphi_i(z) \qquad r\in(0,1) \ \& \ z\in A := \overline{\Pi_{x_0}} \cap S^{d-1}$$

where $(\varphi_i)_{i\geq 0}$ are the Neumann-eigenfunctions of $-\Delta_{S^{d-1}}$, ONB in $L^2(A)$



and β_i : $\quad \beta_i^2 + (d-2)\beta_i - \lambda_i = 0$

λ_i : Neumann-eigenvalue of $-\Delta_{S^{d-1}}$

$f_i := \lim_{r\to 1} (w(r,\cdot), \varphi_i)_{L^2(A)}$.

1st. Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

(3) v is $C^{2}(\overline{\Omega})$;

1st. Lemma

Let v be a weak solution of

$$(\mathrm{TP})\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\gamma_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $\mathcal{C}^{1,1}(\overline{\Omega})$;

(3) v is $\mathcal{C}^{2}(\overline{\Omega})$;

(4) v is quadratic;

One shows that there is a subspace E in $\mathbb{R}^d$ with $\dim E > 0$ such that

$$\Omega \times (E \cap S^{d-1}) \ni (x,e) \mapsto D^2V|_x(e,e) \quad \text{is constant}$$

$$\Longrightarrow V(x) = \frac{\Lambda}{2}\left|\Pi_E(x-x_0)\right|^2 + DV|_{x_0}\left(\Pi_E(x-x_0)\right) + g\left(\Pi_{E^\perp}(x)\right)$$

Now, one proceeds an induction on dimension.

Here, $\Lambda := \sup_{x\in\overline{\Omega},\, e \in S^{d-1}} D^2V|_x(e,e)$.

1st. Lemma

Let v be a weak solution of

$$(TP)\begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\gamma_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^d(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

(3) v is $C^2(\overline{\Omega})$;

(4) v is quadratic;

1st. Lemma

Let v be a weak solution of

$$(TP)\quad \begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^d(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

(3) v is $C^2(\overline{\Omega})$;

(4) v is quadratic;

(5) v is concave;

1$^{\text{st}}$. Lemma

Let v be a weak solution of

$$(TP)\begin{cases} \Delta v + \mu = 0 & \text{in } \Omega \\ D_{\nu_i} v + \gamma_i = 0 & \text{on } \Sigma_i \end{cases}$$

with constants μ, γ_i such that $\sum_{i=1}^{m} \gamma_i \mathcal{H}^{d-1}(\Sigma_i) = \mu \mathcal{H}^{d}(\Omega)$.

Then, the following statements are equivalent.

(1) v is semi-concave;

(2) v is $C^{1,1}(\overline{\Omega})$;

(3) v is $C^{2}(\overline{\Omega})$;

(4) v is quadratic;

(5) v is concave;

(6) Ω is a product of circumsolids.

THE UNIVERSITY OF SYDNEY

Daniel Hauer

Reminder

Let Ω be convex.

- A function $\varphi : \Omega \to \mathbb{R}_+ := (0,\infty)$ is *log-concave* if $V(x) := \log \varphi(x)$ is *concave*

$\iff$ for all $x, \hat{x} \in \Omega$ & $\lambda \in (0,1)$, one has that

$$\varphi(\lambda x + (1-\lambda)\hat{x}) \geq \varphi^{\lambda}(x) \cdot \varphi^{1-\lambda}(\hat{x})$$

Consequence:

φ *log-concave* $\overset{\Longrightarrow}{\cancel{\Longleftarrow}}$ for every $c \in \mathbb{R}_+$ the *super-level set* $E_c := \{x \mid \varphi(x) \geq c\}$ is *convex*.

3 Theorem [Andrews, Clutterbuck, H. '18]

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$.

- If $d=2$ & Ω is not a product of circumsolids, then the 1st Robin eigenfunction φ_α has non-convex superlevel sets for sufficiently small $\alpha>0$.
- If $d\geq 3$ & Ω has boundary points with inconsistent normals ν_i, then the same conclusion holds.

