Is DRO the Only Approach for Optimization Problems with Convex Uncertainty?

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A Static Robust Constraint

Assuming: g is convex in x and Z is convex.

$$g(x, z) \leq 0 \qquad \forall z \in Z,$$

g concave in z is 'easy' (Ben-Tal et al., 2015),

g **convex** in z is 'hard'.

A Distributional Constraint

$$\mathbb{E}_{\mathbb{P}_z}\left[g(x,z)\right] \leq 0 \qquad \forall \mathbb{P}_z \in \mathcal{P},$$

g convex in x, **convex** in z is 'doable'
(e.g., Wiesemann et al., 2014; Postek et al., 2017)

General Idea

1. Reformulate to an equivalent Adjustable Robust Optimization constraint
2. Apply standard ARO techniques:
    - Linear Decision Rules
    - Non-Linear Decision Rules
    - Fourier-Motzkin Elimination
    - etc...

Original Constraint

We are interested in reformulating the robust constraint

$$a(x)^\top z + \|A(x)z + b(x)\|_2 \leq c(x) \qquad\qquad \forall z \in Z,$$

where $a : \mathbb{R}^n \mapsto \mathbb{R}^L, A : \mathbb{R}^n \mapsto \mathbb{R}^{q\times L}, b : \mathbb{R}^n \mapsto \mathbb{R}^q$ are affine,

with polyhedral uncertainty:

$$Z = \left\{ z \in \mathbb{R}^L_+ \ \ : \ \ Dz \leq d \right\},$$

$$D \in \mathbb{R}^{p\times L}, \quad d \in \mathbb{R}^p.$$

Reformulation

We introduce an auxiliary uncertainty set:

$$\forall z \in Z: \qquad a(x)^\top z + \|A(x)z + b(x)\|_2 \leq c(x)$$

$$\iff \forall z \in Z: \quad a(x)^\top z + \max_{w : \|w\|_2 \leq 1} \left\{ w^\top \left(A(x)z + b(x)\right) \right\} \leq c(x)$$

$$\iff \forall w \in W, \forall z \in Z: \qquad a(x)^\top z + w^\top \left(A(x)z + b(x)\right) \leq c(x)$$

$$W = \{ w \in \mathbb{R}^q \ : \ \|w\|_2 \leq 1 \}$$

Reformulation

We find the robust counterpart wrt z:

$$\forall w \in W, \forall z \in Z : \qquad a(x)^\top z + w^\top (A(x)z + b(x)) \leq c(x)$$

$$\iff \forall w \in W : \qquad \max_{z \in Z} \left\{ \left(a(x) + A(x)^\top w \right)^\top z + w^\top b(x) \right\} \leq c(x)$$

$$\iff \forall w \in W : \ w^\top b(x) + \min_{\lambda \geq 0} \left\{ d^\top \lambda \mid D^\top \lambda \geq a(x) + A(x)^\top w \right\} \leq c(x)$$

$$\iff \forall w \in W, \exists \lambda \geq 0 : \qquad \begin{cases} d^\top \lambda + b(x)^\top w \leq c(x) \\ D^\top \lambda \geq a(x) + A(x)^\top w, \end{cases}$$

which is a linear ARO problem.

Linear Decision Rules

Using a linear decision rule

$$\lambda = u + Vw,$$

with $u \in \mathbb{R}^p$, $V \in \mathbb{R}^{p \times q}$ we obtain a safe approximation:

$$\forall w \in W : \quad \begin{cases} d^\top (u + Vw) + b(x)^\top w \leq c(x) \\ D^\top (u + Vw) \geq a(x) + A(x)^\top w \\ u + Vw \geq 0 \end{cases}$$

$$\iff \quad \begin{cases} d^\top u + \|V^\top d + b(x)\|_2 \leq c(x) & \\ a_i(x) + \|A_i(x) - V^\top D_i\|_2 \leq D_i^\top u & i = 1, \ldots, L \\ \|\left(V^\top\right)_j\|_2 \leq u_j & j = 1, \ldots, r. \end{cases}$$

