Inventory Routing under Uncertainty

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Vendor Managed Inventory (I)

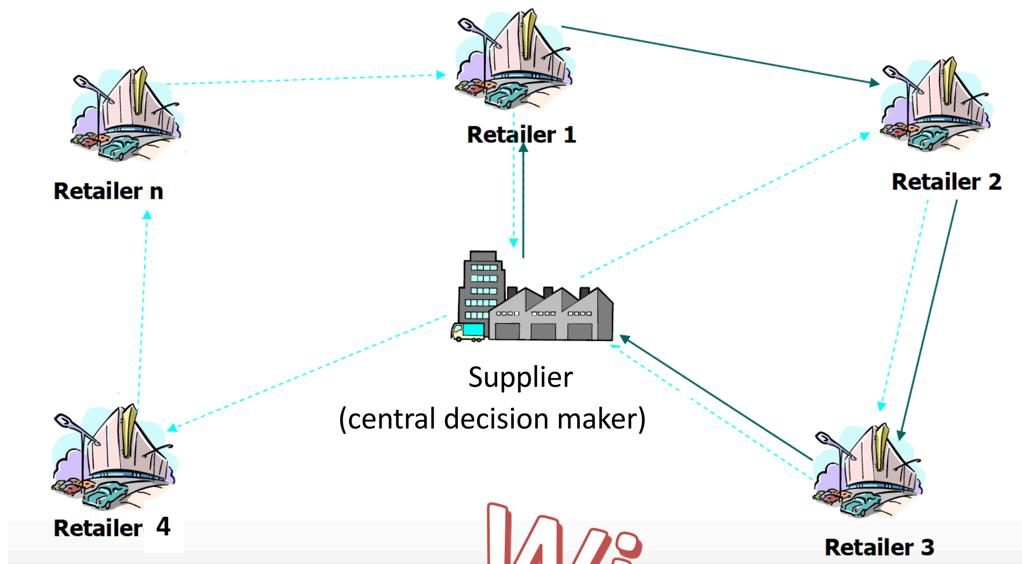
Retailers:

** monitor inventory levels

** place orders (when, how much)

Supplier:

** makes delivery decision



Advantages:

Win-Win

1. Save on distribution cost through coordination
2. Save efforts to inventory control

Vendor Managed Inventory (II)

Supplier (central decision maker)

When to deliver?

How much to deliver?

On which routes?

Inventory management + vehicle routing =

Inventory routing problem (IRP)

Inventory Routing Problem (I)

Applications:

maritime logistics, transportation of gas and oil, groceries, perishable products, blood, bicycle sharing

		Year	Products	Routing aspects	Inventory management aspects
1	Bell et al.	1983	Industrial gases	Restricted routing few visits on each route	Bounds on delivered quantities bounds on number of visits
2	Golden et al.	1984	Industrial gases	Heuristic routing savings-based heuristic	Fixed quantity to deliver
3	Miller	1987	Chemicals	Restricted routing perturbations of existing solution	Inventory balance and capacities
4	Blumenfeld et al.	1987	Automobile components	Flow-based routing	Economic order quantity assumption
5	Christiansen	1999	Ammonia	Exact routing	Inventory balance and capacities
6	Gaur and Fisher	2004	Groceries	Restricted routing few visits on each route	Fixed quantities to deliver
7	Campbell and Savelsbergh	2004	Industrial gases	Restricted routing cluster-based approach	Bounds on delivered quantities
8	Persson and Göthe-Lundgren	2005	Bitumen	Exact routing	Inventory balance and capacities
9	Custódio and Oliveira	2006	Frozen products	Restricted routing greedy heuristics	Economic order quantity assumption
10	Al-Khayyal and Hwang	2007	Oil	Exact routing	Inventory balance and capacities
11	Alegre et al.	2007	Automobile components	Heuristic routing local search	Fixed quantities to deliver
12	Dauzère-Pérès et al.	2007	Calcium carbonate slurry	Restricted routing direct shipping	Inventory balance and capacities
13	Hemmelmayr et al.	2008	Blood	Fixed giant routes skipping customers/ heuristic routing	Inventory balance and capacities

Table 1: Applications of inventory routing problem (Andersson et al. 2010)

Inventory Routing Problem (II)

Reference	Time horizon		Structure			Routing			Inventory policy		Inventory decisions			Fleet composition		Fleet size		
	Finite	Infinite	One-to-one	One-to-many	Many-to-many	Direct	Multiple	Continuous	Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Nonnegative	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Archetti et al. (2007)	✓			✓			✓		✓	✓			✓	✓		✓		
Raa and Aghezzaf (2008)		✓		✓			✓		✓				✓	✓			✓	
Savelsbergh and Song (2008)	✓				✓			✓	✓		✓			✓			✓	
Zhao, Chen, and Zang (2008)		✓		✓			✓		✓				✓	✓				✓
Abdelmaguid, Dessouky, and Ordóñez (2009)	✓			✓			✓		✓			✓			✓		✓	
Boudia and Prins (2009)	✓			✓			✓		✓				✓	✓			✓	
Raa and Aghezzaf (2009)		✓		✓			✓		✓		✓			✓			✓	
Geiger and Sevaux (2011a)	✓			✓			✓		✓		✓			✓				✓
Solyalı and Süral (2011)	✓			✓			✓			✓			✓	✓		✓		
Adulyasak, Cordeau, and Jans (2013)	✓			✓			✓		✓	✓			✓	✓	✓		✓	
Archetti et al. (2012)	✓			✓			✓		✓	✓			✓	✓		✓		
Coelho, Cordeau, and Laporte (2012a)	✓			✓			✓		✓	✓			✓	✓		✓		
Coelho, Cordeau, and Laporte (2012b)	✓			✓			✓		✓	✓			✓	✓			✓	
Michel and Vanderbeck (2012)	✓			✓			✓			✓			✓	✓			✓	
Coelho and Laporte (2013a)	✓			✓			✓		✓	✓			✓	✓	✓		✓	
Coelho and Laporte (2013b)	✓			✓			✓		✓	✓			✓	✓			✓	
Hewitt et al. (2013)	✓				✓			✓	✓				✓		✓		✓	

