The Optimal Design of Wall-Bounded Heat Transport

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Joint work with Charlie Doering (Michigan)

Banff COV Workshop

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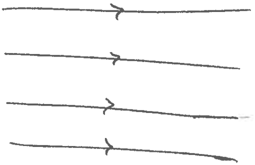
I.T. & C. Doering, *Optimal wall-to-wall transport by incompressible flows*, PRL '17
I.T. & C. Doering, *On the optimal design of wall-to-wall heat transport*, submitted
Andre Souza, I.T., & C. Doering, *Optimal 2D wall-to-wall transport — numerics*, in prep

Heat transport in a fluid with velocity $\mathbf{u}(\mathbf{x}, t)$ occurs by two mechanisms:



advection at rate $||\mathbf{u}||/L$

diffusion at rate κ/L^2

Together, they determine $T(\mathbf{x}, t) =$ temperature through

$$\partial_t T + \operatorname{div}(\mathbf{u}T - \kappa \nabla T) = 0$$

We recognize the *heat flux*

$$\mathbf{J} = \mathbf{u}T - \kappa \nabla T$$

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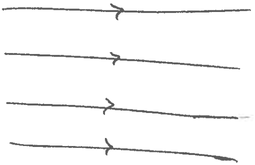

advection

diffusion

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The Péclet number

$$Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{||\mathbf{u}||/L}{\kappa/L^2} \gg 1$$

Heat transport in a fluid layer

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u} + \mathbf{f}$$
$$\text{div } \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$\mathbf{J} = \mathbf{u} T - \nabla T$$

Question: Which forces $\mathbf{f}$ produce the largest transport of heat,

$$\max_{\mathbf{f}} \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle ?$$

Notation for averaging:

$$\langle \cdot \rangle = \limsup_{\tau \to \infty} \frac{1}{\tau |\text{fluid layer}|} \int_0^\tau \int_{\text{fluid layer}} \cdot \, dx dt$$

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To flow $\geq$ not to flow

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The *Nusselt number* is defined as enhancement of heat transport

$$Nu(\mathbf{u}) = \frac{\text{total vertical heat flux}}{\text{conductive vertical heat flux}} = \frac{\langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle}{\langle -\nabla T \cdot \hat{\mathbf{k}} \rangle} \geq 1$$

We seek to **maximize** it... the answer is $+\infty$ w/o constraints

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Enstrophy budget

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$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

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A natural constraint is on the **power** expended to sustain fluid flow

From the momentum eqn.,

$$\langle \mathbf{f} \cdot \mathbf{u} \rangle = \left\langle |\nabla \mathbf{u}|^2 \right\rangle$$

average power expended = average "enstrophy"

The wall-to-wall optimal transport problem

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
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$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$Nu = \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle$$

Problem: Maximize the wall-to-wall heat transport Nu amongst all incompressible flows sat. a given enstrophy budget,

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ \text{b.c.s}}} \quad Nu(\mathbf{u})$$

What do optimizers look like?

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

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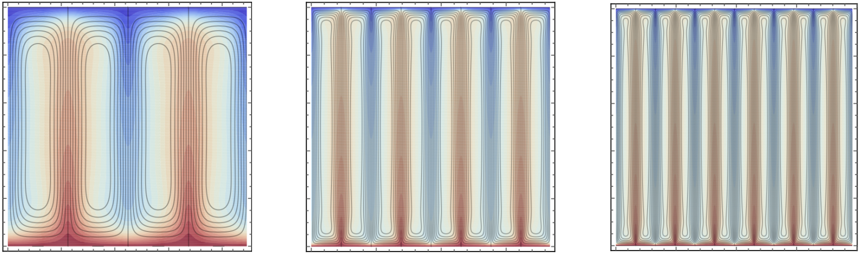


[1]

[1] A. Souza, PhD thesis 2016

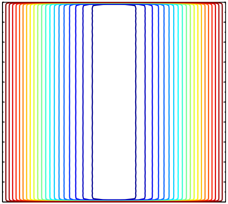
What do optimizers look like?

K.E. budget

$\langle|\mathbf{u}|^2\rangle^{1/2} = Pe$

stress-free b.c.

