A particle Gibbs sampler for Feynman-Kac measures ; Stability and propagation of chaos.

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Workshop, Shape Analysis, Stochastic Geometric Mechanics and Applied Optimal Transport, Banff International Research Station joint work with

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Discrete time case : Del Moral-Kohn-Patras, Annales IHP (16) joint work with

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Aim: to estimate Feynman-Kac integral

$$\eta_t(f) := \frac{1}{Z_t}\mathbb{E}\left[f(X_t)e^{-\int_0^t V_s(X_s)\,ds}\right]$$

where

- X_t is a continuous time Markov process with generator L_t on state space S;
- V_t is a time dependent function on S;
- $Z_t := \mathbb{E}\left[e^{-\int_0^t V_s(X_s)\,ds}\right]$.

We will rather be interested in historical process $\hat{X}_t := (X_s)_{s\le t}$. our aim will be to estimate

$$\mathbb{Q}_t(\mathcal{F}) := \frac{1}{Z_t}\mathbb{E}\left[\mathcal{F}(\hat{X}_t)e^{-\int_0^t V_s(X_s)\,ds}\right].$$

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$$L_{t,\eta_t} f(x) := L_t f(x) + V_t(x) \int_S (f(y) - f(x)) \, d\eta_t(y)$$

with η_t the distribution of $\bar{X}_t$.
Also define the N particle system $\left(\xi_t^{N,i}\right)_{1 \le i \le N}$, each particle evolving independently of the others with generator L_t,
with additional jumps at rate V_t on

$$m(\xi_t^N) := \frac{1}{N} \sum_{j=1}^{N} \delta_{\xi_t^{N,j}}.$$

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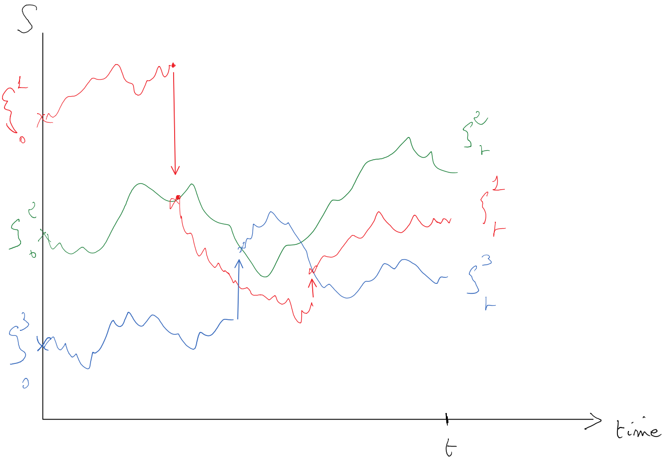

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Figure: The system of particles ξ^N

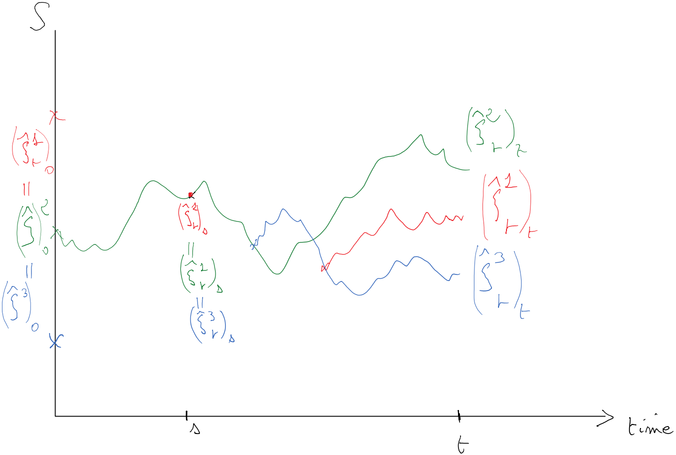

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Figure: The system of historical processes $\hat{\xi}^N$

Proposition 1

Assume $\eta_0 = \mathcal{L}(X_0) = \mathcal{L}(\bar{X}_0) = \mathcal{L}(\xi_0^{N,i})$. Then for all $N \geq 1$,

$$\eta_t(f) = \frac{1}{Z_t}\mathbb{E}\left[f(X_t)e^{-\int_0^t V_s(X_s)\,ds}\right] = \mathbb{E}[f(\bar{X}_t)] = \frac{1}{Z_t}\mathbb{E}\left[m(\xi_t^N)(f)e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\right].$$

Proof

Define $\alpha_t^N(f) := \mathbb{E}\left[m(\xi_t^N)(f)e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\right]$, and $\alpha_t^\infty(f) := \mathbb{E}\left[f(\bar{X}_t)e^{-\int_0^t \eta_s(V_s)\,ds}\right]$.