Table 2: Variation of inventory routing problems (Coelho et al. 2014)

Stochastic Inventory Routing Problem

- Dynamic Programming
  - Campbell et al. (1998); Kleywegt et al. (2002, 2004); Hvattum et al.(2009); Hvattum and Lkketangen (2009)
- Stochastic Programming
  - Adulyasak et al. (2015)
- Robust Optimization
  - Aghezzaf (2008); Solyali et al. (2012); Bertsimas et al. (2016)

Problem Setting (I)

Parameters:

** One supplier: $\{0\}$

** A set of retailers: $[N] = \{1, \ldots, N\}$

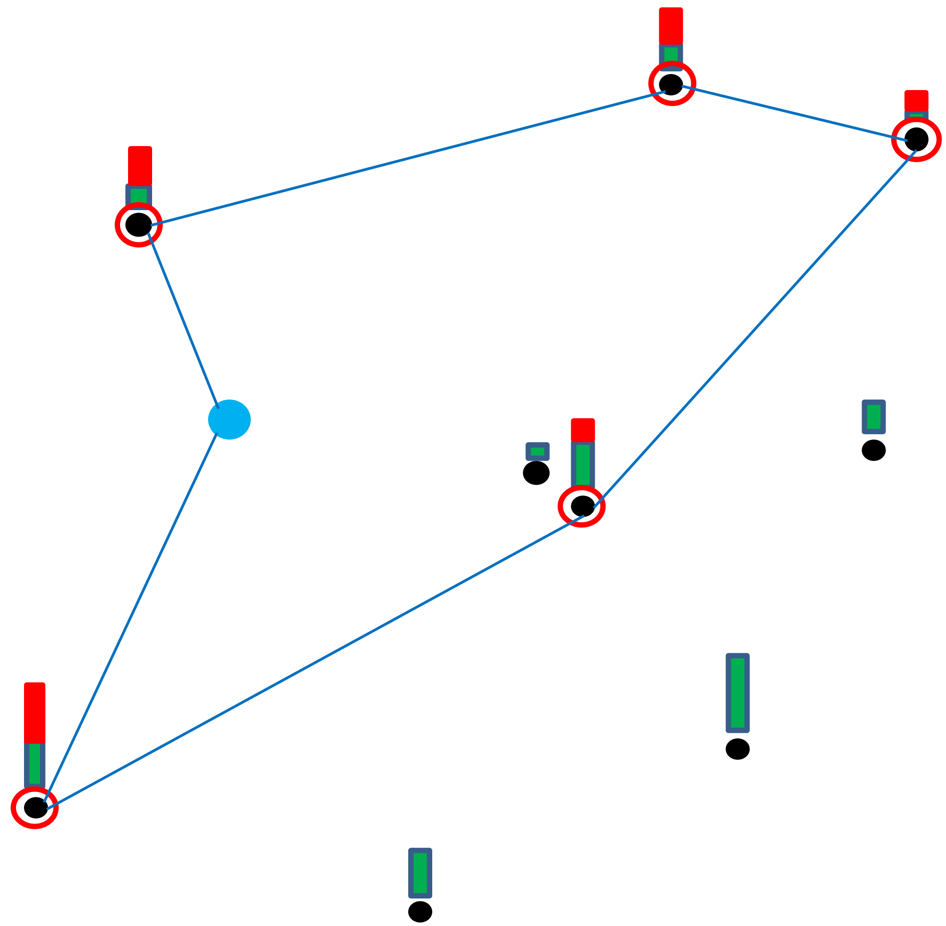
** Planning horizon: $[T] = \{1, 2, \cdots, T\}$

** Travel cost: $c_{ij}, (i,j) \in \mathcal{A}$

** Initial inventory: $x_n^0 = 0$

** Uncertain demand $\tilde{d}_{nt}$

$$\tilde{\boldsymbol{D}}^t \in \Re^{N \times t} \qquad \tilde{\boldsymbol{D}}^T = \tilde{\boldsymbol{D}}$$



Decisions:

** Retailer visit: $y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \in \{0,1\}$

** Routing: $z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \in \{0,1\}$

** Order quantity: $q_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \in \Re_+$

** Inventory level: $x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \in \Re$

Problem Setting (II)

