Fly-By-Night Life Insurance and the NPMLE for Weibull Frailty Models

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Joint work with Jiaying Gu (U. of Toronto)

Carey et al (1992) Medfly Experiment

- 1,203,646 medflies survival times recorded in days
- 167 aluminum mesh cages of roughly 7200 flies each
- Adults were given a diet of sugar and water *ad libitum*
- Sex determined at time of death
- Pupae were sorted into one of five size classes

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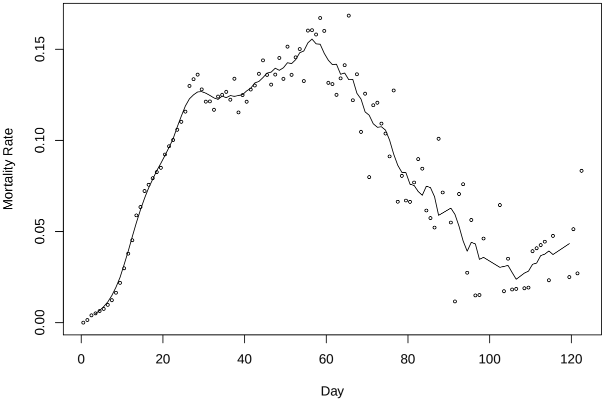
- Mortality rates actually declined at the oldest observed ages, contradicting the view that aging is an inevitable, monotone process of senescence.
- The right tail of the survival distribution was, at least by human standards, remarkably long.
- The experiment provided strong evidence for a crossover in gender specific mortality rates.

Raw Daily Medfly Mortality Rates and MA(7) Smooth



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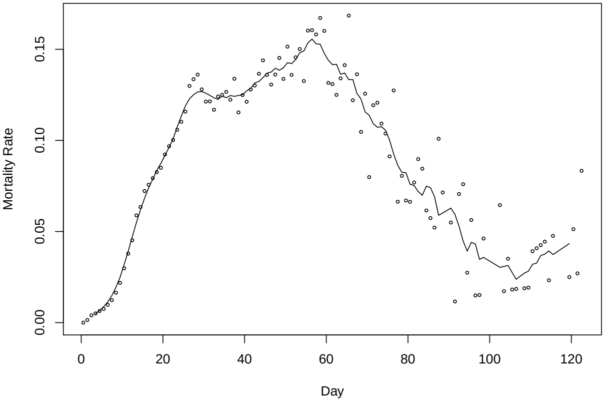


"The observed decline in mortality offered no consolation to the 99.8 percent of the flies that were already dead by age 60, but to the remaining 0.02 percent, more than 2000 less frail ones, it offered some hope of a prolonged retirement. The oldest flies in the experiment expired on day 172."

Mixture Models

There are several approaches to modeling this surprising tail behavior. An early suggestion, by Vaupel and Carey (1998) was to consider mixtures of the classical Weibull or Gompertz survival distributions:

$$g(x) = \int \varphi(x, \theta) dF(\theta),$$

where φ is either the Weibull or Gompertz density, and F is an unknown mixing density. In both cases we consider scale mixtures, and we therefore need to determine an appropriate shape parameter for each distribution.

Estimating the Shape Parameter

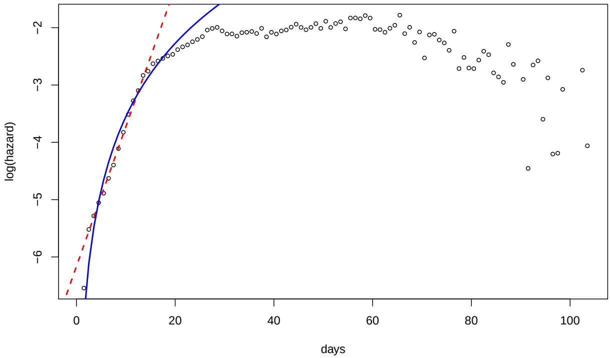


Figure: Estimated Baseline Gompertz and Weibull Hazard Models: Linear (Gompertz) and log linear (Weibull) fits to the initial $k = 15$ observations of daily log mortality rates. This is somewhat analogous to Hill estimation of the Pareto exponent.

Kiefer-Wolfowitz Nonparametric MLE for Mixture Models

Given iid observations, $x_1, \cdots, x_n$ from g, we wish to solve

$$\max_{F \in \mathcal{F}} \left\{ \sum_{i=1}^{n} \log(g(x_i)) \mid g(x) = \int \varphi(x, \theta) dF(\theta) \right\}$$

where $\mathcal{F}$ is the (convex) set of distribution functions.

- Solutions are discrete distributions with fewer than n points of support,
- Laird (1978) proposed solving with the EM algorithm,
- Recent interior point algorithms for additively separable convex programs enable much more efficient and accurate solutions, K and Mizera (2014).