Definition

We say that a point $x_0 \in \overline{\Omega}$ has consistent normals if the outward unit normals $\{\nu_i \mid i \in \mathcal{J}(x_0)\}$ to the tangent cone Π_{x_0} lie in a hyperplane disjoint from the origine

$\iff$ Π_{x_0} is an (unbounded) circumsolid

$\iff$ there is $\gamma \in \mathbb{R}^d$ solving $\gamma \cdot \nu_i = -1 \quad i \in \mathcal{J}(x_0)$.

Otherwise, we say that x_0 has inconsitent normals.

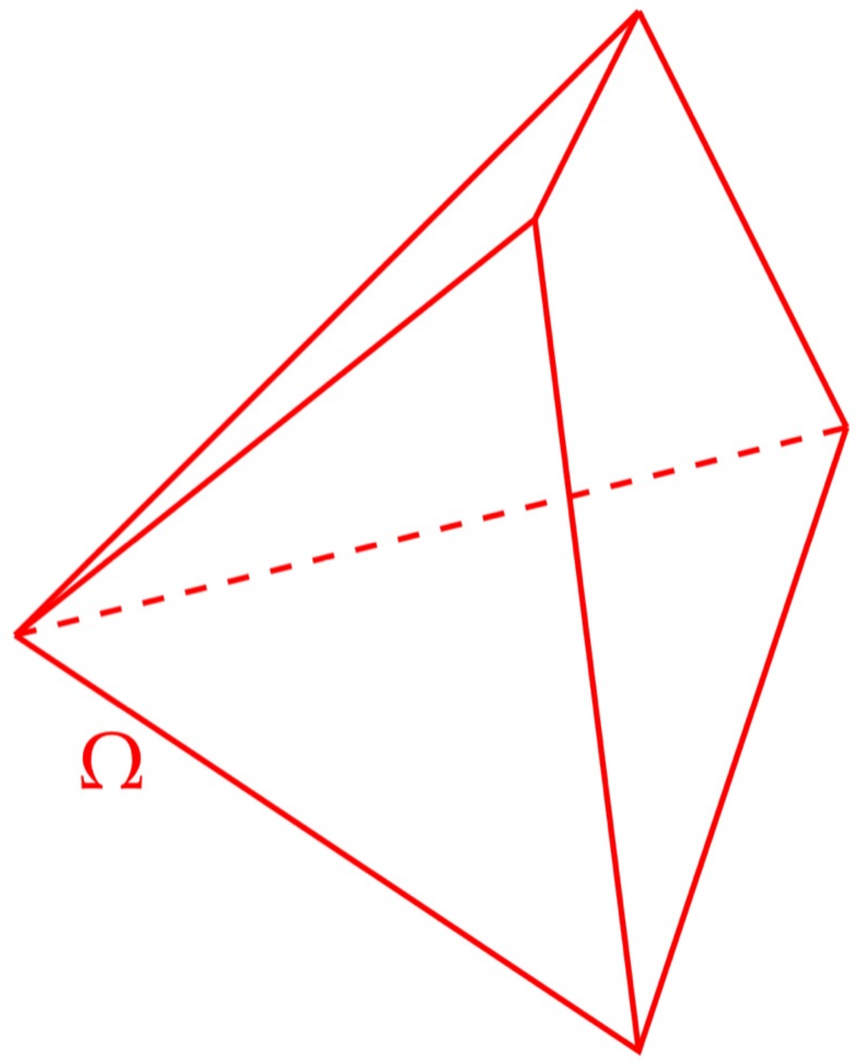
Examples

(1) If Ω is a product of circumsolids, then every boundary point has consistent normals.

Examples

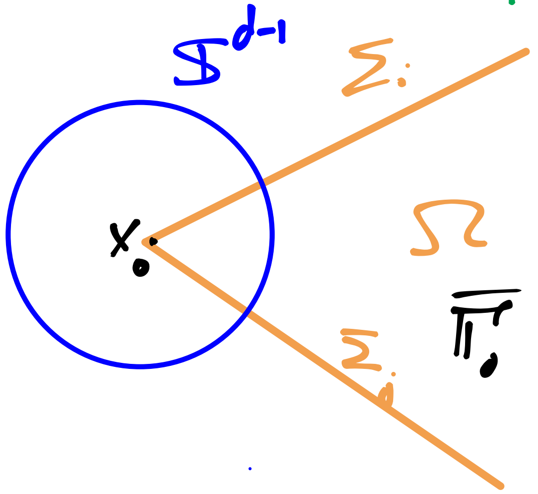
(1) If Ω is a product of circumsolids, then every boundary point has consistent normals.