Result

If there exist $u \in \mathbb{R}^p$, $V \in \mathbb{R}^{p\times q}$ such that

$$\begin{cases} d^\top u + \|V^\top d + b(x)\|_2 \le c(x) & \\ a_i(x) + \|A_i(x) - V^\top D_i\|_2 \le D_i^\top u & i = 1, \ldots, L \\ \|\left(V^\top\right)_j\|_2 \le u_j & j = 1, \ldots, r, \end{cases}$$

it holds that

$$a(x)^\top z + \|A(x)z + b(x)\|_2 \le c(x) \qquad \forall z \in Z.$$

Original Constraint

We are interested in reformulating the robust constraint

$$\boldsymbol{f}\left(\boldsymbol{A}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{b}(\boldsymbol{x})\right) \leq \boldsymbol{0} \qquad \forall z \in Z,$$

where $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{q}} \mapsto \mathbb{R}$ is positively homogeneous, that is:

$$\boldsymbol{\rho f}\left(\frac{\boldsymbol{v}}{\boldsymbol{\rho}}\right) = \boldsymbol{f}(\boldsymbol{v}) \qquad \forall \boldsymbol{\rho} > \boldsymbol{0}$$

with polyhedral uncertainty:

$$Z = \left\{ z \in \mathbb{R}_+^L \; : \; Dz \leq d \right\},$$

$$D \in \mathbb{R}^{p \times L}, \quad d \in \mathbb{R}^p.$$

Reformulation

We introduce an auxiliary uncertainty set W:

$$\forall z \in Z : \quad \boldsymbol{f}\left(\boldsymbol{A}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{b}(\boldsymbol{x})\right) \leq 0$$

$$\iff \forall z \in Z : \quad \max_{\boldsymbol{w} \in \boldsymbol{W}} \left\{ w^\top \left(A(x)z + b(x)\right)\right\} \leq 0$$

$$\iff \forall w \in W, \forall z \in Z : \quad w^\top \left(A(x)z + b(x)\right) \leq 0.$$

$$\boldsymbol{W} = \{\boldsymbol{w} \ : \ \boldsymbol{f}^*(\boldsymbol{w}) \leq \boldsymbol{0}\} \quad \text{(Rockefellar, 1970)}$$

Reformulation

We find the robust counterpart wrt z:

$$\forall w \in W, \forall z \in Z: \quad w^\top (A(x)z + b(x)) \le 0$$

$$\iff \forall w \in W: \quad \max_{z \in Z} \left\{ \left(A(x)^\top w \right)^\top z + w^\top b(x) \right\} \le 0$$

$$\iff \forall w \in W: \min_{\lambda \ge 0} \left\{ d^\top \lambda \,|\, D^\top \lambda \ge A(x)^\top w \right\} + w^\top b(x) \le 0$$

$$\iff \forall w \in W, \exists \lambda \ge 0: \quad \begin{cases} d^\top \lambda + b(x)^\top w \le 0 \\ D^\top \lambda \ge A(x)^\top w, \end{cases}$$

which is a linear ARO problem.

Linear Decision Rules

Using LDRs, we obtain a safe approximation:

$$\lambda = u + Vw,$$

$$\forall w \in W : \quad \begin{cases} d^\top u + w^\top \left(V^\top d + b(x)\right) \leq c(x) \\ -D^\top u + w^\top \left(A(x) - V^\top D\right) \leq 0 \\ u + Vw \geq 0 \end{cases}$$

$$\iff \qquad \begin{cases} d^\top u + \boldsymbol{f}\left(\boldsymbol{V}^\top \boldsymbol{d} + \boldsymbol{b}(\boldsymbol{x})\right) \leq 0 \\ \boldsymbol{f}\left(\boldsymbol{A_i}(\boldsymbol{x}) - \boldsymbol{V}^\top \boldsymbol{D_i}\right) \leq D_i^\top u & i = 1, \ldots, L \\ \boldsymbol{f}\left(\left(\boldsymbol{V}^\top\right)_{\boldsymbol{j}}\right) \leq u_j & j = 1, \ldots, r. \end{cases}$$