Then $\eta_t(f) = \dfrac{\alpha_t^1(f)}{\alpha_t^1(1)}$, $\alpha_0^1(f) = \alpha_0^N(f) = \alpha_0^\infty(f)$ and

$$\begin{aligned}\frac{d}{dt}\alpha_t^N(f) =& \mathbb{E}\Bigg[\Big\{L_t\left(m(\xi_t^N)(f)\right) + \frac{1}{N}\sum_{i=1}^N V_t\left(\xi_t^{N,i}\right)\left(m(\xi_t^N)(f) - f(\xi_t^{N,i})\right) \\ & - m(\xi_t^N)(f)m(\xi_t^N)(V_t)\Big\}e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\Bigg] \\ =& \alpha_t^N\left((L_t - V_t)f\right)\end{aligned}$$

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Remark on jumps: $\widehat{\xi}_t^{N,i}$ jumps at rate $V_t(\xi_t^{N,i})$ on $m(\widehat{\xi}_t^N)$.

We will calculate $\mathbb{Q}_t(\mathcal{F})$ with $\mathbb{E}[\mathcal{F}(\widehat{\bar{X}}_t)] = \frac{1}{Z_t}\mathbb{E}\left[m(\widehat{\xi}_t^N)(\mathcal{F})e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\right]$.

Denote $\mathbb{X}_t$ a random variable with $\mathcal{L}(\mathbb{X}_t|\xi_{[0,T]}^N) = m(\widehat{\xi}_t^N)$.

Define $\zeta_t^N := (\zeta_t^{N,1}, \ldots, \zeta_t^{N,N}) = (\zeta_t^{N,1}, \zeta_t^{N,-})$ by

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Remark on jumps: $\widehat{\xi}_t^{N,i}$ jumps at rate $V_t(\xi_t^{N,i})$ on $m(\widehat{\xi}_t^N)$.

We will calculate $\mathbb{Q}_t(\mathcal{F})$ with $\mathbb{E}[\mathcal{F}(\widehat{\bar{X}}_t)] = \frac{1}{Z_t}\mathbb{E}\left[m(\widehat{\xi}_t^N)(\mathcal{F})e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\right]$.

Denote $\mathbb{X}_t$ a random variable with $\mathcal{L}(\mathbb{X}_t|\xi_{[0,T]}^N) = m(\widehat{\xi}_t^N)$.

Define $\zeta_t^N := (\zeta_t^{N,1}, \ldots, \zeta_t^{N,N}) = (\zeta_t^{N,1}, \zeta_t^{N,-})$ by

- $\zeta_t^{N,1} = X_t$ the frozen trajectory
- each of the $N-1$ particles of $\zeta_t^{N,-} = (\zeta_t^{N,2}, \ldots, \zeta_t^{N,N})$ evolve independently with generator L_t,
- with jumps at rate $(1 - \frac{1}{N})V_t$, on $m(\zeta_t^{N,-})$
- and additional jumps at rate $\frac{2}{N}V_t$ on $\zeta_t^{N,1}$.

Corollary 1

Assume $\eta_0 = \mathcal{L}(X_0) = \mathcal{L}(\bar{X}_0) = \mathcal{L}(\xi_0^{N,i})$. Then for all $N \geq 1$,

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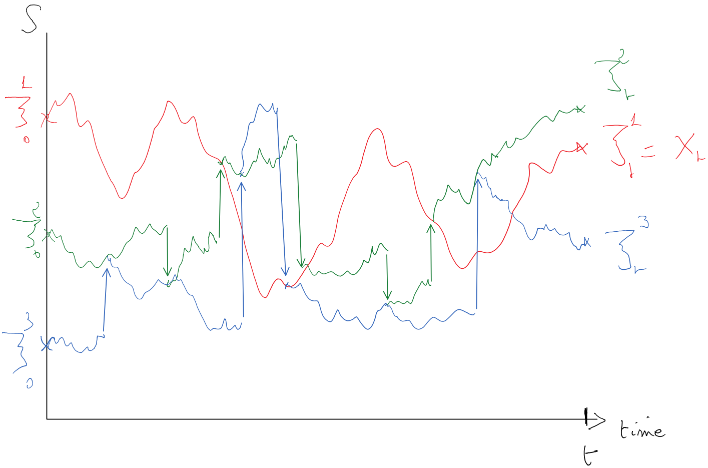