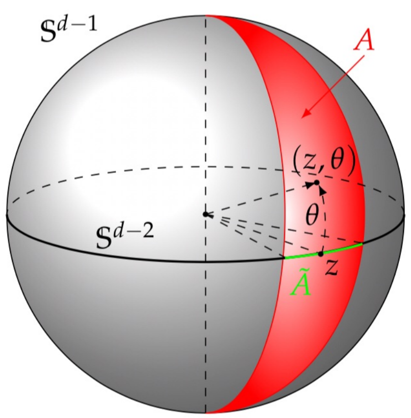
(2)



Proof of Theorem 3

- Let $x_0 \in \partial\Omega$ & choose $r > 0$ sufficiently small such that $B_r(x_0) \cap \Omega = x_0 + r\left(B_1(0) \cap \overline{\Pi}_{x_0}\right)$





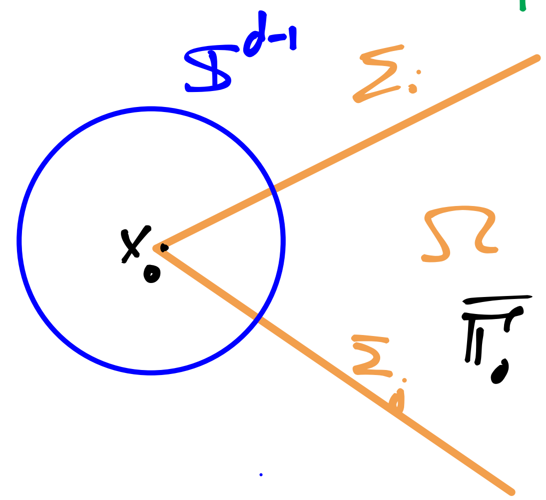
Proof of Theorem 3

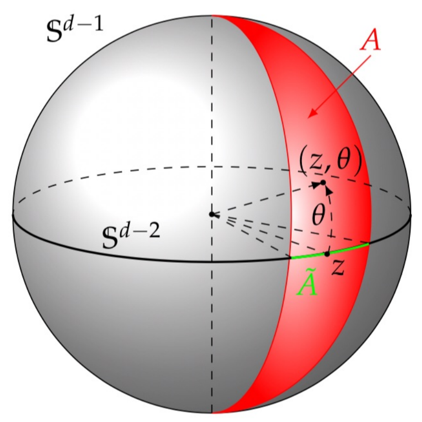
- Let $x_0 \in \partial\Omega$ & choose $r>0$ sufficiently small

  such that $B_r(x_0)\cap\Omega = x_0 + r\left(B_1(0)\cap \overline{\Pi}_{x_0}\right)$

Note

$V \notin C^{1,1}(\overline{\Omega})$

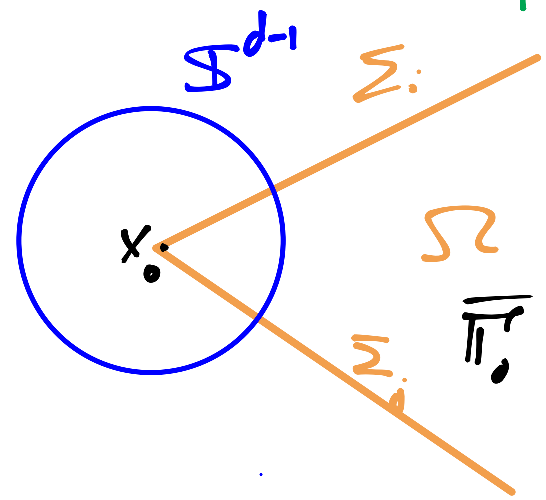


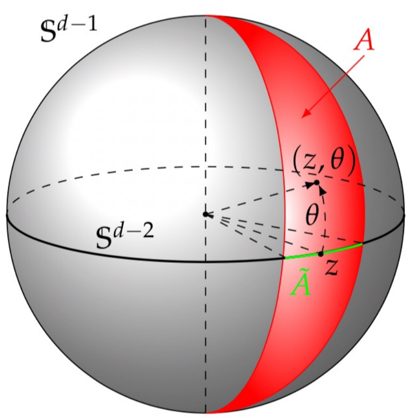


Proof of Theorem 3

- Let $x_0 \in \partial\Omega$ & choose $r > 0$ sufficiently small such that $B_r(x_0) \cap \Omega = x_0 + r\left(B_1(0) \cap \overline{\Pi}_{x_0}\right)$

