The wrinkling of a twisted ribbon

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Outline

An old experiment

Twist a thin ribbon and hold it with moderate tension. It should form wrinkles in the center.

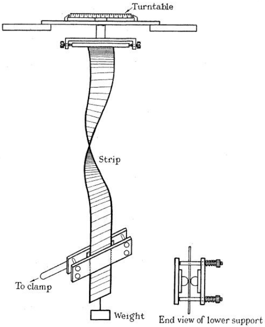


FIG. 3

Figure: Left: A.E. Green, Proc. R. Soc. 1937. Right: Chopin and Kudrolli, PRL 2013 and Chopin et al, J. Elasticity 2015

A family of solutions

1. **Left:** experiments; low tension to high tension.
2. **In this paper** we treat small wrinkles (third and fourth from left).
3. **Right:** experiments; self-intersection

Figure: Chopin and Kudrolli, PRL 2013 and Chopin et al, J. Elasticity 2015

A phase transition diagram

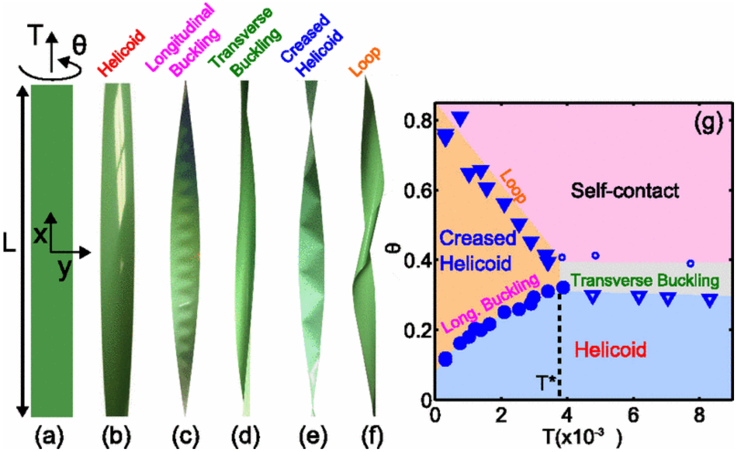


Figure: Chopin and Kudrolli, PRL 2013

Context: the twisted ribbon

- Generally, tension along wrinkles sets the direction of the wrinkling.
- Without tension along the wrinkles systems tend to look more ordered (and this helps mathematically).

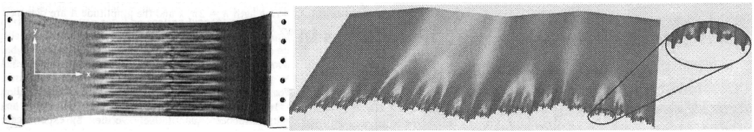


Figure: **Left:** Cerda, Mahadevan [PRL 2003]. **Right:** Audoly, Boudaoud [PRL 2003].

- The twisted ribbon has no tension along the wrinkles, but it is still predictable.

Intuition

Why the ribbon wrinkles:

- Twisting makes the outside edges get longer.
- If you allow the ribbon to compress, but only a little, then the outside is under tension and the inside under compression.
- A one- or two-dimensional object can wrinkle to avoid compression.

Energy minima and scaling laws

We will have an elastic energy functional $E^{(h)}$ representating the state of the ribbon. Of particular importance is the thickness h, which is assumed small.

Goal: prove a scaling law $E_0 + Ch^{4/3} \leq \min E^{(h)} \leq E_0 + Ch^{4/3}$.

The minimum E_0: solve the relaxed problem: set $h = 0$, take the quasiconvexification and minimize. Often there is no closed-form solution, but we have one.

While proving a scaling law we find bounds on the size of the wrinkles.

The form of the energy

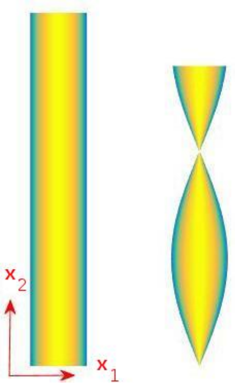
The form of the energy

$$E^{(h)} = \int_\Omega |\mathrm{M}|^2 + h^2|\mathrm{B}|^2 \, \mathrm{d}\boldsymbol{x}$$

- The membrane term M measures the amount of stretching. Specifically, $\langle \boldsymbol{a}, \mathrm{M}(\boldsymbol{x})\boldsymbol{a} \rangle$ is the amount of stretching in a direction $\boldsymbol{a}$ at a point $\boldsymbol{x}$.
- B measures the amount of bending.