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Figure: The system of particles ζ^N

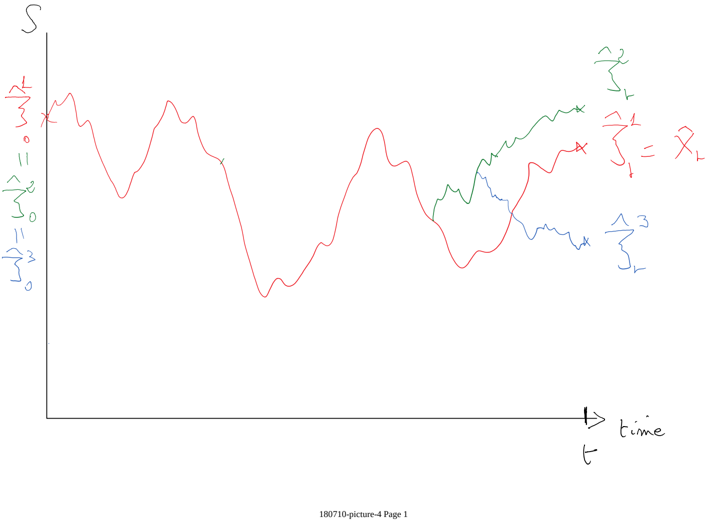

Figure: The system of historical processes $\hat{\zeta}^N$

Theorem 1

Assume that the $\xi_0^{N,i}$, $1 \le i \le N$ (resp. $\zeta_0^{N,i}$, $1 \le i \le N$) are independent with law η_0. Then

$$\mathbb{E}\left[(\mathcal{F}(\mathbb{X}_t, \widehat{\xi}_t^N)e^{-\int_0^t m(\xi_s^N)(V_s)\,ds}\right] = \mathbb{E}\left[(\mathcal{F}(\widehat{X}_t, \widehat{\zeta}_t^N)e^{-\int_0^t (V_s(X_s))\,ds}\right].$$

as soon as $\mathcal{F}$ is symmetric with respect to the $N-1$ last variables.

In particular, letting

$$\mathbb{Q}_t = \frac{1}{Z_t}e^{-\int_0^t V_s(X_s)\,ds}.\mathbb{P}_t^X, \quad \mathbb{Q}_t^\xi := \frac{1}{Z_t}e^{-\int_0^t m(\xi_s)(V_s)\,ds}.\mathbb{P}_t^\xi \quad \text{and} \quad \mathbb{Q}_t^\zeta := \frac{1}{Z_t}e^{-\int_0^t V_s(X_s)\,ds}.\mathbb{P}_t^\zeta$$

(we drop the $N's$ for simplicity) we have

$$\mathbb{Q}_t^\xi = \mathbb{Q}_t^\zeta \quad \text{and} \quad \mathbb{Q}_t^{\zeta^i} = \mathbb{Q}_t.$$

Theorem 1

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(we drop the $N's$ for simplicity) we have

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Let

$$\mathbb{M}_t(z_1, dz_2) := \mathbb{P}\left(\widehat{\zeta}_t \in dz_2 | \widehat{\zeta}_t^1 = z_1\right)$$

$$\mathbb{A}_t(z_2, dz_1) := m(z_2, dz_1)$$

$$\Pi_t(d(z_1, z_2)) := \mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1, dz_2)$$

the forward transition

$$\mathbb{G}_t((z_1, z_2); d(\bar{z}_1, \bar{z}_2)) := \mathbb{M}_t(z_1, d\bar{z}_2)\mathbb{A}(\bar{z}_2, d\bar{z}_1),$$

the backward transition

$$\mathbb{G}_t^-((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) := \mathbb{A}_t(\bar{z}_2, dz_1)\mathbb{M}_t(z_1, dz_2)$$

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Theorem 2

$$\Pi_t(d(z_1,z_2))\mathbb{G}_t((z_1,z_2);d(\bar{z}_1,\bar{z}_2)) = \Pi_t(d(\bar{z}_1,\bar{z}_2))\mathbb{G}_t^-((\bar{z}_1,\bar{z}_2),d(z_1,z_2)).$$