Note
$v \notin \mathcal{C}^{1,1}(\overline{\Omega})$





We define $\hat{v}(x) := v(x_0 + x) + \frac{\mu}{2d}|x|^2$, $x \in B_r(0) \cap \overline{\Pi}_{x_0}$

$\Longrightarrow$ $\hat{v}$ is harmonic satisfying $D_\nu V = -1$ on $B_r(0) \cap \partial\overline{\Pi}_{x_0}$.

$\Longrightarrow$ $\exists$ harmonic function $\hat{w}$ on Γ_{x_0} which

is homogeneous of degree one in radial direction &

$$D_\nu \hat{w} = -1 \text{ on } \partial\Gamma_{x_0}$$

4 Theorem [Andrews, Clutterbuck, H. '18]

For $d \geq 3$, let A be a convex open subset on S^{d-1} with a Lipschitz boundary. Then the first Neumann eigenvalue

$$\lambda_1(A) := \inf_{\substack{\varphi \in C^\infty(\overline{A}) \\ \int_A \varphi \, dV_g = 0}} \frac{\int_A |D\varphi|^2 \, dV_g}{\int_A |\varphi|^2 \, dV_g} \geq d-1 .$$

Moreover, $\lambda_1(A) = d-1$ iff the cone $\Gamma := \{ r z \mid z \in A \} \subseteq \mathbb{R}^d$ has a linear factor $\tilde{\Gamma} \times \mathbb{R}$ for some convex cone $\tilde{\Gamma}$.

In this case, the eigenfunction to $\lambda_1(A)$ is $L(x,z)_{|A} := z_{|A}$, $(x,z) \in \tilde{\Gamma} \times \mathbb{R}$.

Daniel Hauer

$\Longrightarrow$ $\exists$ harmonic function $\hat{w}$ on Γ_{x_0} which

is homogeneous of degree one in radial direction &

$$D_\nu \hat{w} = -1 \text{ on } \partial\Gamma_{x_0}$$

$$\Longrightarrow \quad w(x) := \tilde{v}(x) - \hat{w}(x) \quad \text{on } x \in B_r(0) \cap \overline{\Gamma_{x_0}}$$

is harmonic & $D_\nu w = 0$ on $B_r(0) \cap \partial\Gamma_{x_0}$.

$\Longrightarrow$ $\exists$ harmonic function $\hat{w}$ on Γ_{x_0} which

is homogeneous of degree one in radial direction &

$$D_\nu \hat{w} = -1 \text{ on } \partial\Gamma_{x_0}$$

$\Longrightarrow$ $w(x) := \tilde{V}(x) - \hat{w}(x)$ on $x \in B_r(0) \cap \overline{\Gamma_{x_0}}$

is harmonic & $D_\nu w = 0$ on $B_r(0) \cap \partial\Gamma_{x_0}$.

$$\Longrightarrow \quad V(x_0 + x) = -\frac{\mu}{2d}|x|^2 + \sum_{i=0}^{\infty} \delta_i \, q_i(x) + \hat{w}(x)$$

on $x \in B_r(0) \cap \overline{\Gamma_{x_0}}$,

Daniel Hauer

where:

- $z_i(x) := s^{\beta_i}\varphi_i(z)$, $\quad s>0,\, z\in A := \Gamma_{x_0}\cap \mathbb{S}^{d-1}$

- $(\varphi_i)_{i\geq 0}$ are the Neumann-eigenfunctions of $\Delta^{\mathbb{S}^{d-1}}$, ONB in $L^2(A)$

- β_i : $\quad \beta_i^2 + (d-2)\beta_i - \lambda_i = 0$
- λ_i : Neumann-eigenvalue of $-\Delta^{\mathbb{S}^{d-1}}$.

