Hessenberg Varieties in Combinatorics, Geometry and Representation Theory: 18w5130

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1 Overview of the Field

The topic of this workshop was Hessenberg varieties, which is an exciting area of research lying in the rich intersection of geometry, combinatorics, and representation theory. These varieties were first introduced by De Mari and Shayman in 1988 in the context of computational linear algebra, and more specifically, for applications to the QR algorithm for finding eigenvalues of a complex matrix. Since then, Hessenberg varieties have become important objects in a variety of research areas. The workshop was a great opportunity for mathematicians in disparate fields to share recent results and consider together various interesting open problems and new directions.

In their greatest generality, building on the ideas of De Mari and Shayman, Hessenberg varieties were defined in a 2006 paper by Goresky, Kottwitz and MacPherson [9] for applications to the computation of the orbital integrals that arise in representation theory. Let G be a reductive algebraic group defined over a field $\mathbb{F}$ and let V be an $\mathbb{F}[G]$-module. Fix $s \in V$, a parabolic subgroup P of G, and a P-invariant subspace M of V. The Hessenberg variety $X(s, M, P)$ is defined to be the set of points gP in the partial flag variety G/P such that $g^{-1}(s) \in M$. In most of the discussion below, P will be a Borel subgroup $B \leq G$, V will be the Lie algebra $\mathfrak{g}$ of G with the adjoint action of G, and $M = \mathfrak{m}$ will be a subspace of $\mathfrak{g}$ containing the Lie algebra $\mathfrak{b}$ of B. We write $X(s, \mathfrak{m})$ in this case.

In what follows, we provide a brief overview of past and current work on Hessenberg varieties. We do not claim to be exhaustive.

First, note that $X(s, \mathfrak{b})$ consists of all cosets gB satisfying $sgB \subseteq gB$. The varieties $X(s, \mathfrak{b})$, also called Springer fibers, received much attention from representation theorists during the 1970s and 1980s in the case where s is nilpotent. A highlight of this activity was Springer's description in [16] of irreducible representations of Weyl groups using actions on the cohomology of $X(s, \mathfrak{b})$.

In the original paper on Hessenberg varieties, De Mari and Shayman analyzed $X(s, \mathfrak{m})$ when $G = \mathbf{GL}_n(\mathbb{C})$ and s is a generic (i.e. regular semisimple). The history of and motivation for this enterprise is discussed by De Mari and Shayman in the introduction to [7]. In the same paper, the Betti numbers for such $X(s, \mathfrak{m})$ are determined for a large class of $\mathfrak{m}$. It turns out that in each case the sum of these Betti numbers is $n!$, the order of the Weyl group S_n. The associated Poincaré polynomial is the generating function for a naturally defined statistic on S_n, determined by $\mathfrak{m}$. These permutation statistics interpolate between two well-studied statistics on S_n, the descent number and the length. Here one can see productive interaction between geometry and combinatorics. It is not obvious a priori that the generating functions for the combinatorial statistics in question should be palindromic and unimodal, but this becomes obvious upon applying the Hard Lefschetz Theorem to the Hessenberg varieties. Additional productive interactions followed.

Key results in [7] were generalized by De Mari, Procesi and Shayman in [6]. There, the authors showed that for arbitrary semisimple G, generic $s \in \mathfrak{g}$ and arbitrary $\mathfrak{m}$, the variety $X(s, \mathfrak{m})$ is smooth, the sum of the Betti numbers of $X(s, \mathfrak{m})$ is the order of the Weyl group W, and the Poincaré polynomial is the generating function for a nice statistic on W.

If s is in the subset $\mathfrak{g}^{\text{rs}}$ of regular semisimple elements, then the centralizer $C_G(s)$ is a maximal torus in G and acts on $X := X(s, \mathfrak{m})$. In [21], Tymoczko showed that this torus action satisfies the conditions necessary for the application of the theory developed by Goresky, Kottwitz and MacPherson in [8]. From this, one obtains a combinatorial description of the equivariant and and ordinary cohomology rings of X in terms of the moment graph of X. Using this description, Tymoczko showed that $H^*(X, \mathbb{C})$ admits a representation of W.