Inventory level (backlogging):

$$x_n^t\left(\tilde{\boldsymbol{D}}^t\right) = x_n^{t-1}\left(\tilde{\boldsymbol{D}}^{t-1}\right) + q_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) - \tilde{d}_n^t = \sum_{m=1}^{t}\left(q_n^m\left(\tilde{\boldsymbol{D}}^{m-1}\right) - \tilde{d}_n^m\right)$$

$$\mathcal{Z} = \left\{ \begin{array}{l|ll} & q_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \le M y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), & \forall n \in [N], t \in [T] \ (a) \\ y_0^t, y_n^t \in \mathcal{B}_t, & y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \le y_0^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), & \forall n \in [N], t \in [T] \ (b) \\ q_n^t \in \mathcal{R}_t, & \sum_{j:(n,j)\in\mathcal{A}} z_{nj}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) = y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), & \forall n \in [N]\cup\{0\}, t \in [T] \ (c) \\ z_{ij}^t \in \mathcal{B}_t, & \sum_{j:(j,n)\in\mathcal{A}} z_{jn}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) = y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), & \forall n \in [N]\cup\{0\}, t \in [T] \ (d) \\ \forall n \in [N], t \in [T], & \sum_{(i,j)\in\mathcal{A}(\mathcal{S})} z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \le \sum_{n\in\mathcal{S}} y_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) - y_k^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), & \\ (i,j) \in \mathcal{A} & & \forall \mathcal{S} \subseteq [N], |\mathcal{S}| \ge 2, k \in \mathcal{S}, t \in [T] \ (e) \end{array} \right\}$$

Challenges

- Objective function:
  - **Minimize total expected cost**
  - Cost parameters
  - Expectation of a piecewise linear function
  - Risk neutral criterion

$$\mathbb{E}_{\mathbb{P}}\left[\underbrace{\sum_{t\in[T]}\sum_{n\in[N]}\left(h\left(x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)\right)^+ + b\left(x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)\right)^-\right)}_{\text{Inventory cost}} + \underbrace{\sum_{t\in[T]}\sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)}_{\text{Transportation cost}}\right]$$

Challenges

- Objective function:
  - **Minimize total expected cost**
  - Cost parameters
  - Expectation of a piecewise linear function
  - Risk neutral criterion

Dollars	**1**	**2**	**4**	**…**	**2^N**
Probability	1/2	1/4	1/8	…	$1/2^{N+1}$

$$
\begin{aligned}
\mathrm{E}_{\mathbb{P}}(x) = & \; 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \ldots \\
= & \; \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \\
= & \; \infty
\end{aligned}
$$

St. Petersburg paradox

Challenges

- Objective function:
  - Minimize total expected cost
  - Cost parameters
  - Expectation of a piecewise linear function
  - Risk neutral criterion
- Demand uncertainty:
  - Probability distribution
  - Classical robust setting
- Adaptive decisions
- Computational complexity

Contribution

- Objective function:
  - Minimize total expected cost
  - Cost parameters
  - Expectation of a piecewise linear function
  - Risk neutral criterion
- Demand uncertainty:
  - Probability distribution
  - Classical robust setting
- Adaptive decisions
- Computational complexity

Service level & Service violation index

Distributionally robust optimization

Decision rule

Mixed integer linear programming

Challenges

Our contribution

Objective Function

Inventory | **Transportation**

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{t\in[T]}\sum_{n\in[N]}\left(h\left(x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)\right)^+ + b\left(x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)\right)^-\right) + \sum_{t\in[T]}\sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)\right]$$

Inventory level $x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)$ must be within a requirement window pre-specified as $[\underline{\tau}_n^t, \overline{\tau}_n^t]$

Transportation cost $\sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right)$ must be bounded by B^t

$$\underline{\tau}_n^t \le x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \le \overline{\tau}_n^t$$

too conservative

$$\mathbb{P}\left(x_n^t\left(\tilde{\boldsymbol{D}}^{t-1}\right) \in [\underline{\tau}_n^t, \overline{\tau}_n^t], n\in[N], t\in[T]\right)$$

Non-convex
ignores the magnitude of violation

Risk measure?

Monetary Risk Measure

Definition 1: Monetary Risk Measure

A mapping $\mu(\cdot):\mathcal{V}\to\Re$ is called a *monetary* risk measure if it satisfies the following conditions for all $\tilde{x},\tilde{y}\in\mathcal{V}$:

(P1) Monotonicity: If $\tilde{x}\geq\tilde{y}$, then $\mu(\tilde{x})\geq\mu(\tilde{y})$;

(P2) Translation invariance (equivariance): $\mu(\tilde{x}+c)=\mu(\tilde{x})+c$ for any $c\in\Re$.

- Example
  - Value at Risk (VaR)

Coherent Risk Measure

Definition 2: Coherent Risk Measure*

A mapping $\mu(\cdot):\mathcal{V}\to\Re$ is called a *coherent* risk measure if it satisfies the following conditions for all $\tilde{x},\tilde{y}\in\mathcal{V}$:

(P1) Monotonicity: If $\tilde{x}\geq\tilde{y}$, then $\mu(\tilde{x})\geq\mu(\tilde{y})$;

(P2) Translation invariance (equivariance): $\mu(\tilde{x}+c)=\mu(\tilde{x})+c$ for any $c\in\Re$;

(P3) Convexity: $\mu(\lambda\tilde{x}+(1-\lambda)\tilde{y})\leq\lambda\mu(\tilde{x})+(1-\lambda)\mu(\tilde{y})$ for any $\lambda\in[0,1]$;

(P4) Positive homogeneity: If $k\geq 0$, then $\mu(k\tilde{x})=k\mu(\tilde{x})$.