$\beta_0 = 0$ (since $\lambda_0 = 0$) & $\lambda_i \geq d-1$ for $i\geq 1$
$\Rightarrow \beta_i \geq 1$

$$\boxed{\text{W.l.g., we have } \beta_i > 1 \text{ for } i\geq 1.}$$

$\underline{1^{st}\text{ case:}}$ Ω has a boundary point x_0 with incosistent normals

$\underline{1^{st} \text{ case}}$: Ω has a boundary point x_0 with incosistent normals

$\Longrightarrow$ $\hat{\omega}$ is nonlinear

1^{st} case : Ω has a boundary point x_0 with incosistent normals

$\Longrightarrow$ $\hat{\omega}$ is nonlinear

$\Longrightarrow$ $\hat{\omega}$ admits non-convex super-level sets!

$\underline{1^{st} \text{ case}}$: Ω has a boundary point x_0 with incosistent normals

$\Longrightarrow$ $\hat{\omega}$ is nonlinear

$\Longrightarrow$ $\hat{\omega}$ admits non-convex super-level sets!

Proof. Choose $z \in A$ s.th. $\hat{\omega}(z) \neq 0$ & $D^2\hat{\omega}_{|z} \neq 0$

$\Longrightarrow$ z is a null eigenvector of $D^2\hat{\omega}$ ($D^2\hat{\omega}$ has homog. degree 0)

& $\mathrm{Tr}_{(\mathbb{R}z)^\perp} D^2\hat{\omega}_{|z} = 0$ ($\Delta\hat{\omega} = 0$)

$\Longrightarrow$ $\exists\, \xi \perp z$ s.th. $D^2\hat{\omega}_{|z}(\xi,\xi) > 0$

1^{st} case: Ω has a boundary point x_0 with incosistent normals

Now, set $\eta := \xi - \frac{D\hat{\omega}_{|z}(\xi)}{\hat{\omega}(z)} \cdot z$

$$\Longrightarrow D\hat{\omega}_{|z}(\eta) = D\hat{\omega}_{|z}(\xi) - \frac{D\hat{\omega}_{|z}(\xi)}{\hat{\omega}(z)} \underbrace{D\hat{\omega}_{|z}(z)}_{\substack{=\hat{\omega}(z) \\ \text{by homog. of } \hat{\omega}}}$$

$= 0$

$$\& \; D^2\hat{\omega}_{|z}(\eta,\eta) = D^2\hat{\omega}_{|z}(\xi,\xi) > 0$$

$\underline{1^{st}\text{ case}}$: Ω has a boundary point x_0 with incosistent normals

Now, set $\eta := \xi - \frac{D\hat{\omega}_{|z}(\xi)}{\hat{\omega}(z)} \cdot z$

$$\Longrightarrow \quad D\hat{\omega}_{|z}(\eta) = D\hat{\omega}_{|z}(\xi) - \frac{D\hat{\omega}_{|z}(\xi)}{\hat{\omega}(z)} \underbrace{D\hat{\omega}_{|z}(z)}_{\substack{=\hat{\omega}(z) \\ \text{by homog. of } \hat{\omega}}} = 0$$

$$\& \quad D^2\hat{\omega}_{|z}(\eta,\eta) = D^2\hat{\omega}_{|z}(\xi,\xi) > 0$$

$$\Longrightarrow \quad S_{\hat{\omega}(z)} := \{x \mid \hat{\omega}(x) > \hat{\omega}(z)\} \text{ is not convex near } z$$

Since $\hat{\omega}(z \pm s\eta) > \hat{\omega}(z)$ for $s \neq 0$ small,

$\underline{1^{st}\text{ case:}}$ Ω has a boundary point x_0 with incosistent normals

i.e., $z \pm s\xi \in S_{\hat{w}(z)}$ for $s \neq 0$ small but $z \notin S_{\hat{w}(z)}$?