Recent work has allowed the determination of the cohomology representations in question when $G = \mathbf{GL}_n(\mathbb{C})$. Shareshian and Wachs conjectured in [14, 15] that the Frobenius characteristic of the graded S_n-representation $\sum_{j \geq 0} H^{2j}(X, \mathbb{C})t^j$ tensored with the sign character, is the chromatic quasisymmetric function $F_{\mathfrak{m}}(t)$ of a graph determined by $\mathfrak{m}$. The chromatic quasisymmetric functions refine Stanley's chromatic symmetric functions, which are generating functions for proper colorings of graphs. (By setting $t = 1$, one gets the chromatic symmetric function originally defined by Stanley in [17].) The conjecture was proved first by Brosnan and Chow in [3] by studying the sheaves on $\mathfrak{g}$ obtained from the cohomology groups $H^*(X(s, \mathfrak{m}))$ by varying s. (This sheaf is a local system on the set $\mathfrak{g}^{\text{rs}}$, and the attached monodromy representation factors through S_n.) Another completely independent proof was given by Guay-Paquet in [10] who used a beautiful theorem of Aguiar, Bergeron and Sottile on Hopf algebras and quasisymmetric functions. When combined with the Brosnan-Chow-Guay-Paquet (BC-G-P) theorem, combinatorial results of Shareshian-Wachs in [15] and Athanasiadis in [2] provide formulas for both the irreducible decomposition and character values of the cohomology representations.

The BC-G-P Theorem gives the possibility of attacking a longstanding combinatorial conjecture using geometry. The Stanley-Stembridge conjecture in [17] asserts that the symmetric function obtained from $F_{\mathfrak{m}}(t)$ by setting $t = 1$ is e-positive, that is, a nonnegative integer combination of elementary symmetric functions. (This conjecture arose from an equivalent conjecture of Stanley and Stembridge in [18] on immanants.) Shareshian and Wachs conjectured in [14, 15] that in fact the coefficient of each t^j in $F_{\mathfrak{m}}(t)$ is e-positive. Given the BC-G-P Theorem, this is equivalent to the conjecture that the representation of S_n on $H^{2j}(X, \mathbb{C})$ arises from a permutation representation in which each point stabilizer is a Young subgroup. Attempts to settle the Stanley-Stembridge and Shareshian-Wachs Conjectures on e-positivity using this equivalence are ongoing. Connections of all of this work with the representation theory of the type A Hecke algebra are described by Haiman in [11] and by Clearman, Hyatt, Shelton and Skandera in [5].

For generic $s \in \mathfrak{g}$, the varieties $X(s, \mathfrak{m})$ are paved by affines. This is easily seen using the theorem of Bialynicki-Birula, and it is the key to the computation of the Betti numbers in [6]. In her 2003 PhD thesis, Tymoczko generalized this result to all s in the case $G = \mathbf{GL}_n$ and used the result to compute the Betti numbers of all the Hessenberg varieties $X(s, \mathfrak{m})$ for $\mathbf{GL}_n$ [20]. This result, which is a crucial input to [3], was generalized recently by Precup to arbitrary reductive groups (for a large class of elements s) [12]. We note that, while the structure of $H^*(X(s, \mathfrak{m}))$ as a graded vector space is essentially known by the above results of Tymoczko and Precup, the ring structure is still somewhat mysterious. Recently, much work has been done in this direction by H. Abe, Harada, Horiguchi and Masuda [1]. Moreover, Horiguchi, Masada, T. Abe, Murai, and Sato have proved that the following three rings are isomorphic: the cohomology of a Hessenberg variety for regular nilpotent s, the Weyl group invariants of the cohomology of a generic s, and the quotient of the polynomial ring by the ideal coming from the logarithmic derivation module of certain hyperplane arrangements. The isomorphism between the first two rings generalizes Theorem A of [1] where it was first proved for $G = \mathbf{GL}_n$. (Theorem 2.1 of [1] was a major influence on the work of Brosnan-Chow who generalized it in a different direction.)

The varieties considered by Goresky, Kottwitz and MacPherson are smooth for generic s, but are not in general paved by affines even when s is generic. Their cohomology is, however, conjectured to be motivated (in the sense used by Arapura) by hyperelliptic curves; this means that for any such Hessenberg variety X, there should exist a surjective morphism $Y \to X$ where Y is a product of hyperelliptic curves. In particular, the Hodge structure on the cohomology of the Hessenberg varieties considered by GKM is supposed to lie in the full tensor subcategory generated by the cohomology of hyperelliptic curves. This has been proved in an interesting special case by Chen, Vilonen and Xue (CVX) [4]. Moreover, CVX compute the sheaves obtained by varying the element s. This is analogous to the work of Brosnan-Chow. However, while the monodromy of the sheaves considered by Brosnan and Chow factor through the symmetric group, the CVX sheaves have infinite monodromy.