$\partial_z u = w = 0$



[1]

$\max Nu \sim Pe$

$l_{bulk} \sim Pe^{-1/2}$

enstrophy budget

$\langle|\nabla\mathbf{u}|^2\rangle^{1/2} = Pe$

stress-free b.c.

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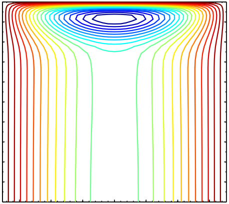


[1]

$Nu \sim Pe^{0.58}$

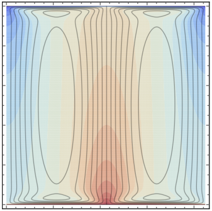
$l_{bulk} \sim Pe^{-0.36}$

enstrophy budget

$\langle|\nabla\mathbf{u}|^2\rangle^{1/2} = Pe$

no-slip b.c.

$u = w = 0$



[2]

$Nu \sim Pe^{0.54}$

$l_{bulk} \sim Pe^{-0.37}$

[1] P. Hassanzadeh, G. P. Chini, & C. R. Doering, JFM 2014

[2] A. Souza, PhD thesis 2016

What must optimizers obey?

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

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$$Nu = \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle$$

Theorem (Souza & Doering, '16)

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ b.c.s}} Nu(\mathbf{u}) \leq C\, Pe^{2/3}$$

$\exists$ multiple proofs:

- a modification of the "background method" (C. Doering & P. Constantin, Phys Rev E '96)
- an elementary "conservation law" argument (C. Seis, JFM '15)

So what's the optimal rate?

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

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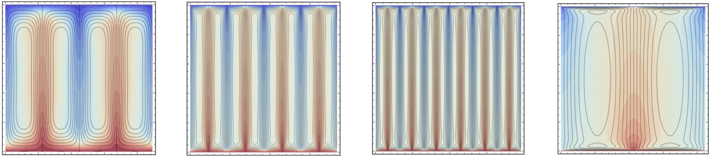


$\cdots$??

$Nu \sim Pe^{1/2}$

$Nu \sim Pe^{0.54}$

$Pe = 4 \times 10^2$ $\quad 5 \times 10^3$ $\quad 1.3 \times 10^4$ $\quad 4 \times 10^4$

Main result

Theorem (T. & Doering, '17)

Up to logarithmic corrections, the optimal rate of heat transport satisfies

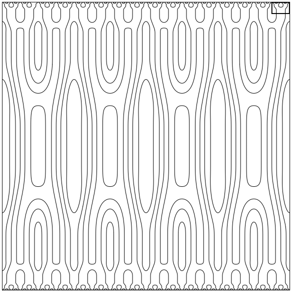
$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle|\nabla\mathbf{u}|^2\rangle^{1/2}=Pe \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \sim Pe^{2/3} \quad \text{as } Pe \to \infty.$$

More precisely, there exist constants C, C' depending only on the domain such that

$$C\frac{Pe^{2/3}}{\log^{4/3} Pe} \leq \max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle|\nabla\mathbf{u}|^2\rangle^{1/2}=Pe \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \leq C' Pe^{2/3}$$

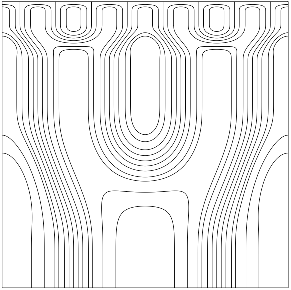
for $Pe \gg 1$.

What do our flows look like?



$$\mathbf{u} = \nabla^{\perp}\psi$$

$$l_k \lesssim \delta_k$$
$$l_{bl} \sim \delta_{bl}$$



Streamlines refine self-similarly from bulk to boundary layer

In the kth stage of refinement,

$$\psi(x,z) = f\left(\frac{z-z_k}{\delta_k}\right)\cdot l_k \Psi\left(\frac{x}{l_k}\right) + g\left(\frac{z-z_k}{\delta_k}\right)\cdot l_{k+1}\Psi\left(\frac{x}{l_{k+1}}\right)$$

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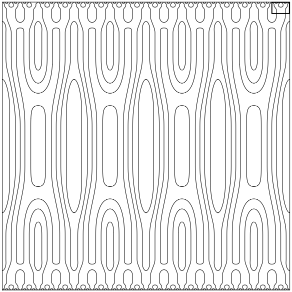