By the homog. of $\hat{w}$ $\Rightarrow$ $S_{\lambda\hat{w}(z)} := \{x \mid \hat{w}(x) > \lambda\hat{w}(z)\}$

is not convex near λz

for all $\lambda > 0$

1^{st} case : Ω has a boundary point x_0 with incosistent normals

i.e., $z \pm s\xi \in S_{\hat{w}(z)}$ for $s \neq 0$ small but $z \notin S_{\hat{w}(z)}^{\circ}$

By the homog. of $\hat{w}$ $\Rightarrow$ $S_{\lambda\hat{w}(z)} := \{x \mid \hat{w}(x) > \lambda\hat{w}(z)\}$ is not convex near λz for all $\lambda > 0$

$\Rightarrow$ V has some non-convex super-level sets.

1^{st} case: Ω has a boundary point x_0 with incosistent normals

i.e., $z \pm s z \in S_{\hat{\omega}(z)}$ for $s \neq 0$ small but $z \notin S_{\hat{\omega}(z)}$!

By the homog. of $\hat{\omega} \Rightarrow S_{\lambda\hat{\omega}(z)} := \{x \mid \hat{\omega}(x) > \lambda \hat{\omega}(z)\}$

is not convex near λz

for all $\lambda > 0$

$\Longrightarrow V$ has some non-convex super-level sets.

<u>Proof.</u> $\exists\, x_1, x_2 \in \Pi_{x_0}' \cap S_{\hat{\omega}(z)}$ but $\frac{x_1 + x_2}{2} \notin \overline{S_{\hat{\omega}(z)}}$

$\Longrightarrow \exists\, \varepsilon > 0$ s.th. $\hat{\omega}(x_i) > \hat{\omega}(z) + \varepsilon$ for $i = 1, 2$

& $\hat{\omega}\left(\frac{x_1 + x_2}{2}\right) < \hat{\omega}(z) - \varepsilon$

$\underline{1^{st}\text{ case:}}$ Ω has a boundary point x_0 with incosistent normals

$$\Longrightarrow V(x_0+\lambda x_j) = V(x_0) + \lambda \hat{\omega}(x_j) - \frac{\mu\lambda^2}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i} q_i(x_j)$$

$\underline{1^{st} \text{ case}}$: Ω has a boundary point x_0 with incosistent normals

$$\Longrightarrow V(x_0+\lambda x_j) = V(x_0) + \lambda \hat{\omega}(x_j) - \frac{\mu\lambda^2}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i} q_i(x_j)$$

$$= V(x_0) + \lambda\left(\hat{\omega}(x_j) - \frac{\mu\lambda}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i - 1} q_i(x_j)\right)$$

1^{st} case: Ω has a boundary point x_0 with incosistent normals

$$\Longrightarrow V(x_0+\lambda x_j) = V(x_0) + \lambda \hat{\omega}(x_j) - \frac{\mu\lambda^2}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i} \varphi_i(x_j)$$

$$= V(x_0) + \lambda \Big(\hat{\omega}(x_j) \underbrace{- \frac{\mu\lambda}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i - 1} \varphi_i(x_j)}_{\longrightarrow 0 \text{ as } \lambda \to 0+} \Big)$$

THE UNIVERSITY OF SYDNEY

Daniel Hauer

1^{st} case: Ω has a boundary point x_0 with incosistent normals

$$\Longrightarrow V(x_0+\lambda x_j) = V(x_0) + \lambda\hat{\omega}(x_j) - \frac{\mu\lambda^2}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i} q_i(x_j)$$

$$= V(x_0) + \lambda\Big(\hat{\omega}(x_j) \underbrace{- \frac{\mu\lambda}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i - 1} q_i(x_j)}_{\longrightarrow 0 \text{ as } \lambda \to 0+}\Big)$$

$$> V(x_0) + \lambda\big(\hat{\omega}(x_j) - \varepsilon\big)$$

$$> V(x_0) + \lambda\,\hat{\omega}(z) \text{ for } \lambda > 0 \text{ small.}$$

1^{st} case: Ω has a boundary point x_0 with incosistent normals

$$\Longrightarrow V(x_0+\lambda x_j) = V(x_0) + \lambda \hat{\omega}(x_j) - \frac{\mu\lambda^2}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i} z_i(x_j)$$

$$= V(x_0) + \lambda \underbrace{\left(\hat{\omega}(x_j) - \frac{\mu\lambda}{2d}|x_j|^2 + \sum_{i=1}^{\infty} f_i \lambda^{\beta_i - 1} z_i(x_j)\right)}_{\longrightarrow 0 \text{ as } \lambda \to 0+}$$