2 Recent Developments and Open Problems

As already noted, the workshop was an excellent opportunity for interaction between mathematicians with disparate interests and expertise. The workshop was successful in facilitating this interaction.

Some of the most exciting recent developments were already summarized in the previous section. Additionally, the following are some specific open problems, in part motivated by these recent developments, that currently drive this research area. Some of these were brought up during a problem session held during the workshop.

1. There are basic geometric questions about Hessenberg varieties which are still open. For example, during our Open Problems Session, Erik Insko asked to describe the singular locus of $X(N, \mathfrak{m})$ in the important case that N is a regular nilpotent.

2. While the theorem of Brosnan-Chow and Guay-Paquet opens the door for a geometric approach to the Stanley-Stembridge and Shareshian-Wachs Conjectures on e-positivity, this has yet to be carried out successfully.

3. Almost all of the combinatorial and geometric work on the varieties $X(s, \mathfrak{m})$ has been in the Lie type A case. The problem of formulating and proving the right conjectures for other simple Lie algebras is currently wide open. For example, there is no known analogue of the chromatic quasisymmetric function $F_{\mathfrak{m}}(t)$ for other Lie types. There is a natural analogue of the Stanley-Stembridge Conjecture that should be investigated. Work of Stembridge in [19] shows that the most obvious analogue of the more general Shareshian-Wachs Conjecture is false, but it is possible that something of this nature is true in many cases. In particular, in the Open Problem Session, Shareshian asked: when is the Weyl group action on the cohomology of a Hessenberg variety a permutation module?

   In connection with this, we mention that, recently, Hiraku Abe and Naoki Fujita have announced results which can be described as a "Weyl character formula for Hessenberg varieties".

4. The representation of W on the cohomology of $X(s, \mathfrak{m})$ described above is determined by an action of W on the moment graph provided by GKM theory. Except in a few specific cases, notably the case of toric Hessenberg varieties, it is not known if this representation arises from an action of W on the variety itself.

   On the other hand, while e-positivity is known in the toric case one would like to have an explicit basis in the equivariant cohomology groups of the Hessenberg variety which is permuted by the Weyl group. This was the subject of a conjecture stated by Chow in the Open Problems Session: he gave a combinatorially defined subset of the equivariant cohomology (in the type A toric case) which is manifestly permuted by the symmetric group and asked whether or not it forms a basis.

5. While the beautiful results of Chen, Vilonen and Xue compute the motive of GKM Hessenberg varieties in one interesting example, the conjecture that all such smooth varieties have cohomology motivated by hyperelliptic curves still seems very hard. Schoen has defined a numerical invariant τX based on the Mumford-Tate groups of a variety X with the following property: if X is dominated by a product of curves then $\tau X \leq 1$ [13]. It may be interesting to compute τX for GKM Hessenberg varieties as a way of gaining evidence for the conjecture. It would also be interesting to try to come up with a conjectural description of the sheaves obtained from the cohomology groups $H^*(s, M, P)$ by varying s. This is done explicitly by CVX in their special case, where the sheaves give rise to representations of the braid group. Perhaps there is a description of these sheaves analogous to the combinatorial description provided by Brosnan-Chow in their proof of the Shareshian-Wachs conjecture.

3 Presentation Highlights

To start the week, Dr. Hiraku Abe gave a beautiful overview talk on the subject of Hessenberg varieties, setting the stage for the week to come. His main goal was to convey that Hessenberg varieties are a subject lying at an exciting intersection of geometry, algebra, and combinatorics, and that they can be studied from multiple different perspectives. Delving into more detail, he explained how the study of Hessenberg varieties touches upon subjects such as the representations of symmetric groups, hyperplane arrangements, the Stanley-Stembridge conjecture, Schubert polynomials, toric degenerations, integrable systems, (holomorphic) symplectic/Poisson geometry, and the Toda lattice.