Result

If there exist $u \in \mathbb{R}^p$, $V \in \mathbb{R}^{p\times q}$ such that

$$\begin{cases} d^\top u + \boldsymbol{f}\left(\boldsymbol{V}^\top \boldsymbol{d} + \boldsymbol{b}(\boldsymbol{x})\right) \leq 0 & \\ \boldsymbol{f}\left(\boldsymbol{A_i}(\boldsymbol{x}) - \boldsymbol{V}^\top \boldsymbol{D_i}\right) \leq D_i^\top u & i = 1, \ldots, L \\ \boldsymbol{f}\left(\left(\boldsymbol{V}^\top\right)_{\boldsymbol{j}}\right) \leq u_j & j = 1, \ldots, r. \end{cases}$$

it holds that

$$f\left(A(x)z + b(x)\right) \leq 0 \qquad \forall z \in Z,$$

Original Constraint

We are interested in reformulating the robust constraint

$$f\left(A(x)z + b(x)\right) \leq 0 \qquad\qquad \forall z \in Z,$$

where $f : \mathbb{R}^p \mapsto \mathbb{R}$ is **convex** and with polyhedral uncertainty:

$$Z = \left\{ z \in \mathbb{R}^L_+ \ \ : \ \ Dz \leq d \right\},$$

$$D \in \mathbb{R}^{p \times L}, \quad d \in \mathbb{R}^p.$$

We introduce an auxiliary variable to make f positively homogeneous:

$$f\left(A(x)z + b(x)\right) \leq 0 \qquad \forall z \in Z$$

$$\iff \begin{cases} \boldsymbol{\alpha f}\left(\frac{\boldsymbol{A(x)z+b(x)}}{\boldsymbol{\alpha}}\right) \leq 0 & \forall z \in Z \\ \boldsymbol{\alpha = 1} \end{cases}$$

Reformulation

We introduce an auxiliary uncertainty set:

$$\forall z \in Z : \qquad \boldsymbol{\alpha f}\left(\frac{\boldsymbol{A(x)z + b(x)}}{\boldsymbol{\alpha}}\right) \leq 0$$

$$\iff \forall z \in Z : \qquad \max_{\begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix} \in W} \left\{ \begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\alpha} \\ A(x)z + b(x) \end{pmatrix} \right\} \leq 0$$

$$\iff \forall \begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix} \in W, \forall z \in Z : \qquad \boldsymbol{w_0 \alpha} + w^\top (A(x)z + b(x)) \leq 0$$

$$\boldsymbol{W} = \left\{ \begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix} \; : \; \boldsymbol{w_0} + f^*(w) \leq 0 \right\}$$

Reformulation

We find the robust counterpart wrt z:

$$\forall \begin{pmatrix} \boldsymbol{w_0} \\ \boldsymbol{w} \end{pmatrix} \in W, \forall z \in Z : \qquad \boldsymbol{w_0}\boldsymbol{\alpha} + w^\top (A(x)z + b(x)) \leq 0$$

$$\Longleftrightarrow \forall \begin{pmatrix} \boldsymbol{w_0} \\ \boldsymbol{w} \end{pmatrix} \in W : \qquad \max_{z \in Z} \left\{ \boldsymbol{w_0}\boldsymbol{\alpha} + w^\top b(x) + \left(A(x)^\top w\right)^\top z \right\} \leq 0$$

$$\Longleftrightarrow \forall \begin{pmatrix} \boldsymbol{w_0} \\ \boldsymbol{w} \end{pmatrix} \in W : \min_{\lambda \geq 0} \left\{ d^\top \lambda \,|\, D^\top \lambda \geq A(x)^\top w \right\} + w^\top b(x) + \boldsymbol{w_0}\boldsymbol{\alpha} \leq 0$$

$$\Longleftrightarrow \forall \begin{pmatrix} \boldsymbol{w_0} \\ \boldsymbol{w} \end{pmatrix} \in W, \exists \lambda \geq 0 : \qquad \begin{cases} d^\top \lambda + b(x)^\top w + \boldsymbol{w_0} \leq 0 \\ D^\top \lambda \geq A(x)^\top w, \end{cases}$$

which is a linear ARO problem.