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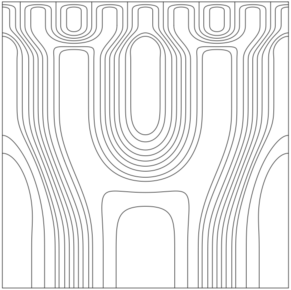
$$\ell(z)$$



Horizontal lengthscales satisfy

$$l_{bl} \sim \frac{\log^{1/3} Pe}{Pe^{2/3}} \qquad l_{bulk} \sim \frac{\log^{1/6} Pe}{Pe^{1/3}}$$

and

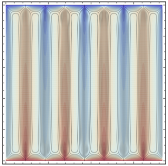
$$\ell(z) \sim \frac{\log^{1/6} Pe}{Pe^{1/3}}(1-z)^{1/2}$$

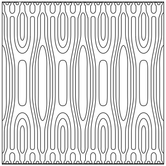
Brief sketch of the proof

Main challenges

As $Pe \to \infty$, our designs feature

- increasingly fine lengthscales
- an increasing number of distinct lengthscales





Simplify by taking $\mathbf{u}(\mathbf{x})$ indpt. of time (and why should time-dependence help?)

Main goals: Motivate our "branched" flow designs, and estimate their heat transport Nu in the advection-dominated limit $Pe \to \infty$.

Punchline: The analysis of optimal heat transport is analogous to pattern formation in micromagnetics, elasticity theory, etc.

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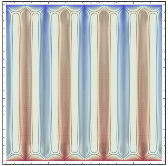

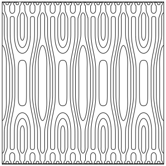

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Step 1: Obtain a general variational principle for heat transport

A non-local Dirichlet principle for heat transport

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} \cdot \nabla T = \Delta T$$

$$\text{div } \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

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$$Nu = \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle$$

Lemma

There exist dual variational principles for heat transport by a steady divergence-free flow.

$$Nu(\mathbf{u}) - 1 = \min_{\eta : \eta|_{\partial\Omega} = 0} \fint |\nabla \eta|^2 + |\nabla \Delta^{-1}(-w + \mathbf{u} \cdot \nabla \eta)|^2$$

$$= \max_{\xi : \xi|_{\partial\Omega} = 0} \fint 2w\xi - |\nabla \Delta^{-1} \mathbf{u} \cdot \nabla \xi|^2 - |\nabla \xi|^2$$

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Step 2: Recognize optimal heat transport as "energy-driven pattern formation"

...what plays the role of "free energy"?

A useful change of variables

Consider the general class of steady wall-to-wall problems,

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u})$$

where, e.g.,

$$||\mathbf{u}||^2 = \fint_\Omega |\mathbf{u}|^2 \quad \text{in the energy-constrained case}$$

$$||\mathbf{u}||^2 = \fint_\Omega |\nabla \mathbf{u}|^2 \quad \text{in the enstrophy-constrained case}$$

Now we know the variational principle

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u}) = 1 + \max_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \left\{ \fint 2w\xi - |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 - |\nabla \xi|^2 \right\}$$

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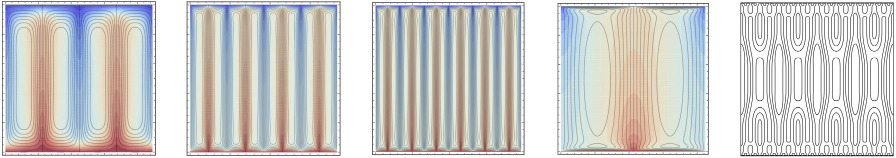
A useful change of variables

<u>Application:</u> The enstrophy-constrained wall-to-wall problem

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} \quad Nu(\mathbf{u})$$

is **equivalent** to solving

$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ \fint_\Omega w\xi = 1 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0},\xi|_{\partial\Omega}=0}} \quad \fint_\Omega |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$



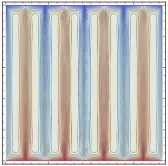
$Nu \sim Pe^{1/2}$ $\qquad$ $Nu \sim Pe^{0.54}$ $\qquad$ $Nu \sim Pe^{2/3}$

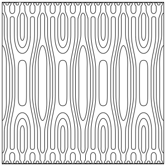
Step 3: The heat transport of branched flow designs