Dr. Julianna Tymoczko discussed some of the state-of-the-art techniques for describing and computing the cohomology and K-theory rings of Hessenberg varieties. Two of the techniques which she discussed were the famous Goresky-Kottwitz-MacPherson theory, as well as some known generalizations of the Tanisaki ideal (which describes the cohomology rings of type A Springer varieties as quotient rings). In addition, she discussed her recent work joint with Erik Insko and Alexander Woo, which gives an explicit formula – in terms of Schubert polynomials – for the cohomology and K-theory class of regular Hessenberg varieties in the cohomology of the flag variety.

Dr. Chow presented his recent results joint with Dr. Brosnan which proved the Shareshian-Wachs conjecture. This is a conjecture which links the famous Stanley-Stembridge conjecture in combinatorics, stating that the chromatic symmetric polynomial X_G of an indifference graph G is e-positive. (In fact, Stanley and Stembridge's original conjecture was stated more generally, but Guay-Paquet reduced it to this case.) In the setting of positivity conjectures of this type, it is natural to ask whether X_G is in fact $\mathrm{ch}\rho$ of a naturally occurring representation ρ, where ch is the standard characteristic map. Shareshian and Wachs had conjectured several years ago that (essentially) the desired representation is the symmetric group representation on the cohomology ring of a regular semisimple Hessenberg variety defined by Tymoczko. The main result of Dr. Chow's recent work, joint with Dr. Brosnan, is a proof of this Shareshian-Wachs conjecture, and Dr. Chow explained the basic ideas of their proof, the most crucial part of which is the identification of the fixed subspaces of ρ by a Young subgroup with the cohomology of a 'smaller' regular Hessenberg variety.

Dr. Guay-Paquet presented his new insights into the role that Hopf algebras play in the theory of the symmetric group representations on the cohomology rings of regular semisimple Hessenberg varieties, and in particular, can explain certain linear relations arising between q-chromatic symmetric functions. Specifically, he explained that there is a (graded connected) Hopf algebra constructed from Dyck paths, and that there exists a natural graded Hopf algebra map from it to the Hopf algebra of quasisymmetric functions. He then explained how this map agrees with the map considered by Shareshian-Wachs and Brosnan-Chow in connection to the Stanley-Stembridge conjecture.

To start the discussions on the second day, Dr. Martha Precup gave an overview of the known results on the Betti numbers of Hessenberg varieties. She explained how combinatorial formulas for the Betti numbers, given in the language of permutations, can be obtained through strategically chosen affine pavings of Hessenberg varieties. As an example application, she presented her theorem that the Betti numbers of regular Hessenberg varieties are palindromic; this is a property of the Betti numbers which is relevant in the study of the Stanley-Stembridge conjecture. Finally, she presented her joint work with Dr. Harada on the cohomology rings of abelian Hessenberg varieties, which gives inductive formulas for the symmetric group representations which appear. In particular, Dr. Precup explained how this result yields a proof of the graded Stanley-Stembridge conjecture in the abelian case.

Dr. Erik Insko followed with a talk whose theme was the study of the singularities of Hessenberg varieties. Much of this study is based on the foundational work of Tymoczko on paving Hessenberg varieties by affines. Building on this, Insko and Yong described the singular locus of Peterson varieties and showed that it is a local complete intersection. Dr. Insko also sketched the follow-up work of Abe-DeDieu-Galetto-Harada showing that regular nilpotent Hessenberg varieties are also local complete intersections. Finally, Dr. Inkso explained his recent joint work with M. Precup which explores the smoothness and the irreducible components of semisimple (not necessarily regular) Hessenberg varieties for the special case when $h(i) = i + 1$. In particular, they are able to show that the only singularities that occur are at the intersections of the irreducible components.

In the afternoon, Dr. Mikiya Masuda told us about his joint work with H. Abe and T. Horiguchi on a presentation, by generators and relations, of the cohomology rings of regular semisimple Hessenberg varieties in type A of the form $h = (h(1), n, n, \ldots, n)$. Dr. Masuda began his discussion by reminding the audience of the well-known Borel presentation for the cohomology rings $H^*(Flags)$ of flag varieties, and also stating that, in general, the ring $H^*(Flags)$ does not surject onto the cohomology of the Hessenberg variety. Dr. Masuda then recalled the framework of Goresky-Kottwitz-MacPherson theory, which describes in an explicit combinatorial fashion the equivariant cohomology of flag and Hessenberg varieties. Using this perspective, Dr. Masuda explained how to construct the necessary 'extra classes' which generate the cohomology ring, and proceeded to derive the correct relations among them.