Linear Decision Rules

Using LDRs, we obtain a safe approximation:

$$\lambda = u + Vw + rw_0,$$

$$\forall w \in W: \quad \begin{cases} d^\top u + \begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix}^\top \begin{pmatrix} \mathbf{1} + \boldsymbol{d}^\top \boldsymbol{r} \\ V^\top d + b(x) \end{pmatrix} \leq 0 \\ -D^\top u + \begin{pmatrix} \boldsymbol{w_0} \\ w \end{pmatrix}^\top \begin{pmatrix} -\boldsymbol{D}^\top \boldsymbol{r} \\ A(x) - V^\top D \end{pmatrix} \leq 0 \\ u + Vw + \boldsymbol{r}\boldsymbol{w_0} \geq 0 \end{cases}$$

Result

If there exist $u \in \mathbb{R}^p, V \in \mathbb{R}^{p\times q}, r \in \mathbb{R}^p$ such that

$$\begin{cases} d^\top u + \left(\mathbf{1} + \boldsymbol{d}^\top \boldsymbol{r}\right) f\left(\frac{V^\top d + b(x)}{\mathbf{1}+\boldsymbol{d}^\top \boldsymbol{r}}\right) \le 0 & \\ \mathbf{1} + \boldsymbol{d}^\top \boldsymbol{r} > \mathbf{0} & \\ -D_i^\top u + \left(-\boldsymbol{D_i}^\top \boldsymbol{r}\right) f\left(\frac{A_i(x) - V^\top D_i}{-\boldsymbol{D_i}^\top \boldsymbol{r}}\right) \le 0 & \\ -\boldsymbol{D_i}^\top \boldsymbol{r} > \mathbf{0} & i = 1, \ldots, L \\ u_i + \boldsymbol{r_i} f\left(\frac{V_i^\top}{\boldsymbol{r_i}}\right) \le 0 & \\ \boldsymbol{r_i} > \mathbf{0} & i = 1, \ldots, q, \end{cases}$$

it holds that

$$f\left(A(x)z + b(x)\right) \le 0 \qquad \forall z \in Z,$$

Progressive Approximation

A progressive approximation for an SOC constraint:

$$a(x)^\top z^{(k)} + \|A(x)z^{(k)} + b(x)\|_2 \leq c(x) \qquad k = 1, \ldots, K,$$

where

$$z^{(1)}, \ldots, z^{(K)} \in Z.$$

(Hadjiyiannis, 2011): take $z^{(1)}, \ldots, z^{(K)}$ as critical scenarios for feasible solution $\hat{x}$.

Robust Regression

$$\min_{x \in \mathbb{R}^n, \tau} \quad \tau$$

$$\text{s.t.} \quad \|(A + \Delta)x - b\|_2 \leq \tau \qquad \forall \Delta \in Z,$$

with a budget uncertainty set Z with control parameter Γ.

Γ: The number of features that exhibit uncertainty.

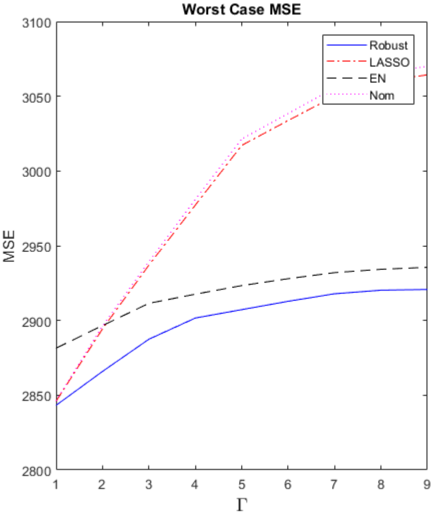
Robust Regression: Results

Diabetes dataset from Efron et al. (2004), 442 observations, 10 features.