Kiefer Wolfowitz Estimated Mixing Distributions

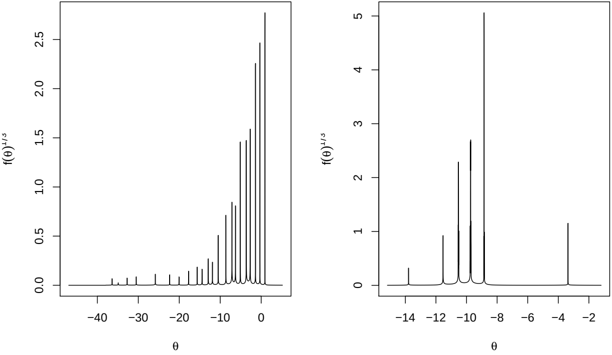


Figure: Estimated Mixing Distributions for the Gompertz (left) and Weibull (right) Models

Estimated (Mixture) Hazard Functions

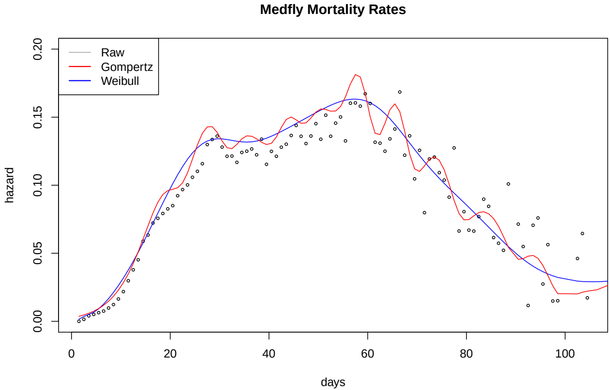


Figure: Hazard Functions for the Estimated Gompertz and Weibull Models

Gender Specific Estimated Hazard Rates

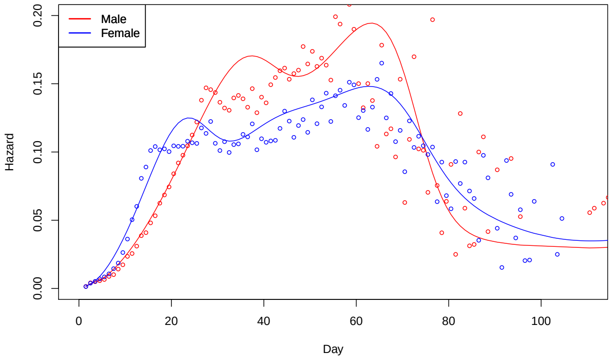


Figure: Gender Specific Hazard Functions for the Weibull Mixture Model: Raw daily mortality rates are plotted in black for males and grey for females, superimposed are the estimated hazard functions for the Weibull mixture models.

Gender Crossover

Several interesting features:

- Until about age 25 female mortality is higher than for males.
- But after age 25 female mortality is substantially below that of males.
- Hazard crossover implies survival function crossover at about age 36.
- Reverses human pattern in which males are more frail than females.
- The second hazard crossover at 75 shouldn't be taken very seriously since it is quite imprecise.

The Cage Density Controversy

A controversial aspect of the original experiment was the effect of cage density on mortality; critics argued that high density would make flies unhappy, and lead to earlier mortality.

- Let the baseline Weibull scale take the form $\theta_0 \exp(d_i \beta)$ where d_i is the initial cage density,
- Evaluate the profile KW mixture likelihood on a equally spaced grid $\beta \in [-1, 1]$,
- This yields point estimate of $\hat{\beta} \approx -0.5$ implying higher cage density shifts the survival distribution to the right, thus prolonging lives, and contradicting the critics.
- The classical Wilks $2 \log \lambda \rightsquigarrow \chi_1^2$ interval for β is quite precise.

Profile Likelihood for Cage Density Effect

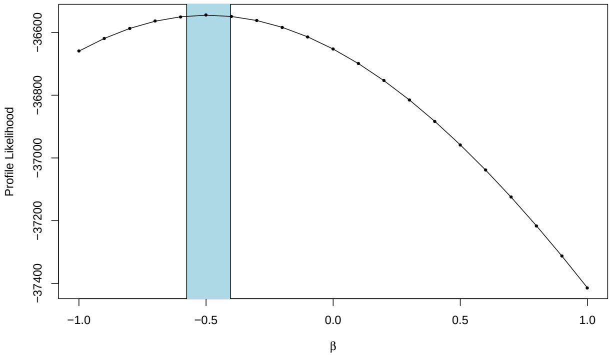
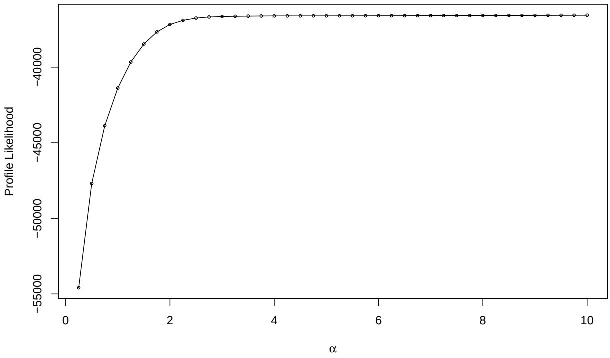


Figure: Profile Likelihood for the Initial Cage Density Effect in the Weibull Mixture Model and Wilks confidence interval.