Building on the above work of Abe-Harada-Horiguchi-Masuda (AHHM) on a generators-and-relations presentation of the cohomology ring of regular nilpotent Hessenberg varieties in type A, Dr. Horiguchi presented his results which interpret the generators in the AHHM presentation as a linear combination of Schubert polynomials.

Dr. James Carrell's talk was about the cohomology and equivariant cohomology groups of varieties X equipped with a faithful action of the Borel subgroup B of SL_2. A beautiful theorem of Carrell and Akyildiz says the following: If X is a smooth, projective complex variety and if B acts on X with a unique fixed point, then the fixed point scheme X^B is just the affine scheme $\mathrm{Spec}\,\mathrm{H}^*X$ associated to the cohomology groups of X (with complex coefficients). Later M. Brion and Carrell used this result along with a deep surjectivity result of D. Peterson to compute the cohomology of regular nilpotent Hessenberg varieties. This result played a large role in many of the talks in the workshop. In particular, part of M. Precup's talk consisted of an extensive generalization of the Brion—Carrell theorem.

In talks delivered on Wednesday, October 24, Andy Wilson, Jim Haglund, and Mark Skandera discussed connections between regular semisimple Hessenberg varieites of type A and various combinatorial phenomena. These connections are realized through examination of the chromatic quasisymmetric functions of unit interval orders, which were conjectured by Shareshian and Wachs, and proved by Brosnan and Chow (also independently by Guay-Paquet) to be (essentially) Frobenius characteristics of representations of symmetric groups on the cohomology of the varieties in question.

Drs. Haglund and Wilson discussed LLT polynomials, which were introduced in a 1997 paper of Leclerc, Lascoux and Thibon. These polynomials play a considerable role in the very active study of Macdonald polynomials. Moreover, certain LLT polynomials are closely related to chromatic quasisymmetric functions. Indeed, the chromatic quasisymmetric function of a unit interval graph is a generating function for proper colorings of the graph with the positive integers, while some LLT polynomials are generating functions for *all* colorings of unit interval graphs. In addition, there are LLT analogues of key questions about chromatic quasisymmetric functions of unit interval graphs (or, equivalently, representations of symmetric groups on the cohomology of regular semisimple Hessenberg varieties). The longstanding Stanley-Stembridge conjecture, already discussed above, also has an LLT analogue, which states that the LLT polynomials under consideration become nonnegative integer combinations of elementary symmetric functions after a simple linear change of variables.

Dr. Skandera discussed his work with Clearman, Hyatt and Shelton on characters of type A Hecke algebras. In this work, the authors address the problem of evaluating such characters on elements of the Kazhdan-Lusztig basis. This basis is indexed by permutations. The main result of Clearman et al. solves this problem for basis elements indexed by permutations avoiding certain patterns. The key result is a combinatorial formula for character values when the permutation avoids the pattern 312. There is a nice bijection between such permutations in S_n and regular semisimple Hessenberg varieties of type A contained in the flag variety $GL_n(\mathbb{C})/B$. It turns out that knowing the values of the irreducible characters of the Hecke algebra on a Kazhdan-Lusztig basis element C'_w, with $w \in S_n$ 312-avoiding, is the same as knowing the irreducible decomposition of the representation of S_n on the cohomology of the corresponding Hessenberg variety. Clearman et al. obtain their results using combinatorial objects called "descending star networks". These descending star networks inspired some informal discussion among the workshop participants, to be detailed in the next section.

In a pair of coordinated talks, Drs. Satoshi Murai and Takuro Abe explained the broader context of the recent work of Abe-Horiguchi-Masuda-Murai-Sato on Hessenberg varieties and hyperplane arrangements. It was shown by Sommers and Tymoczko that the Poincaré polynomials of certain regular nilpotent Hessenberg varieties admit a factorization, the factors of which are parametrized by certain exponents of the Hessenberg ideal. As Drs. Murai and Abe explained, this can be understood in the broader context of free hyperplane arrangements and the Terao factorization of the Poincaré polynomial of the complements of free hyperplane arrangements. They also explained that the cohomology ring of regular nilpotent Hessenberg varieties can be identified with the Solomon-Terao algebra of the hyperplane arrangement corresponding to the Hessenberg ideal. This allows them to derive several interesting consequences, including a computation of the volume polynomial of the Hessenberg variety; moreover, it was pointed out that it seems natural in this context to expect generalizations of these ideas to Schubert varieties and other related varieties.