- Example
  - Conditional/Average Value at Risk (CVaR)

$$\tilde{x},\tilde{y}\quad\text{vs.}\quad[\underline{\tau},\overline{\tau}]$$

$$\mu(\tilde{x}-\overline{\tau})-\mu(\tilde{y}-\overline{\tau})=(\mu(\tilde{x})-\overline{\tau})-(\mu(\tilde{y})-\overline{\tau})=\mu(\tilde{x})-\mu(\tilde{y})$$

* Arzner et al., 1999

Service Violation Index (I)

Definition 3: Service Violation Index $(\tilde{x}\ vs.\ [\underline{\tau}, \overline{\tau}])$

A class of function $\rho_{\underline{\tau},\overline{\tau}}(\cdot) : \mathcal{V} \to [0, \infty]$ is a Service Violation Index (SVI) if for all $\tilde{x}, \tilde{y} \in \mathcal{V}$,

it satisfies the following properties:

(P1) Monotonicity: $\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) \geq \rho_{\underline{\tau},\overline{\tau}}(\tilde{y})$ if $\max\{\tilde{x} - \overline{\tau}, \underline{\tau} - \tilde{x}\} \geq \max\{\tilde{y} - \overline{\tau}, \underline{\tau} - \tilde{y}\}$.

(P2) Satisficing: $\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = 0$ if $\tilde{x} \in [\underline{\tau}, \overline{\tau}]$, $\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \infty$ if $\tilde{x} \notin [\underline{\tau}, \overline{\tau}]$.

(P3) Convexity: $\rho_{\underline{\tau},\overline{\tau}}(\lambda\tilde{x} + (1-\lambda)\tilde{y}) \leq \lambda\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) + (1-\lambda)\rho_{\underline{\tau},\overline{\tau}}(\tilde{y})$, for any $\lambda \in [0, 1]$.

(P4) Positive homogeneity: $\rho_{\lambda\underline{\tau},\lambda\overline{\tau}}(\lambda\tilde{x}) = \lambda\rho_{\underline{\tau},\overline{\tau}}(\tilde{x})$ for any $\lambda \geq 0$.

Theorem 1: representation from risk measure

A function $\rho_{\underline{\tau},\overline{\tau}}(\cdot) : \mathcal{V} \to [0, \infty]$ is a SVI if and only if it has the representation as

$$\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \mu\left(\frac{\max\{\tilde{x} - \overline{\tau}, \underline{\tau} - \tilde{x}\}}{\alpha}\right) \leq 0\right\},$$

$$\mu(\tilde{x}) = \inf_{\eta \in \Re}\left\{\eta \,\middle|\, \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u(\tilde{x} - \eta)\right] \leq 0\right\}$$

where $\mu(\cdot) : \mathcal{V} \to \Re$ is a convex risk measure. Conversely, given a SVI $\rho_{\underline{\tau},\overline{\tau}}(\cdot)$, the underlying convex risk measure is given by

$$u(x) = \max_{k \in [K]} \{a_k x + b_k\}$$

$$\mu(\tilde{x}) = \min\{a \,|\, \rho_{-\infty,0}(\tilde{x} - a) \leq 1\}.$$

Service Violation Index (II)

Definition 4: Utility-based SVI

$$\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k\left(\frac{\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\}}{\alpha}\right)+b_k\right\}\right) \leq 0\right\}$$

Theorem 1: representation from risk measure

A function $\rho_{\underline{\tau},\overline{\tau}}(\cdot) : \mathcal{V} \to [0,\infty]$ *is a SVI if and only if it has the representation as*

$$\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \mu\left(\frac{\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\}}{\alpha}\right) \leq 0\right\},$$

$$\mu(\tilde{x}) = \inf_{\eta\in\Re}\left\{\eta \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u(\tilde{x}-\eta)\right] \leq 0\right\}$$

where $\mu(\cdot) : \mathcal{V} \to \Re$ *is a convex risk measure. Conversely, given a SVI* $\rho_{\underline{\tau},\overline{\tau}}(\cdot)$*, the underlying convex risk measure is given by*

$$u(x) = \max_{k\in[K]}\{a_k x + b_k\}$$

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Service Violation Index (II)

Definition 4: Utility-based SVI

$$\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k\left(\frac{\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\}}{\alpha}\right)+b_k\right\}\right)\leq 0\right\}$$

For example: $u(x) = \max\{-1, x\}$

$$\rho^* = \rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max\left\{-1, \frac{\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\}}{\alpha}\right\}\right)\leq 0\right\}$$

We guarantee

$$\mathbb{P}\left(\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\} > \phi\right) \leq \frac{1}{1+\phi/\rho^*}, \quad \forall \phi > 0, \mathbb{P}\in\mathcal{P}$$

Service Violation Index (III)

Definition 4: Utility-based SVI

$$\rho_{\underline{\tau},\overline{\tau}}(\tilde{x}) = \inf\left\{\alpha > 0 \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k\left(\frac{\max\{\tilde{x}-\overline{\tau}, \underline{\tau}-\tilde{x}\}}{\alpha}\right)+b_k\right\}\right)\leq 0\right\}$$

$$\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \inf\left\{\sum_{i\in[I]}\alpha_i \,\middle|\, \begin{array}{l} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k\left(\frac{\max\{\tilde{x}_i-\overline{\tau}_i, \underline{\tau}_i-\tilde{x}_i\}}{\alpha_i}\right)+b_k\right\}\right)\leq 0, \\ \alpha_i > 0, \forall i\in[I] \end{array}\right\}$$

$$\tilde{\boldsymbol{x}} \in \mathcal{V}^I$$

$$\begin{array}{ll} \inf & \sum_{t\in[T]}\sum_{n\in[N]}\alpha_n^t + \sum_{t\in[T]}\beta^t, \\ \text{s.t.} & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k \max\left\{\underline{\tau}_n^t - \tilde{x}_n^t(\cdot), \tilde{x}_n^t(\cdot)-\overline{\tau}_n^t\right\} + b_k\alpha_n^t\right\}\right)\leq 0, \ \forall n\in[N], t\in[T] \\ & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k\left(c^t(\cdot)-B^t\right)+b_k\beta^t\right\}\right)\leq 0, \qquad \forall t\in[T] \end{array}$$