Profile Likelihood for the Weibull Shape Parameter

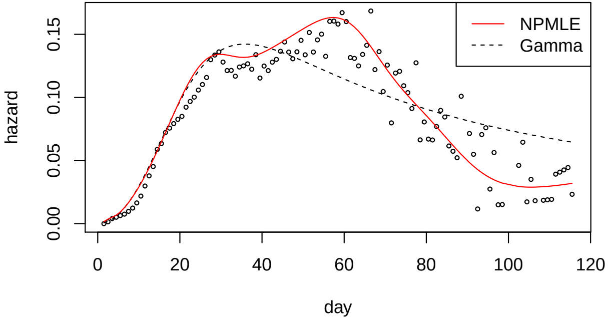


Sometimes profile likelihood is less pleasant Ishwaran (1999) shows that in Weibull models with shape parameter α_0, one can always find Weibull mixture models arbitrarily close (in Hellinger distance) for any $\alpha > \alpha_0$.

Five Medfly Life Lessons

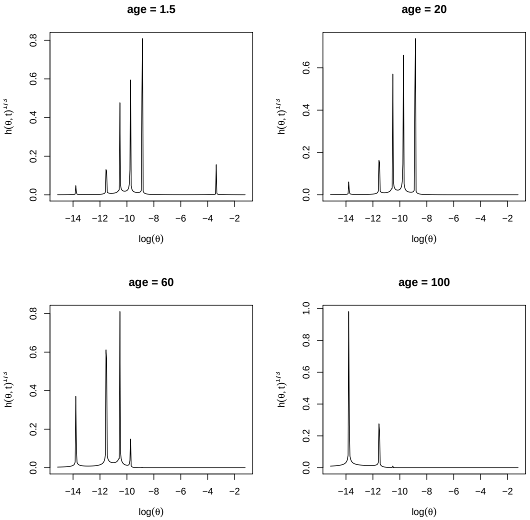
Males are tough ...	but only until 25.
Bigger is better ...	but only before 18.
Small is beautiful...	after 25.
Crowds are good ...	especially of guys.
Life gets safer ...	but only after 60.

Parametric Gamma vs. NPMLE Hazards



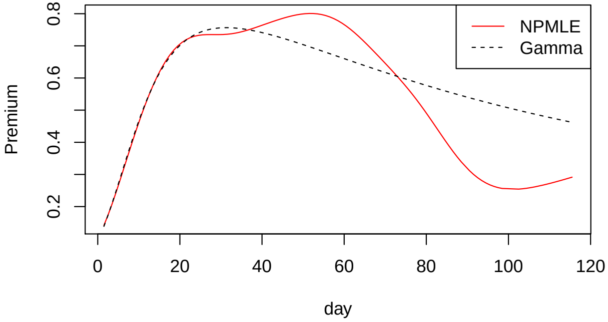
The NPMLE offers a more flexible specification compared to more traditional parametric models that assume gamma frailty.

Conditional Frailty at Various Ages



Conditioning on age we obtain different frailty distributions. Note the we have plotted the cube root of the density to accentuate smaller masses.

Fly-by-Nite Life Insurance Premia by Age



Life insurance rates for 10-day policies written for flies by their age, comparing the NPMLE and the parametric gamma solutions.

Five Medfly Statistical Lessons

- Nonparametric mixture models can be easily estimated by the Kiefer-Wolfowitz MLE
- Finite dimensional mixture models are considerably more difficult to estimate
- Profile likelihood can be used to estimate semiparametric extended models
- Sometimes semiparametric inference with profile likelihood is viable, van der Vaart(1996)
- Sometimes profile likelihood behaves badly.

NPMLEs and Shape Constraints

Shape constrained density estimation:

$$\max_{f \in \mathcal{F}} \left\{ \sum_{i=1}^{n} \log f(X_i) \mid D^2(\log f) \geqslant 0 \right\}$$

NPMLE for mixture models:

$$\max_{g \in \mathcal{G}} \left\{ \sum_{i=1}^{n} \log f(X_i) \mid f = Ag, \; g \geqslant 0 \right\}$$

Rather than constraining derivatives of a transformed density to be non-negative, the NPMLE constrains the mixing distribution to have non-negative mass points. Both yield sparse solutions with a small number of discrete jumps.

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