The branching construction

Recall: Our main result states that

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \sim Pe^{2/3} \quad \text{up to logs}$$

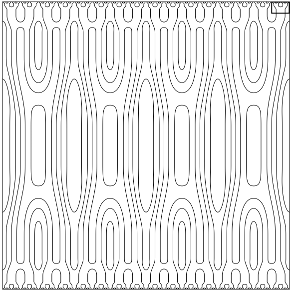




We just showed: It is equivalent to prove

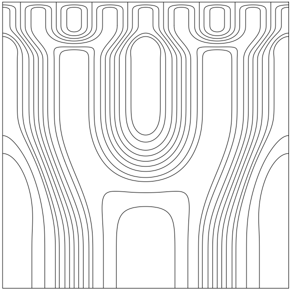
$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ \fint_\Omega w\xi = 1 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0},\xi|_{\partial\Omega}=0}} \fint_\Omega |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2 \sim \frac{1}{Pe^{2/3}}$$

The branching construction



$$\mathbf{u} = \nabla^{\perp}\psi$$
$$\xi = w$$

$$l_k \lesssim \delta_k$$
$$l_{bl} \sim \delta_{bl}$$



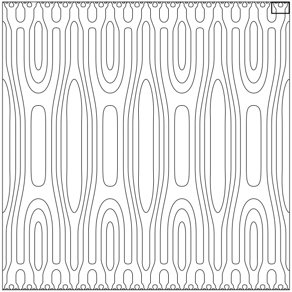
Claim: Constructions such as above can be made to satisfy

the "net flux" constraint $\displaystyle \fint_{\Omega} w\xi = 1$

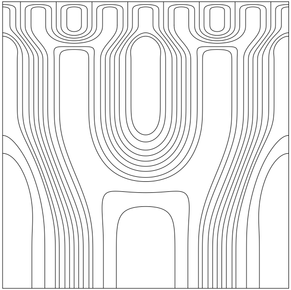
and to achieve

$$\fint_{\Omega} |\nabla\Delta^{-1}\mathrm{div}\,\mathbf{u}\xi|^2 + \frac{1}{Pe^2}\fint_{\Omega}|\nabla\mathbf{u}|^2 \cdot \fint_{\Omega}|\nabla\xi|^2 \lesssim \frac{\log^{4/3} Pe}{Pe^{2/3}}$$

The branching construction



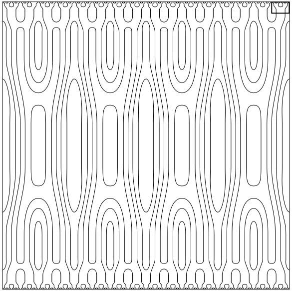
$$\mathbf{u} = \nabla^{\perp}\psi$$
$$\xi = w$$
$$\ell(z)$$



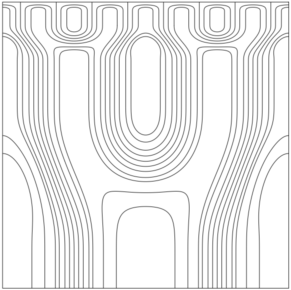
$$\fint_{\Omega} |\nabla\Delta^{-1}\mathrm{div}\,\mathbf{u}\xi|^2 + \frac{1}{Pe^2}\fint_{\Omega}|\nabla\mathbf{u}|^2 \cdot \fint_{\Omega}|\nabla\xi|^2$$
$$\lesssim l_{bl} + \int_{z_{bulk}}^{z_{bl}} (\ell')^2\,dz + \frac{1}{Pe^2}\left(\frac{1}{l_{bulk}^2} + \int_{z_{bulk}}^{z_{bl}} \frac{1}{\ell^2}\,dz + \frac{1}{l_{bl}}\right)^2$$

where $\ell = \ell(z) =$ horizontal lengthscale

The branching construction



$$\mathbf{u} = \nabla^{\perp}\psi$$
$$\xi = w$$
$$\ell(z)$$



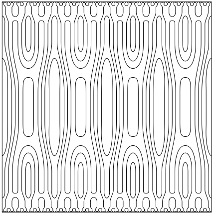
$$\min_{\substack{\ell(z) \\ \ell(z_{bulk})=l_{bulk} \\ \ell(z_{bl})=l_{bl}}} \left\{ l_{bl} + \int_{z_{bulk}}^{z_{bl}} (\ell')^2\, dz + \frac{1}{Pe^2}\left(\frac{1}{l_{bulk}^2} + \int_{z_{bulk}}^{z_{bl}} \frac{1}{\ell^2}\, dz + \frac{1}{l_{bl}}\right)^2 \right\} \sim \frac{\log^{4/3} Pe}{Pe^{2/3}}$$