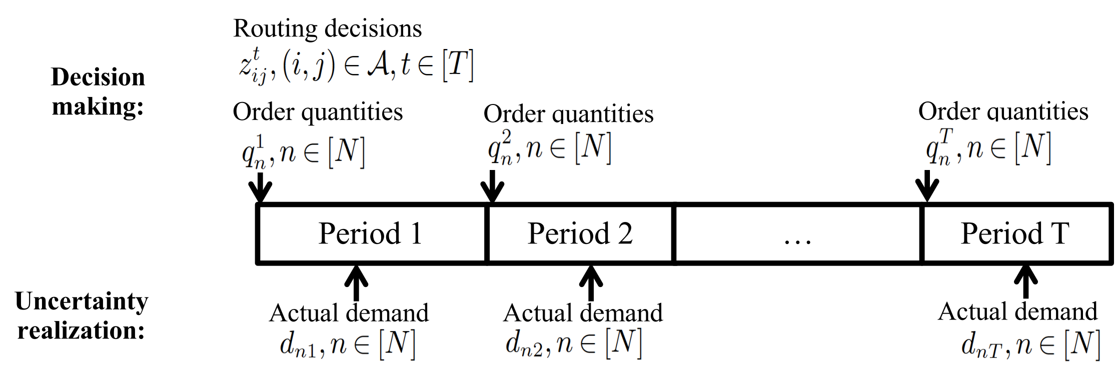
Adaptive Decisions

- Linear decision rule
  - Order quantity: $q_n^t\left(\tilde{\boldsymbol{D}}\right) = q_{n0}^t + \langle \boldsymbol{Q}_n^t, \tilde{\boldsymbol{D}} \rangle, \quad n \in [N], t \in [T]$

    $q_{n0}^t \in \mathbb{R}, \ \boldsymbol{Q}_n^t \in \mathbb{R}^{N \times T} \qquad \left(\boldsymbol{Q}_n^t\right)_l = \boldsymbol{0}, \ l \geq t$
  - Inventory level:

    $$x_n^t\left(\tilde{\boldsymbol{D}}\right) = \sum_{m=1}^{t}\left(q_n^m(\tilde{\boldsymbol{D}}) - \tilde{d}_n^m\right) = \sum_{m=1}^{t}\left(q_{n0}^m + \langle \boldsymbol{Q}_n^m, \tilde{\boldsymbol{D}} \rangle - \tilde{d}_n^m\right) = x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}} \rangle$$

    $$x_{n0}^t = \sum_{m=1}^{t} q_{n0}^m, \ \boldsymbol{X}_n^t = \sum_{m=1}^{t}\left(\boldsymbol{Q}_n^m - \boldsymbol{E}_n^m\right)$$
  - Routing decision: $z_{ij}^t = z_{ij}^t\left(\tilde{\boldsymbol{D}}^{t-1}\right), t \in [T]$
- Extended decision rule*



*Bertsimas et al., 2017

Model (I)

$$\inf \sum_{t\in[T]} \sum_{n\in[N]} \alpha_n^t,$$

$$\text{s.t.} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left(\max_{k\in[K]} \left\{ a_k \max\left\{ \underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle - \overline{\tau}_n^t \right\} + b_k \alpha_n^t \right\} \right) \le 0, \forall n\in[N], t\in[T], \quad \text{(a)}$$

SVI

$$q_{n0}^t + \langle \boldsymbol{Q}_n^t, \boldsymbol{D}\rangle \ge 0, \quad \forall \boldsymbol{D}\in\mathcal{W}, n\in[N], t\in[T], \text{ (b)}$$

$$q_{n0}^t + \langle \boldsymbol{Q}_n^t, \boldsymbol{D}\rangle \le M y_n^t, \quad \forall \boldsymbol{D}\in\mathcal{W}, n\in[N], t\in[T], \text{ (c)}$$

$$x_{n0}^t = \sum_{m=1}^t q_{n0}^m \quad \forall n\in[N], t\in[T]$$

$$\boldsymbol{X}_n^t = \sum_{m=1}^t (\boldsymbol{Q}_n^m - \boldsymbol{E}_n^m) \quad \forall n\in[N], t\in[T]$$

$$(\boldsymbol{Q}_n^t)_l = \boldsymbol{0}, \quad \forall n\in[N], l\ge t, l,t\in[T]$$

Inventory

$$\alpha_n^t \ge \epsilon, \quad \forall n\in[N], t\in[T],$$

$$(y_0^t, y_n^t, z_{ij}^t, n\in[N], t\in[T], (i,j)\in\mathcal{A}) \in \mathcal{Z}_R$$