$$> V(x_0) + \lambda\left(\hat{\omega}(x_j) - \varepsilon\right)$$

$$> V(x_0) + \lambda\, \hat{\omega}(z) \text{ for } \lambda > 0 \text{ small.}$$

Similarly, $V\left(x_0 + \lambda \frac{x_1+x_2}{2}\right) < V(x_0) + \lambda\, \hat{\omega}(z)$

$$\Longrightarrow S_{V(x_0)+\lambda\hat{\omega}(z)} := \{x \mid V(x) > V(x_0) + \lambda\,\hat{\omega}(z)\} \text{ is not convex.}$$

2nd case: Ω has boundary points with consistent normals

THE UNIVERSITY OF SYDNEY

2^{nd} case: Ω has boundary points with consistent normals

$$\Longrightarrow \exists\, \vartheta \in \mathbb{R}^d \setminus \{0\} \text{ s.th. } \hat{w}(x) = \vartheta \cdot x \quad (\text{linear})$$

2^{nd} case: Ω has boundary points with consistent normals

$$\Rightarrow \exists\, \vartheta \in \mathbb{R}^d \setminus \{0\} \text{ s.th. } \hat{w}(x) = \vartheta \cdot x \quad (\text{linear})$$

Since Ω is not a product of circumsolids

$$\Rightarrow \exists\, x_0 \in \partial\Omega \;\&\; 1 < \beta_1 < 2 \quad \text{s.th.}$$

$$V(x_0+x) = V(x_0) - \frac{\mu}{2d}|x|^2 + f_1 \, z_1(x) + \sum_{i>1}^{\infty} f_i \, z_i(x) + \hat{w}(x)$$

on $B_r(0) \cap \Pi'_{x_0}$

$$z_i(x) = s^{\beta_i} \varphi_i(z), \quad s>0,\ z \in A.$$

$\underline{2^{nd}\text{ case:}}$ Ω has boundary points with consistent normals

To see that v has a non-convex super-level set,

show: $\exists x$ s.th. $x_0+x \in \Omega$ & $\xi \in \mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi)=0 \quad \& \quad D^2v_{|x_0+x}(\xi,\xi)>0$$

2^{nd} case: Ω has boundary points with consistent normals

To see that v has a non-convex super-level set,

show: $\exists x$ s.th. $x_0+x \in \Omega$ & $\xi \in \mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi) = 0 \quad \& \quad D^2v_{|x_0+x}(\xi,\xi) > 0$$

Note:

$$Dv_{|x_0+\lambda x}(\xi) = \gamma \cdot \xi - \frac{\mu\lambda}{d} x \cdot \xi + \sum_{i\geq 1} f_i \, \lambda^{\beta_i - 1} D q_i{}_{|x}(\xi)$$

2^{nd} case: Ω has boundary points with consistent normals

To see that v has a non-convex super-level set,

show: $\exists\, x$ s.th. $x_0 + x \in \Omega$ & $\xi \in \mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi) = 0 \quad \& \quad D^2 v_{|x_0+x}(\xi,\xi) > 0$$

Note:

$$Dv_{|x_0+\lambda x}(\xi) = \gamma \cdot \xi - \underbrace{\frac{\mu\lambda}{d}\, x \cdot \xi + \sum_{i \geq 1} f_i\, \lambda^{\beta_i - 1} D\varphi_i|_x(\xi)}_{\longrightarrow 0 \text{ as } \lambda \to 0+}$$

2^{nd} case: Ω has boundary points with consistent normals

To see that v has a non-convex super-level set, show: $\exists x$ s.th. $x_0+x\in\Omega$ & $\xi\in\mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi)=0 \quad \& \quad D^2v_{|x_0+x}(\xi,\xi)>0$$

Note:

$$D^2v_{|x_0+\lambda x}(\xi,\eta)=-\frac{\mu}{d}\,\xi\cdot\eta+f_1\cdot\lambda^{\beta_1-2}D^2\varphi_{1|x}(\xi,\eta)+\sum_{i>1}f_i\,\lambda^{\beta_i-2}D^2\varphi_{i|x}(\xi,\eta)$$