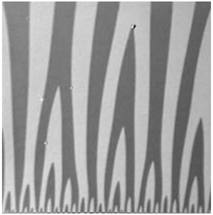
$$\ell(z) \sim \frac{\log^{1/6} Pe}{Pe^{1/3}} (1-z)^{1/2}$$

$$l_{bulk} \sim \frac{\log^{1/6} Pe}{Pe^{1/3}} \qquad l_{bl} \sim \frac{\log^{1/3} Pe}{Pe^{2/3}}$$

Concluding remarks

- For enstrophy-constrained transport $\max Nu \sim Pe^{2/3}$ up to logs
- Extensive 2D numerics finds $Nu \sim Pe^{0.54} \approx Pe^{6/11}$
- Proof combines
  1. The old *a priori* upper bound $\max Nu \lesssim Pe^{2/3}$
  2. A new functional analytic framework for optimal heat transport
  3. A new branching construction achieving $Nu \gtrsim Pe^{2/3-}$
- We were inspired by the analysis of branching in materials science, e.g., micromagnetics





[1]

[1] A. Hubert & R. R. Schäfer, *Magnetic Domains* 1998

Other examples of branching flows in fluid dynamics?

An old scientific question...

Does nature achieve optimal transport?

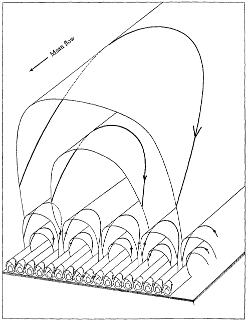
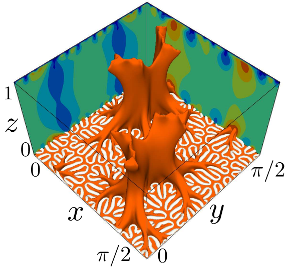


FIGURE 3. Qualitative sketch of the boundary-layer region of the vector field yielding maximum transport of momentum.

F. H. Busse, *Bounds for turbulent shear flow*, JFM '70



S. Motoki, G. Kawahara, & M. Shimizu, *Maximal heat transfer between two parallel plates*, arxiv 1801.04588

Thanks for listening

A non-local Dirichlet principle for heat transport

Lemma

For divergence-free $\mathbf{u}(\mathbf{x})$*,*

$$Nu(\mathbf{u}) - 1 = \min_{\eta:\eta|_{\partial\Omega}=0} \fint |\nabla\eta|^2 + |\nabla\Delta^{-1}(-w + \mathbf{u}\cdot\nabla\eta)|^2$$

$$= \max_{\xi:\xi|_{\partial\Omega}=0} \fint 2w\xi - |\nabla\Delta^{-1}\mathbf{u}\cdot\nabla\xi|^2 - |\nabla\xi|^2$$

Proof: Let $T = 1 - z + \theta$ and consider

$$\pm\mathbf{u}\cdot\nabla\theta_\pm = \Delta\theta_\pm + w$$

In the new variables

$$\xi = \frac{1}{2}(\theta_+ + \theta_-) \quad \text{and} \quad \eta = \frac{1}{2}(\theta_+ - \theta_-)$$

these become

$$\mathbf{u}\cdot\nabla\eta = \Delta\xi + w$$

$$\mathbf{u}\cdot\nabla\xi = \Delta\eta$$

A non-local Dirichlet principle for heat transport

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For divergence-free $\mathbf{u}(\mathbf{x})$,

$$Nu(\mathbf{u}) - 1 = \min_{\eta:\eta|_{\partial\Omega}=0} \fint |\nabla\eta|^2 + |\nabla\Delta^{-1}(-w + \mathbf{u}\cdot\nabla\eta)|^2$$
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$$\mathbf{u}\cdot\nabla\eta = \Delta\xi + w$$
$$\mathbf{u}\cdot\nabla\xi = \Delta\eta$$

A non-local Dirichlet principle for heat transport

Equivalently,

$$\mathbf{u} \cdot \nabla \Delta^{-1} \mathbf{u} \cdot \nabla \eta = \Delta \eta + \mathbf{u} \cdot \nabla \Delta^{-1} w$$
$$\mathbf{u} \cdot \nabla \Delta^{-1} \mathbf{u} \cdot \nabla \xi = \Delta \xi + w$$

and these are symmetric!