Routing

$$\mathcal{Z}_R = \left\{ \begin{array}{l|ll} & \sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ij}^t \le B^t, & \forall t\in[T] \\ y_0^t, y_n^t, z_{ij}^t \in\{0,1\}, & y_n^t \le y_0^t, & \forall n\in[N], t\in[T] \\ \forall n\in[N], t\in[T], & \sum_{j:(n,j)\in\mathcal{A}} z_{nj}^t = y_n^t, & \forall n\in[N]\cup\{0\}, t\in[T] \\ (i,j)\in\mathcal{A} & \sum_{j:(j,n)\in\mathcal{A}} z_{jn}^t = y_n^t, & \forall n\in[N]\cup\{0\}, t\in[T] \\ & \sum_{(i,j)\in\mathcal{A}(\mathcal{S})} z_{ij}^t \le \sum_{n\in\mathcal{S}} y_n^t - y_k^t, & \forall \mathcal{S}\subseteq[N], |\mathcal{S}|\ge 2, k\in\mathcal{S}, t\in[T] \end{array} \right\}$$

Uncertainty Description (I)

- Distributional uncertainty set (moment)

$$\mathcal{P} = \left\{ \mathbb{P} \;\middle|\; \begin{array}{l} \mathbb{P}\left(\underline{\boldsymbol{D}} \le \tilde{\boldsymbol{D}} \le \overline{\boldsymbol{D}}\right) = 1, \\ \mathbb{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{D}}\right) = \boldsymbol{\mu}, \\ \mathbb{E}_{\mathbb{P}}\left[f_l \left(\sum_{(n,t)\in\mathcal{S}_h} \frac{\tilde{d}_{nt} - \mu_{nt}}{\sigma_{nt}} \right) \right] \le \epsilon_{lh}, \forall l \in [L], h \in [H] \end{array} \right\}$$

$$f_l(x) = \max_{k\in[K_l]} \{o_{lk}x + p_{lk}\}$$

Examples: $f_1(x) = \max\{x, -x\} = |x|$

$\mathcal{S}_1 = \{(n,t)\}$ and $\epsilon_{11} = 1$	$\mathbb{E}_{\mathbb{P}}\left(\left\lvert \tilde{d}_{nt} - \mu_{nt} \right\rvert\right) \le \sigma_{nt}$
$\mathcal{S}_2 = \{(n,t)_{n\in[N]}\}$	$\mathbb{E}_{\mathbb{P}}\left(\left\lvert \sum_{n\in[N]} \left(\tilde{d}_{nt} - \mu_{nt}\right) / \sigma_{nt} \right\rvert\right) \le \epsilon_{12}$
$\mathcal{S}_3 = \{(n,t)_{t\in[T]}\}$	$\mathbb{E}_{\mathbb{P}}\left(\left\lvert \sum_{t\in[T]} \left(\tilde{d}_{nt} - \mu_{nt}\right) / \sigma_{nt} \right\rvert\right) \le \epsilon_{13}$

Uncertainty Description (I)

- Distributional uncertainty set (moment)

$$\mathcal{P} = \left\{ \mathbb{P} \left| \begin{array}{l} \mathbb{P}\left(\underline{\boldsymbol{D}} \leq \tilde{\boldsymbol{D}} \leq \overline{\boldsymbol{D}}\right) = 1, \\ \mathbb{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{D}}\right) = \boldsymbol{\mu}, \\ \mathbb{E}_{\mathbb{P}}\left[f_l \left(\sum_{(n,t)\in\mathcal{S}_h} \frac{\tilde{d}_{nt} - \mu_{nt}}{\sigma_{nt}} \right) \right] \leq \epsilon_{lh}, \forall l \in [L], h \in [H] \end{array} \right. \right\}$$

$$f_l(x) = \max_{k\in[K_l]} \{o_{lk} x + p_{lk}\}$$

Proposition 1

With the above distributional uncertainty set, the constraint

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left(\max_{k\in[K]} \left\{ a_k \max\left\{ \underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}} \rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}} \rangle - \overline{\tau}_n^t \right\} + b_k \alpha_n^t \right\} \right) \leq 0$$

can be formulated as a set of linear constraints.

Uncertainty Description (II)

- Distributional uncertainty set (Kantorovich Metric)

$$\mathbb{P}^\dagger\left(\tilde{\boldsymbol{D}}^\dagger = \hat{\boldsymbol{D}}^{(m)}\right) = \frac{1}{M_r}, \qquad \forall m \in [M_r].$$

$$\mathcal{P} = \left\{\mathbb{P} \in \mathcal{P}_0(\mathcal{W}) \middle| d_W(\mathbb{P}, \mathbb{P}^\dagger) \le \theta\right\}.$$

$$\begin{aligned} d_W(\mathbb{P}, \mathbb{P}^\dagger) := \inf\ & \mathbb{E}_{\overline{\mathbb{P}}}\left(\left\|\tilde{\boldsymbol{D}} - \tilde{\boldsymbol{D}}^\dagger\right\|\right) \\ \text{s.t. }\ & (\tilde{\boldsymbol{D}}, \tilde{\boldsymbol{D}}^\dagger) \sim \overline{\mathbb{P}} \\ & \Pi_{\tilde{\boldsymbol{D}}}\overline{\mathbb{P}} = \mathbb{P} \\ & \Pi_{\tilde{\boldsymbol{D}}^\dagger}\overline{\mathbb{P}} = \mathbb{P}^\dagger \\ & \overline{\mathbb{P}}\left((\tilde{\boldsymbol{D}}, \tilde{\boldsymbol{D}}^\dagger) \in \mathcal{W} \times \mathcal{W}\right) = 1 \end{aligned}$$

Proposition 2

With the above uncertainty set, the constraint

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k \in [K]}\left\{a_k \max\left\{\underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle - \overline{\tau}_n^t\right\} + b_k \alpha_n^t\right\}\right) \le 0$$

can be formulated as a set of linear constraints.