Dr. Peter Crooks touched upon a different aspect of Hessenberg theory, namely, in its relation with the theory of integrable systems. Completely integrable systems are Hamiltonian systems which exhibit a maximal number of symmetries (in a sense which can be made precise); the symmetries allow us to reduce the number of variables and to build explicit solutions. Dr. Crooks described recent joint work with H. Abe, in which they show how a famous holomorphic integrable system, the Kostant-Toda lattice, can be related to a family of Hessenberg varieties, using Mischenko-Fomenko theory. This work raises the question of whether, and how, this Abe-Crooks construction can be related or altered to a construction of *real* integrable systems on single Hessenberg varieties (as opposed to families thereof).

Dr. Ting Xue's talk was based on joint work with K. Vilonen and T-H. Chen, which proves an analogue of the Springer correspondence for the symmetric pair $(\mathrm{SL}(N), \mathrm{SO}(N))$. The result can be applied to the family of Hessenberg varieties over the space $\mathfrak{p}$ of trace-free symmetric $N \times N$ matrices: it shows that the monodromy action factors through the Hecke algebra at $q = -1$. This gives an efficient way to compute the cohomology of Hesssenberg varieties associated to the symmetric pair.

4 Scientific Progress Made and Outcomes of the Meeting

The small size of the workshop meant that all the participants had the opportunity to interact with one another in substantial ways, and many productive informal conversations took place as a result. Below we describe a small sample of some of the mathematical developments that occurred as a result of these interactions. Two are described in some detail in Section 4.1 and 4.2, while Section 4.3 briefly summarizes additional developments.

4.1 Hessenberg varieties over finite fields

Brosnan and Shareshian began work on a conjecture relating point counting for regular semisimple Hessenberg varieties, defined over finite fields, and representations of Weyl groups on the cohomology of such varieties defined over $\mathbb{C}$. To explain the conjecture, we need some preparation. Given a field $\mathbb{F}$, a Hessenberg variety defined over $\mathbb{F}$ is determined by the following data: a reductive algebraic group G defined over $\mathbb{F}$ (and therefore over the algebraic closure $\overline{\mathbb{F}}$); a Borel subgroup B of G; a subspace $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G that contains the Lie algebra $\mathfrak{b}$ of B and is $Ad(B)$-invariant; and an element x of $\mathfrak{g}$ such that $ad(x)$ has entries in $\mathbb{F}$ with respect to some basis for $\mathfrak{g}$. Note that all of these ingredients other than x can be defined uniformly, independent of $\mathbb{F}$. Indeed, one can choose a root datum and from this, one obtains $G = G_{\mathbb{F}}$, $\mathfrak{g}$, B and $\mathfrak{b}$ by then choosing $\mathbb{F}$. Moreover, $\mathfrak{h}$ is determined by an appropriate choice of negative roots in the associated root system, which does not depend on $\mathbb{F}$. Assume that the root datum and negative roots have been fixed. When $\mathbb{F} = \mathbb{C}$ and $x \in \mathfrak{g}$ is assumed to be regular semisimple, the isomorphism type of the associated Hessenberg variety does not depend on the particular choice of x. Thus, given a prime power q, we may assume that x has been chosen to have coordinates (say, with respect to a Chevalley basis for $\mathfrak{g}$ arising from our root datum) in a ring R of algebraic integers with an ideal I satisfying $R/I \cong \mathbb{F}_q$. Now we can reduce x modulo I and use the resulting x_q to define a Hessenberg variety over $\mathbb{F}_q$.

We have now defined (smooth, projective) regular semisimple Hessenberg varieties $X_{\mathbb{C}}$ and X_q, over $\mathbb{C}$ and $\mathbb{F}_q$ respectively. On the one hand, we can consider Tymoczko's "dot action" representation of the Weyl group W of $G_{\mathbb{C}}$ on $H^*(X_{\mathbb{C}})$, as already mentioned above. On the other hand, wne can count, for each finite extension $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$, the number N_r of $\mathbb{F}_{q^r}$ points on X_q and store this information in the zeta function $Z(X_q; t) := \exp(\sum_{r \geq 1} N_r \frac{t^r}{r})$. According to the the celebrated Weil Conjectures (as proved by Dwork, Grothendieck and Deligne), $Z(X_q; t)$ is determined by the eigenvalues in $\overline{\mathbb{Q}_\ell}$ of the Frobenius map σ_q on the ℓ-adic cohomology of the $\overline{\mathbb{F}_q}$-variety X_q. Moreover, the eigenvalues of σ_q on $H^i(X_q; \mathbb{Q}_\ell)$ are algebraic integers of norm $q^{\frac{i}{2}}$. The connected component T of the identity in the centralizer of x_q under the adjoint action of $G_{\overline{\mathbb{F}_q}}$ is a σ_q-invariant maximal torus. There is a standard bijection between the set of conjugacy classes of such tori and the set of conjugacy classes in W. Pick w in the conjugacy class of W corresponding to the class of T in $G_{\overline{\mathbb{F}_q}}$. Assume that w has order n and fix any isomorphism between the group of n^{th} roots of unity in $\overline{\mathbb{Q}_\ell}$ and the group of complex n^{th} roots of unity. We can now state the conjecture.