As a consequence, $\mathbb{Q}_t$ is reversible with respect to $\mathbb{K}_t$

$$\Pi_t(d(z_1,z_2)) \sim \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{\mathbb{M}(z_1,d\bar{z}_2)}{\longrightarrow} \begin{pmatrix} z_1 \\ \bar{z}_2 \sim \left(\widehat{\zeta}_t \middle| \widehat{\zeta}_t^1 = z_1\right) \end{pmatrix} \overset{\mathbb{A}(\bar{z}_2,d\bar{z}_1)}{\longrightarrow} \begin{pmatrix} \bar{z}_1 \sim m(\bar{z}_2) \\ \bar{z}_2 \end{pmatrix} \sim \Pi_t(d(\bar{z}_1,\bar{z}_2))$$

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$$\Pi_t(d(z_1,z_2))\mathbb{G}_t((z_1,z_2);d(\bar{z}_1,\bar{z}_2)) = \Pi_t(d(\bar{z}_1,\bar{z}_2))\mathbb{G}_t^-((\bar{z}_1,\bar{z}_2),d(z_1,z_2)).$$

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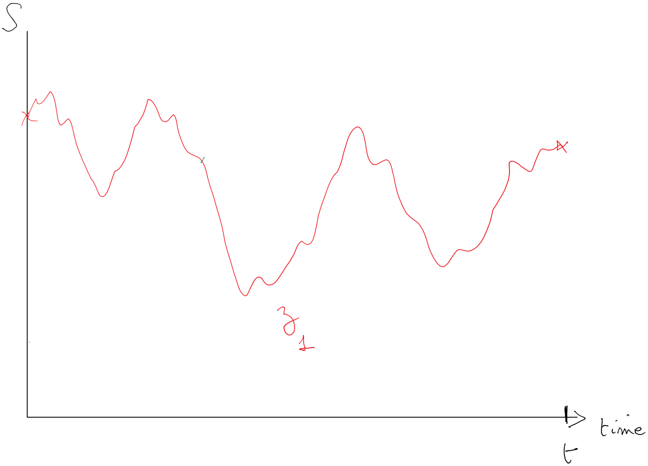

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Figure:

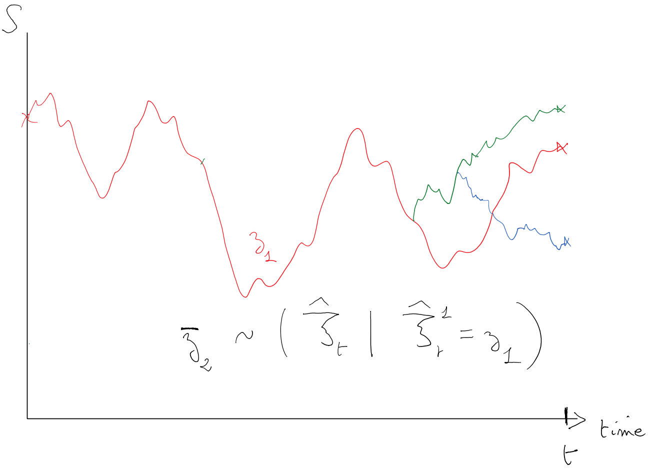

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Figure:

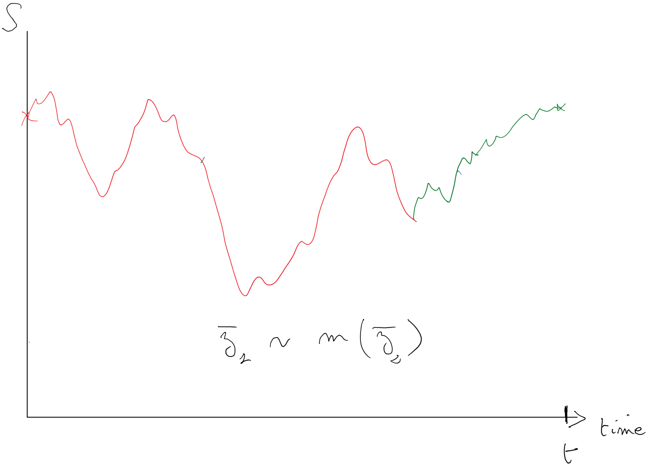

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Figure:

Proof From $\mathbb{Q}_t^\xi = \mathbb{Q}_t^\zeta$ we get
$\mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1, dz_2) = \Pi_t(d(z_1, z_2)) = \mathbb{Q}_t^\xi(dz_2)\mathbb{A}_t(z_2, dz_1)$.
Moreover

$$\begin{aligned}
\Pi_t(d(z_1, z_2))\mathbb{G}_t((z_1, z_2); d(\bar{z}_1, \bar{z}_2)) &= \mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1, dz_2)\mathbb{M}_t(z_1, d\bar{z}_2)\mathbb{A}_t(\bar{z}_2, d\bar{z}_1) \\
&= \mathbb{Q}_t^\xi(d\bar{z}_2)\mathbb{A}_t(\bar{z}_2, dz_1)\mathbb{M}_t(z_1, dz_2)\mathbb{A}_t(\bar{z}_2, d\bar{z}_1) \\
&= \Pi_t(d(\bar{z}_1, \bar{z}_2))\mathbb{G}_t^-((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) \\
&= \mathbb{Q}_t(d\bar{z}_1)\mathbb{M}_t(\bar{z}_1, d\bar{z}_2)\mathbb{G}_t^-((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) \\
&= \mathbb{Q}_t(d\bar{z}_1)\mathbb{M}_t(\bar{z}_1, d\bar{z}_2)\mathbb{A}_t(\bar{z}_2, dz_1)\mathbb{M}_t(z_1, dz_2).
\end{aligned}$$

Integrating with respect to z_2 and $\bar{z}_2$ we get

$$\mathbb{Q}_t(dz_1)\mathbb{K}_t(z_1, d\bar{z}_1) = \mathbb{Q}_t(d\bar{z}_1)\mathbb{K}_t(\bar{z}_1, dz_1)$$

with

$$\mathbb{K}_t = \mathbb{M}_t\mathbb{A}_t : \qquad \mathbb{K}_t(f)(z_1) = \mathbb{E}\left[m(\widehat{\zeta}_t)(f) | \widehat{\zeta}_t^1 = z_1\right]. \quad \square$$

Proof From $\mathbb{Q}_t^{\xi} = \mathbb{Q}_t^{\zeta}$ we get
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Moreover

$$\begin{aligned}
\Pi_t(d(z_1, z_2))\mathbb{G}_t((z_1, z_2); d(\bar{z}_1, \bar{z}_2)) &= \mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1, dz_2)\mathbb{M}_t(z_1, d\bar{z}_2)\mathbb{A}_t(\bar{z}_2, d\bar{z}_1) \\
&= \mathbb{Q}_t^{\xi}(d\bar{z}_2)\mathbb{A}_t(\bar{z}_2, dz_1)\mathbb{M}_t(z_1, dz_2)\mathbb{A}_t(\bar{z}_2, d\bar{z}_1) \\
&= \Pi_t(d(\bar{z}_1, \bar{z}_2))\mathbb{G}_t^{-}((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) \\
&= \mathbb{Q}_t(d\bar{z}_1)\mathbb{M}_t(\bar{z}_1, d\bar{z}_2)\mathbb{G}_t^{-}((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) \\
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\end{aligned}$$

Integrating with respect to z_2 and $\bar{z}_2$ we get

$$\mathbb{Q}_t(dz_1)\mathbb{K}_t(z_1, d\bar{z}_1) = \mathbb{Q}_t(d\bar{z}_1)\mathbb{K}_t(\bar{z}_1, dz_1)$$

with

$$\mathbb{K}_t = \mathbb{M}_t\mathbb{A}_t : \qquad \mathbb{K}_t(f)(z_1) = \mathbb{E}\left[m(\widehat{\zeta}_t)(f) | \widehat{\zeta}_t^1 = z_1\right]. \quad \square$$

Proof From $\mathbb{Q}_t^\xi = \mathbb{Q}_t^\zeta$ we get
$\mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1, dz_2) = \Pi_t(d(z_1, z_2)) = \mathbb{Q}_t^\xi(dz_2)\mathbb{A}_t(z_2, dz_1)$.
Moreover

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\Pi_t(d(z_1,z_2))\mathbb{G}_t((z_1,z_2);d(\bar z_1,\bar z_2)) &= \mathbb{Q}_t(dz_1)\mathbb{M}_t(z_1,dz_2)\mathbb{M}_t(z_1,d\bar z_2)\mathbb{A}_t(\bar z_2,d\bar z_1)\\
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&= \Pi_t(d(\bar z_1,\bar z_2))\mathbb{G}_t^-((\bar z_1,\bar z_2),d(z_1,z_2))\\
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Theorem 3