Model (II)

$$\inf \sum_{t\in[T]} \sum_{n\in[N]} \alpha_n^t,$$

$$\text{s.t.} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left(\max_{k\in[K]} \left\{ a_k \max\left\{ \underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle - \overline{\tau}_n^t \right\} + b_k \alpha_n^t \right\} \right) \le 0, \forall n\in[N], t\in[T], \quad \text{(a)}$$

Linear!

$$q_{n0}^t + \langle \boldsymbol{Q}_n^t, \boldsymbol{D}\rangle \ge 0, \quad \forall \boldsymbol{D}\in\mathcal{W}, n\in[N], t\in[T], \text{(b)}$$

$$q_{n0}^t + \langle \boldsymbol{Q}_n^t, \boldsymbol{D}\rangle \le M y_n^t, \quad \forall \boldsymbol{D}\in\mathcal{W}, n\in[N], t\in[T], \text{(c)}$$

$$x_{n0}^t = \sum_{m=1}^{t} q_{n0}^m \quad \forall n\in[N], t\in[T]$$

Linear!

$$\boldsymbol{X}_n^t = \sum_{m=1}^{t} (\boldsymbol{Q}_n^m - \boldsymbol{E}_n^m) \quad \forall n\in[N], t\in[T]$$

$$\left(\boldsymbol{Q}_n^t\right)_l = \boldsymbol{0}, \quad \forall n\in[N], l\ge t, l, t\in[T]$$

$$\alpha_n^t \ge \epsilon, \quad \forall n\in[N], t\in[T],$$

$$(y_0^t, y_n^t, z_{ij}^t, n\in[N], t\in[T], (i,j)\in\mathcal{A}) \in \mathcal{Z}_R$$

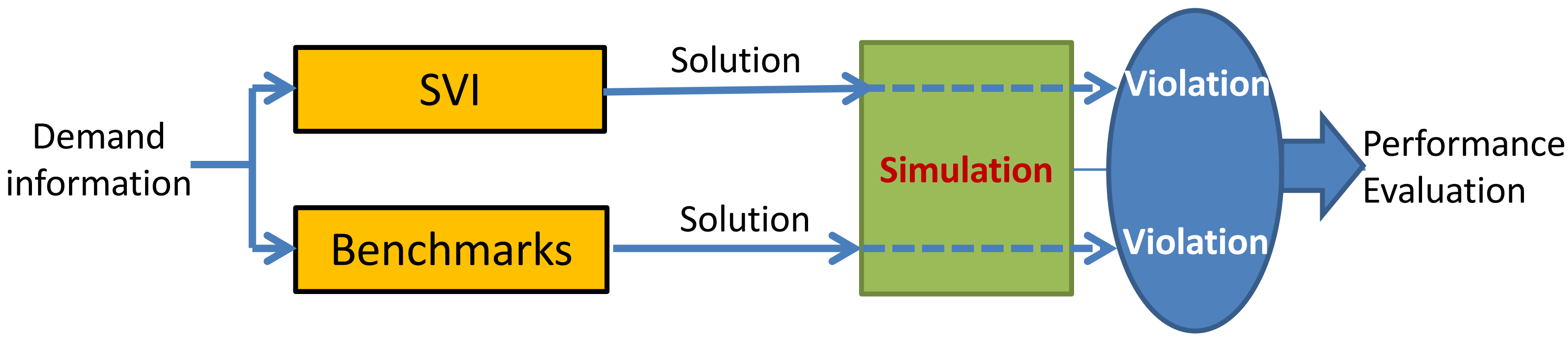
Linear!

The overall problem can be formulated as a mixed-integer linear programming problem.

$$\mathcal{Z}_R = \left\{ \begin{array}{l|ll} & \sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ij}^t \le B^t, & \forall t\in[T] \\ y_0^t, y_n^t, z_{ij}^t \in\{0,1\}, & y_n^t \le y_0^t, & \forall n\in[N], t\in[T] \\ \forall n\in[N], t\in[T], & \sum_{j:(n,j)\in\mathcal{A}} z_{nj}^t = y_n^t, & \forall n\in[N]\cup\{0\}, t\in[T] \\ (i,j)\in\mathcal{A} & \sum_{j:(j,n)\in\mathcal{A}} z_{jn}^t = y_n^t, & \forall n\in[N]\cup\{0\}, t\in[T] \\ & \sum_{(i,j)\in\mathcal{A}(\mathcal{S})} z_{ij}^t \le \sum_{n\in\mathcal{S}} y_n^t - y_k^t, & \forall \mathcal{S}\subseteq[N], |\mathcal{S}|\ge 2, k\in\mathcal{S}, t\in[T] \end{array} \right\}$$

Computational Study

- T=4; N=8
- Demand: 10 + 30 * Beta(2, 4)
- Budget: 0.7TSP; 0.8TSP
- Inventory lower bound $\underline{\tau}_n^t$: 0
- Inventory upper bound $\overline{\tau}_n^t$: 25, 30, 35, 40, 45



Performance of SVI-AR

Cost Budget	Inventory upper	Violation Prob.	Expected violation	Standard Deviation	Conditional Expected Violation	VaR @99%	Run time
TSP|0.7|0.7|0.7	25	16.7%	22.19	3.40	4.21	12.04	112
	30	12.8%	16.77	3.21	4.03	12.04	188
	35	7.2%	8.78	2.77	3.33	8.56	164
	40	2.3%	2.83	2.26	2.10	3.56	68
	45	0.7%	0.67	1.84	0.61	0	138
TSP|0.8|0.8|0.8	25	14.7%	19.11	3.18	4.13	11.37	195
	30	10.6%	14.35	2.95	4.08	11.41	203
	35	6.3%	7.51	2.55	3.09	7.89	76
	40	1.9%	2.42	2.07	1.88	2.94	167
	45	0.6%	0.57	1.69	0.52	0	141