Conjecture 1. *Under this identification, the multiplicity of a complex n^{th} root α as an eigenvalue of w in the dot action on $H^i(X_\mathbb{C})$ is the multiplicity of $q^{\frac{i}{2}}\alpha$ as an eigenvalue of σ_q on $H^i(X_q;\mathbb{Q}_\ell)$.*

4.2 Permutation bases for the dot action

As explained in previous sections, we know from the Brosnan-Chow-Guay-Paquet proof of the Shareshian-Wachs conjecture that, in order to prove the Stanley-Stembridge conjecture, it is sufficient to show that the dot-action representation on the cohomology of regular semisimple Hessenberg varieties is a permutation representation in which each point stabilizer is a Young subgroup. This then motivates the natural question: if it is a permutation representation, then can we explicit build a permutation basis for it?

The work of Harada and Precup on abelian Hessenberg varieties shows that, in these cases, the dot action representation is indeed a permutation representation of the appropriate type. Based on these ideas and the question above, Harada, Precup, and Tymoczko began working on the following problem during the BIRS workshop:

Problem 2. *Let $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be an abelian Hessenberg function. Find an explicit basis of the cohomology $H^*_T(\mathcal{H}ess(S, h))$ that is permuted by the dot action. In addition, for h an arbitrary Hessenberg function, find an explicit basis of the S_n-invariant subspace of $H^*_T(\mathcal{H}ess(S, h))$ (i.e. the trivial subrepresentation).*

One of the reasons this problem is non-trivial is that the dot action does not permute cohomology bases that satisfy upper-triangular vanishing conditions (which are natural in terms of the Goresky-Kottwitz-MacPherson description of the equivariant cohomology). Thus, any basis permuted by the dot action represents an unusual and interesting new basis for the equivariant cohomology of Hessenberg varieties. Moreover, an explicit such basis will also provide new intuition for proving the Stanley-Stembridge conjecture in the non-abelian case.

As is stated in Problem 2, our goal has two components: namely, the construction of a complete basis for the full representation $H^*_T(\mathcal{H}ess(S, h))$ in the *abelian case*, and, a construction of a basis for the S_n-invariant subspace (i.e. the "trivial part of the representation") $H^*_T(\mathcal{H}ess(S, h))^{S_n}$ in the general case. We address these separately below.

We know that GKM theory gives an explicit description of the T-equivariant cohomology $H^*_T(\mathcal{H}ess(S, h))$ using the *moment graph* for the T-action. Specifically, in this theory, an equivariant cohomology class in $H^*_T(\mathcal{H}ess(S, h))$ is obtained by assigning a polynomial in $\mathbb{C}[t_1, \ldots, t_n]$ (satisfying certain conditions) to each vertex of the moment graph, which is a permutation. Given $w \in S_n$, the dot action of w maps the polynomial $f_y(t_1, \ldots, t_n)$ assigned to $y \in S_n$ to $f_{wy}(t_{w(1)}, \ldots, t_{w(n)})$.