Γ	1	2	3	4	5	6	7	8	9
LDR	938.9	942.6	946.2	948.5	949.4	950.4	951.2	951.6	951.7
Time(s)	0.3	0.5	0.3	0.3	0.3	0.4	0.4	0.3	0.2
%Gap	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

LDR: Approximated objective values
Time: Computation time
Gap: Optimality Gap

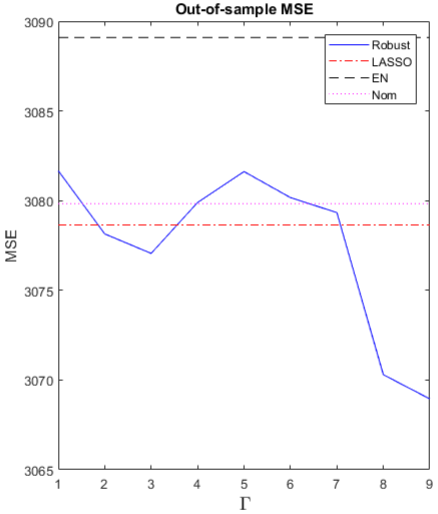
Robust Regression: Results



EN: Elastic Net

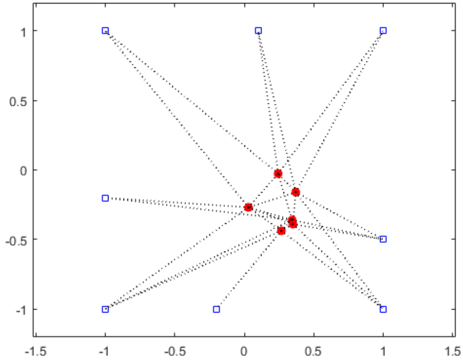
Nom: Nominal Solution

Robust Regression: Results



EN: Elastic Net
Nom: Nominal Solution

Robust Sensor Placement



Nominal solution for a sensor placement problem.
Squares: Sensors, Dots: Modules, Dashed Lines: Links.

Robust Sensor Placement

$$\min_{x,y(\zeta,\xi),\tau} \quad \tau$$

$$s.t. \quad \forall(\zeta,\xi) \in Z : \begin{cases} \sum_{(i,j)\in A} \|y_i(\zeta,\xi) - y_j(\zeta,\xi)\|_2 \leq \tau \\ y_i(\zeta,\xi) = \bar{a}_i + \hat{a}_i\zeta_i & \forall i \in M \\ y_i(\zeta,\xi) = x_i & \forall i \in H, \end{cases}$$

$$Z = \left\{ (\zeta,\xi) : \ \zeta \leq \xi, \ -\zeta \leq \xi, \ \xi \leq 1, \sum_{i=1}^{2|M|} \xi_i \leq \Gamma \right\}$$

Robust Sensor Placement: Results

$|M|$ fixed sensors with uncertain location, data from (Boyd and Vandenberghe, 2004).

$\lvert M\rvert$	10	20	30	40	50	60	70
LDR	24.4	70.8	149.5	257.0	392.0	564.1	762.9
LB	24.3	70.8	149.4	256.9	391.9	564.1	762.9
Time(s)	0.35	2.4	10.4	30.6	59.4	102.5	193.3

$|M|$: Number of fixed sensors

LDR: Approximated objective values

LB: Lower bound from 1 scenario

Time: Computation time

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Progressive Approximation

For example, given a feasible solution $\left(\hat{x}, \hat{u}, \hat{V}\right)$:

$$\bar{w} = \text{argmax}_{w \in W} \left\{ d^\top \left(\hat{v} + \hat{V} w \right) + b(\hat{x})^\top w \right\} = \frac{\hat{V}^\top d + b(\hat{x})}{\| \hat{V}^\top d + b(\hat{x}) \|_2}.$$

From this, recover original critical scenario $\bar{z}$:

$$\bar{z} = \text{argmax}_{z \in Z} \left\{ a(\hat{x})^\top z + (\bar{w})^\top \left(A(\hat{x}) z + b(\hat{x}) \right) \right\}.$$