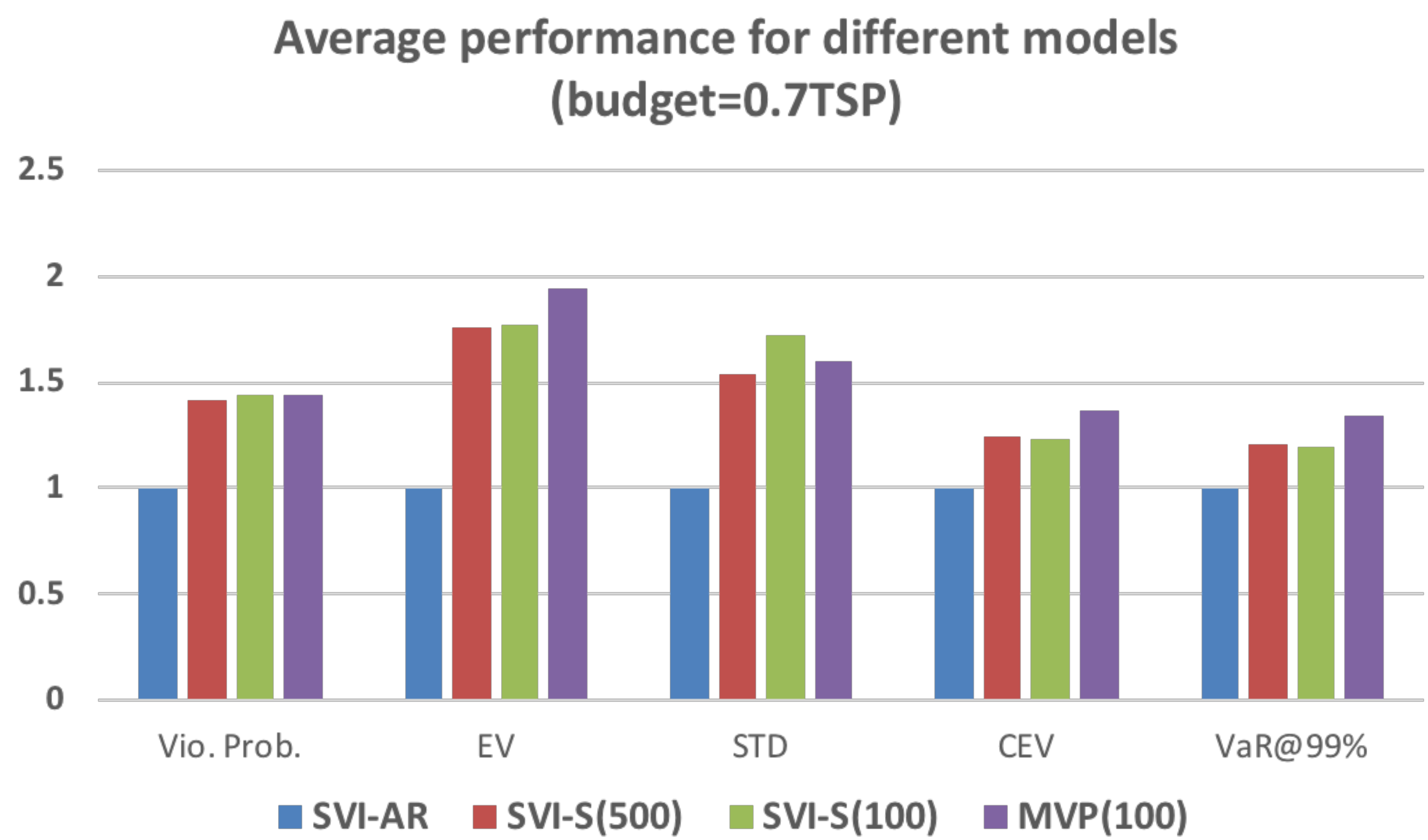
Benchmarks:

Minimize Violation Probability (MVP): $\min \sum_{t\in[T]} \sum_{n\in[N]} \left(1 - \mathbb{P}\left(\underline{\tau}_n^t \le \tilde{x}_n^t \le \overline{\tau}_n^t\right)\right)$

SVI-sampling (SVI-S):

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k \max\left\{\underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \tilde{\boldsymbol{D}}\rangle - \overline{\tau}_n^t\right\} + b_k \alpha_n^t\right\}\right)$$
$$= \frac{1}{M_s}\sum_{m\in[M_s]}\left(\max_{k\in[K]}\left\{a_k \max\left\{\underline{\tau}_n^t - x_{n0}^t - \langle \boldsymbol{X}_n^t, \boldsymbol{D}^{(m)}\rangle, x_{n0}^t + \langle \boldsymbol{X}_n^t, \boldsymbol{D}^{(m)}\rangle - \overline{\tau}_n^t\right\} + b_k \alpha_n^t\right\}\right)$$

Performance Comparison



Conclusion

- Conclusions
  - Propose Service Violation Index to measure service level.
  - Describe uncertain demand with distributionally robust approach.
  - Formulate a mixed-integer linear programming problem to solve the stochastic IRP.

- Extensions
  - Lost-sale case
  - Algorithms

Thank you very much !