Tymoczko already has a conjectured method for building an explicit basis, as follows. Let X be a matrix with a single nilpotent Jordan block. In [1] it is shown that the cohomology ring of $\mathcal{H}ess(X, h)$ is isomorphic to the S_n-invariants in the cohomology ring of $\mathcal{H}ess(S, h)$. Moreover, work of Mbirika shows that the cardinality of the set of monomials

$$\{t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} : 0 \leq \alpha_i \leq h(i) - i \text{ for all } i\}$$

gives the Betti numbers of the Hessenberg variety $\mathcal{H}ess(X, h)$ when X consists of a single nilpotent Jordan block. Braden observed that for each $\alpha = (\alpha_1, \ldots, \alpha_n)$ we obtain a GKM cohomology class p^α by setting

$$p^\alpha_e = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \quad \text{and} \quad p^\alpha_w = t_{w(1)}^{\alpha_1} t_{w(2)}^{\alpha_2} \cdots t_{w(n)}^{\alpha_n}$$

for each permutation w. Moreover, the class p^α is S_n-invariant by definition. Thus, we know that these classes are a set of S_n-invariant cohomology classes. If we restrict to those α with $0 \leq \alpha_i \leq h(i) - i$ for each i there are exactly the expected number of them, in each degree. This leads us to the following concrete conjecture.

Conjecture 3. *The set* $\{p^\alpha : \alpha = (\alpha_1, \ldots, \alpha_n),\ 0 \leq \alpha_i \leq h(i) - i \text{ for all } i\}$ *forms an equivariant permutation basis for the trivial representations appearing in the dot action representation on* $H_T^*(\mathcal{H}ess(S, h))$.

What remains is to prove that the above set of classes is linearly independent; this is one of the main goals of the project undertaken by Harada, Precup, and Tymoczko.

We now describe the second part of the solution to Problem 2. In the abelian case, the only nontrivial permutation representation which appear in the decomposition of the dot-action representation are the M^λ's where λ has exactly two parts. Combining Conjecture 3 and the inductive description of Harada and Precup yields the necessary tools for defining an explicit permutation basis in the abelian case. The first step is to decompose the moment graph for the T-action on $H_T^*(\mathcal{H}ess(S, h))$ in order to obtain a decomposition analogous to the one given by Harada and Precup. We begin by defining a subset of permutations associated to each maximum independent subset of vertices in Γ_h denoted by W_V for $V \in I_2(\Gamma_h)$. The work of Harada and Precup proves $W_V \simeq S_{n-2}$ and that the induced subgraph of the moment graph corresponding to W_V can be identified with the moment graph for the smaller Hessenberg variety $H_T^*(\mathcal{H}ess(S, h_V))$.

The next important step is to define a permutation basis of equivariant cohomology classes in $H_T^*(\mathcal{H}ess(S, h))$ using this inductive structure. Indeed, as in the case of the trivial part of the representation as discussed above, we already have a candidate basis for the permutation representations corresponding to partitions of at most two parts. What remains to be shown is that this basis is linearly independent, and is also linearly independent when considered together with the permutation basis for the trivial representations given in Conjecture 3 above. Although computations of this form can be non-trivial, the Lie theoretic tools developed by Harada and Precup give us new tools with which to attack this problem.

Harada, Precup, and Tymoczko will work on this problem in June-July 2019 through their participation in the MSRI Summer Research for Women in Mathematics program.

4.3 Other interactions

One idea for further study, discussed at the workshop by Skandera and Shareshian, is to try to define analogues of the "descending star networks", discussed by Dr. Skandera in his talk, for Weyl groups other than symmetric groups. Regular semsimple Hessenberg varieties are defined in all Lie types. The main obstacle to developing a combinatorial approach to such varieties is the lack of an analogue to the chromatic quasisymmetric function. One can hope that appropriately defined descending star networks will stand in for chromatic quasisymmetric functions in the development of such an approach. In addition, such networks might shed light on connections between Hessenberg varieties and Hecke algebra representations in arbitrary Lie type.

In another development, Dr. Tymoczko asked during the meeting whether or not every Schubert variety in the flag variety is a Hessenberg variety. Since the meeting, Drs. Shareshian and Precup, together with their collaborator Dr. Laura Escobar, have shown that this question has a positive answer in Lie type A.

Finally, based on Dr. Horiguchi's presentation on his work relating Schubert classes and the elements f_{ij} defining the ideal in the AHHM presentation of the cohomology rings of regular nilpotent Hessenberg varieties, Martha Precup gave an conjecture for an explicit formula – in terms of Schubert classes – for these generators f_{ij}. The advantage of Dr. Precup's conjecture over the known formula is that, firstly, it would generalize to other Lie types, and secondly, it gives an interpretation of the relations f_{ij} in terms of Schubert calculus. If her conjecture can be proven, it would open the door for more investigations into the relationship between Schubert calculus and the cohomology rings of Hessenberg varieties.

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