2^{nd} case: Ω has boundary points with consistent normals

To see that v has a non-convex super-level set,

show: $\exists\, x$ s.th. $x_0 + x \in \Omega$ & $\xi \in \mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi) = 0 \quad \& \quad D^2 v_{|x_0+x}(\xi,\xi) > 0$$

Note:

$$D^2 v_{|x_0+\lambda x}(\xi,\eta) = -\frac{\mu}{d}\,\xi\cdot\eta + f_1\cdot\lambda^{\beta_1-2} D^2 z_{1|x}(\xi,\eta) + \underbrace{\sum_{i>1} f_i\,\lambda^{\beta_i-2} D^2 z_{i|x}(\xi,\eta)}_{\longrightarrow\, 0 \text{ as } \lambda \to 0+}$$

2^{nd} case: Ω has boundary points with consistent normals

To see that v has a non-convex super-level set,

show: $\exists x$ s.th. $x_0 + x \in \Omega$ & $\xi \in \mathbb{R}^d$ s.th.

$$Dv_{|x_0+x}(\xi) = 0 \quad \& \quad D^2v_{|x_0+x}(\xi,\xi) > 0$$

Note:

$$D^2v_{|x_0+\lambda x}(\xi,\eta) = -\frac{\mu}{d}\,\xi\cdot\eta + f_1\cdot \underbrace{\lambda^{\beta_1-2} D^2\varphi_{1|x}(\xi,\eta)}_{\to +\infty \text{ as } \lambda\to 0+} + \sum_{i>1} f_i\, \underbrace{\lambda^{\beta_i-2} D^2\varphi_{i|x}(\xi,\eta)}_{\to 0 \text{ as } \lambda \to 0+}$$

2nd case: Ω has boundary points with consistent normals

Lemma

Suppose that the restriction of $\mathcal{V}_1$ to the hyperplane $L := \{x \in \Pi_{x_0} \mid \gamma \cdot x = |\gamma|\}$ is not concave.

Then v admits a non-convex super-level set.

In $d=2$ the hypothesis of this Lemma holds.

THE UNIVERSITY OF SYDNEY

Daniel Hauer

Open problem

Can one prove the fundamental Gap conjecture for Robin eigenvalues without using the log-concavity of the first eigenfunction?

Thank You!

References

[1] B. Andrews, J. Clutterbuck, Proof of the fundamental gap conjecture, J. Amer. Math. Soc., Vol. 24, No. 3, 899–916, 2011.

[2] M.S. Ashbaugh & R. Benguria, Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials, Proc. Amer. Math. Soc., Vol. 105, No. 2, 419–424, 1989.

[3] R. Bañuelos & P. Kröger, Gradient estimates for the ground state Schrödinger eigenfunction and application, Comm. Math. Phys., Vol. 224, No. 2, 545–550, 2001.

[4] R. Bañuelos & P.J. Méndez-Hernández, Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps, J. Funct. Anal., Vol. 176, No. 2, 368–399, 2000.

[5] M. van der Berg, On condensation in the free-Boson gas and the spectrum of the Laplacian, J. Statist. Phys., Vol. 31, No. 3, 623–637, 1983.

[6] B. Davis, On the spectral gap for fixed membranes, Ark. Mat., Vol. 39, No. 1, 65–74, 2001.

[7] M. Horváth, On the first two eigenvalues of Sturm-Liouville operators, Proc. Amer. Math. Soc., Vol. 131, No. 4, 1215–1224, 2003.

[8] R. Lavine, The eigenvalue gap for one-dimensional convex potentials, Proc. Amer. Math. Soc., Vol. 121, No. 3, 815–821, 1994.

[9] J. M. Singer, B. Wong, S.-T. Yau & Stephen S.-T. Yau, An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4), Vol. 12, No. 2, 319–333, 1985.

[10] S.-T. Yau, Nonlinear analysis in geometry, Monographies de l'Enseignement Mathématique, Vol. 33, Genova, Série de Conférences de l'Union Mathématique Internationale No. 8, 1986.

[11] Q. H. Yu & J. Q. Zhong, Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator, Trans. Amer. Math. Soc., Vol. 294, No. 1, 341–349, 1986.