They express optimality for the dual variational principles

$$\min_{\eta : \eta|_{\partial\Omega}=0} \fint |\nabla \eta|^2 + |\nabla \Delta^{-1}(-w + \mathbf{u} \cdot \nabla \eta)|^2$$
$$\max_{\xi : \xi|_{\partial\Omega}=0} \fint 2w\xi - |\nabla \Delta^{-1} \mathbf{u} \cdot \nabla \xi|^2 - |\nabla \xi|^2$$

After i.b.p., one finds that the optimal values $= Nu(\mathbf{u}) - 1$. □

Turbulent heat transport
Rayleigh-Bénard Convection

Rayleigh-Bénard Convection

$$T = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta u + \hat{\mathbf{k}} Ra T$$

$$\text{div } \mathbf{u} = 0$$

$$T = 1$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$

$$Nu = 1 + \langle Tw \rangle$$

Question: What is the dependence of

$$Nu = Nu(Pr, Ra)$$

in the turbulent regime, $Ra \gg 1$?

Scaling laws vs. bounds

$$T = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta u + \hat{\mathbf{k}} Ra T$$

$$\text{div } \mathbf{u} = 0$$

$$T = 1$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$

$$Nu = 1 + \langle Tw \rangle$$

Scaling law predictions:

Malkus, '54	$Nu \sim Ra^{1/3}$	
Kraichnan '62, Spiegel '71	$Nu \sim Pr^{1/2} Ra^{1/2}$	"ultimate scaling"

Rigorous bounds:

Howard, '63	$Nu \lesssim Ra^{1/2}$	with stat. hypotheses
Doering & Constantin, '96	$Nu \lesssim Ra^{1/2}$	fully rigorous, 3D
Whitehead & Doering, '11	$Nu \lesssim Ra^{5/12}$	2D + stress-free b.c.

A new bounding method

$$T = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta u + \hat{\mathbf{k}} Ra T$$
$$\text{div } \mathbf{u} = 0$$

$$T = 1$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$Nu = 1 + \langle Tw \rangle$$

Wall-to-wall transport gives new bounds on RBC:[1]

The momentum eqn. implies

$$\left\langle |\nabla \mathbf{u}|^2 \right\rangle = Ra \cdot \langle Tw \rangle = Ra \cdot (Nu - 1)$$

[1] P. Hassanzadeh, G. Chini, C. Doering JFM 2014

A new bounding method

$$T = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\text{div } \mathbf{u} = 0$$

$$T = 1$$

$$F(Pe) = \max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}}} Nu(\mathbf{u})$$

Choosing Pe by

$$Pe^2 = Ra \cdot (Nu - 1),$$

one concludes for RBC

$$Nu \leq F(Ra \cdot (Nu - 1))$$

E.g.,

$$Nu \lesssim \frac{Pe^{2/3}}{\log^{\alpha} Pe} \text{ for wall-to-wall} \implies Nu \lesssim \frac{Ra^{1/2}}{\log^{3\alpha/2} Ra} \text{ for RBC}$$

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Our result

$$F(Pe) \gtrsim \frac{Pe^{2/3}}{\log^{4/3} Pe}$$

limits improvements to

$$Nu \lesssim Ra^{1/2}$$

by this method to *logarithmic corrections*

Regarding 2D RBC

$$T = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta u + \hat{\mathbf{k}} Ra T$$
$$\text{div } \mathbf{u} = 0$$

$$T = 1$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$Nu = 1 + \langle Tw \rangle$$

The Whitehead-Doering bound states

$$Nu \lesssim Ra^{5/12} \ll Ra^{1/2} \quad \text{in 2D w/ stress-free b.c.}$$

Our result is that

$$\max Nu \sim Ra^{1/2} \quad \text{up to logs}$$

Thus: 2D RBC achieves *strongly* sub-optimal heat transport