Under some regularity assumptions, for all functions f with oscillations bounded by 1,

$$|\mathbb{E}\left[m(\xi_t)(f)\right] - \eta_t(f)|_{\text{tv}} \leq \frac{ct}{N}$$

for an explicit c not depending on f, t and N.
Similarly, a.s. uniformly

$$\left|\mathbb{E}\left[m(\widehat{\zeta}_t^-)(\mathcal{F})|\widehat{\zeta}_t^1\right] - \mathbb{Q}_t(\mathcal{F})\right|_{\text{tv}} \leq \frac{ct}{N}.$$

As a consequence, for all $n \geq 1$, $t \geq 0$, $N \geq 1$, for all probability measure μ on càdlàg paths from $[0, t]$ to S,

$$\left|\mu\mathbb{K}_t^n - \mathbb{Q}_t\right|_{\text{tv}} \leq \left(\frac{c(t \vee 1)}{N}\right)^n$$

The regularity condition is satisfied if S is a compact manifold, L_t is an elliptic diffusion with jumps, V is bounded.
More generally, it is satisfied under the condition: $\exists h > 0$ s.t. $\forall t \geq 0,\ x \in S$,

$$\rho(h)\mu_{t,h}(dy) \leq P_{t,t+h}(x, dy) \leq \rho(h)^{-1}\mu_{t,h}(dy),$$

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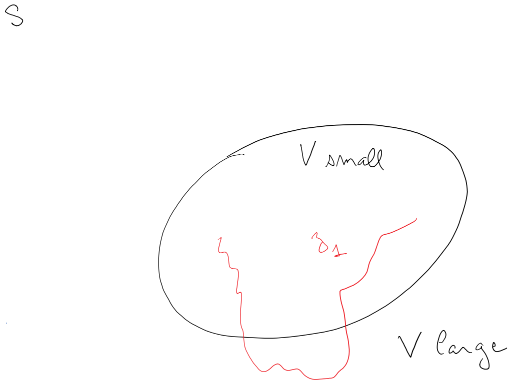

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Figure:

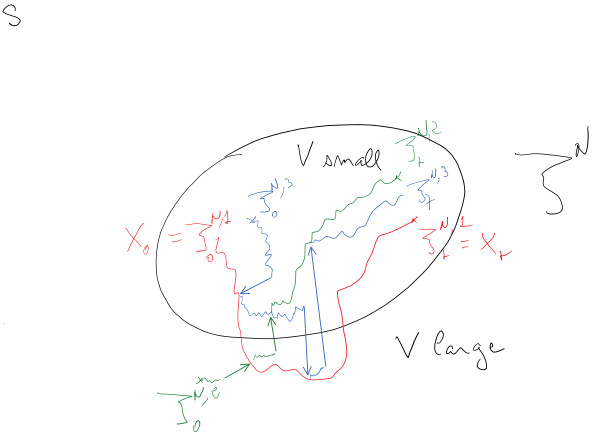

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Figure:

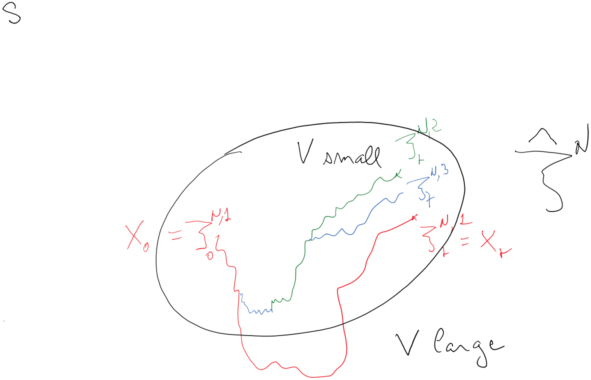

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Figure:

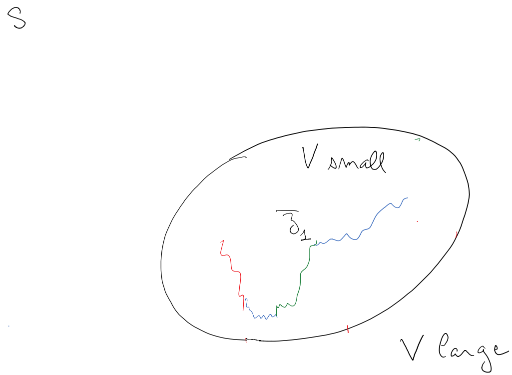

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Figure:

Idea of proof of $|\mathbb{E}\left[m(\xi_t)(f)\right] - \eta_t(f)|_{\mathrm{tv}} \leq \frac{ct}{N}$

Let $\phi_{s,t}(\eta)$ be the law at time t of the nonlinear diffusion $\bar{X}^{\eta}_{s,t}$ started at time s with law η.
Then

$$\phi_{0,t}(m(\xi_0)) = \phi_{0,t}(\eta_0) = \eta_t \quad \text{and} \quad \phi_{t,t}(m(\xi_t)) = m(\xi_t)$$

so we investigate the interpolation $s \mapsto \phi_{s,t}(m(\xi_s)) =: Y_s$,and prove that it is a semimartingale with drift bounded in absolute value by c/N.
So

$$|\mathbb{E}[m(\xi_t)(f)] - \eta_t(f)| \leq \frac{ct}{N}.$$

Similarly

$$\left|\mathbb{E}[m(\widehat{\zeta}_t^{-})(\mathcal{F})|\hat{\zeta}_t^1] - \eta_t(\mathcal{F})\right| \leq \frac{ct}{N}$$

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- $Y_t = Y_t^{\eta_0}$ a Brownian motion with non linear drift $b_t(\eta_t, Y_t) := \int_M \eta_t(dx) b_t(x,y)$ in a manifold (M, g)
- $\phi_t(\eta_0) = \eta_t$, the law of Y_t
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$$B_t(z_1,z_2) := \begin{pmatrix} \nabla_y b_t(z_2,z_1) & \nabla_x b_t(z_1,z_2) \\ \nabla_x b_t(z_2,z_1) & \nabla_y b_t(z_1,z_2) \end{pmatrix}, \quad C_t = \frac{1}{2}(B_t + B_t')$$

Theorem 4

Under condition

$$(H1) : C_t(x,y) - \frac{1}{2}\mathrm{Ric}_{M\times M}(x,y) \le -\lambda_1 g_{M\times M}(x,y)$$

we have

$$\mathbb{W}_2(\phi_t(\mu_0), \phi_t(\mu_1)) \le e^{-\lambda_1 t}\mathbb{W}_2(\mu_0,\mu_1).$$

If $b_t(x,y) = -\nabla(F\circ\rho_x)(y) - \nabla U(y)$, with F a real function, ρ_x distance to x,

$$(H_1) : \nabla^2 U_{M\times M}(x,y) + \nabla^2(F\circ\rho)(x,y) + \frac{1}{2}\mathrm{Ric}_{M\times M}(x,y) \ge \lambda_1 g_{M\times M}(x,y).$$

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$$(H1) : C_t(x,y) - \frac{1}{2}\mathrm{Ric}_{M\times M}(x,y) \le -\lambda_1 g_{M\times M}(x,y)$$

we have

$$\mathbb{W}_2(\phi_t(\mu_0), \phi_t(\mu_1)) \le e^{-\lambda_1 t}\mathbb{W}_2(\mu_0,\mu_1).$$

If $b_t(x,y) = -\nabla(F\circ\rho_x)(y) - \nabla U(y)$, with F a real function, ρ_x distance to x,

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Sketch of proof : infinitesimal parallel coupling

- $\varepsilon \mapsto Y_0^\varepsilon$ optimal between $Y_0^0 \sim \mu_0$ and $Y_0^1 \sim \mu_1$,
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Theorem 5

Assume $\mathrm{Ric}_M \geq \kappa$. Under condition $(H2)$: $C_t(x,y) \leq -\lambda_2 g_{M\times M}(x,y)$ we have $\mathbb{E}\left[\rho^2(\zeta_t^1,\xi_t^1)\right]^{1/2} \leq \frac{2}{2\lambda_2+\kappa}\left(1-e^{-\frac{(2\lambda_2+\kappa)t}{2}}\right)\sqrt{\frac{\beta_t(\mu)}{N}}$
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$$\beta_t(\mu) := \frac{1}{N}\int \phi_t(\mu)(dx)\,\|b_t(x,x)-b_t(\phi_t(\mu),x)\|^2 + \left(1-\frac{1}{N}\right)\int \phi_t(\mu)(dx)\phi_t(\mu)(dy)\,\|b_t(x,y)-b_t(\phi_t(\mu),y)\|^2.$$

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Thanks for your attention