Variables for a twisted ribbon

- The **domain** $\Omega = (-1/2, 1/2) \times (0, l)$ is a rectangle. Points are parameterized by $(x_1, x_2) \in \Omega$.
- The **tangential displacement** $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ and **normal displacement** $v : \Omega \to \mathbb{R}$.
- **Twist** per unit length ω.
- **Displacement of the top:** $-\frac{1}{2}\omega^2\xi^2$. Assume: $\xi < 1/2$.
- **Wrinkled zone:** for $|x_1| < \xi$, the ribbon is compressed in its reference state.



This energy is from Chopin et al, J. Elasticity 2015.

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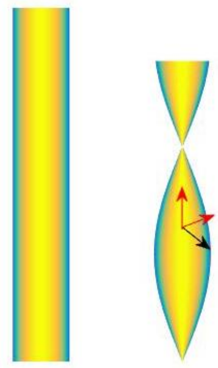

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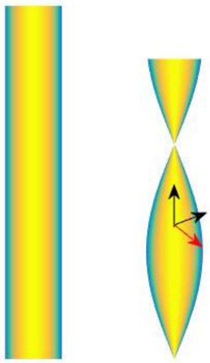

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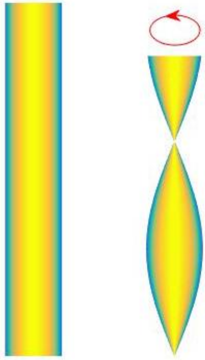

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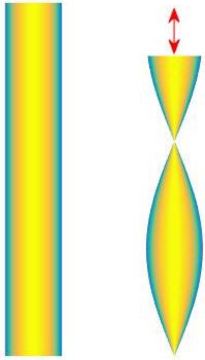

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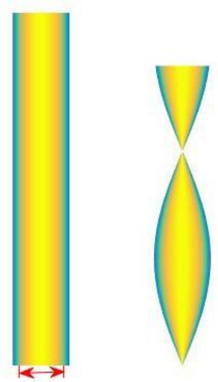


Nonlinear energy of an elastic sheet

Let $\boldsymbol{y}^{(h)} : \Omega_h \to \mathbb{R}^3$ be the position of the sheet. We start with the energy

$$E_{NL}^{(h)}(\boldsymbol{y}^{(h)}) = \int_{\Omega_h} \left| \sqrt{(\nabla \boldsymbol{y}^{(h)})^T (\nabla \boldsymbol{y}^{(h)})} - \mathrm{Id} \right|^2 + h^2 \left| A_{\boldsymbol{y}^{(h)}} \right|^2 \mathrm{d}\boldsymbol{x}.$$

- Interpret this as membrane plus bending energy $\int |\mathbf{M}|^2 + |\mathbf{B}|^2 \, \mathrm{d}\boldsymbol{x}$, but
- the resemblence to a Landau theory (lower order non-convex energy regularized by small, higher-order term) is not so clear.

We follow [Chopin et al, J Elas 2015] and linearize around a helicoid.

Small-slope energy of a twisted ribbon

We want to find out how the minimum of the energy scales with h:

$$E^{(h)}(\boldsymbol{u}, v) = \int_\Omega |\mathbf{M}(\boldsymbol{u}, v)|^2 + h^2 |\mathbf{B}(\boldsymbol{u}, v)|^2$$

$$\mathbf{M}(\boldsymbol{u}, v) = \mathrm{e}(\boldsymbol{u}) + \frac{1}{2}\begin{pmatrix} \partial_1 v \\ \partial_2 v + \omega x_1 \end{pmatrix} \otimes \begin{pmatrix} \partial_1 v \\ \partial_2 v + \omega x_1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \omega v \\ \omega v & \omega^2 \xi^2 \end{pmatrix}$$

$$\mathbf{B}(\boldsymbol{u}, v) = \nabla\nabla v + \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$$

with boundary data:

$$\boldsymbol{u}(x_1, 0) = \boldsymbol{u}(x_1, l) = 0$$
$$v(x_1, 0) = v(x_1, l) = 0$$

Heuristics

Vertical stretching:

$$m_{22} = \partial_2 u_2 + \frac{1}{2}(\partial_2 v + \omega x_1)^2 - \frac{1}{2}\omega^2\xi^2$$
$$= \partial_2 u_2 + \omega x_1 \partial_2 v + \frac{1}{2}\left((\partial_2 v)^2 - \omega^2\left(\xi^2 - x_1^2\right)\right)$$

Red: Mean-0 in x_2. Blue: Positive. Green: Sign depends on x_1.

- Vertical lines are stretched if $|x_1| > \xi$ and (in the reference state) compressed if $|x_1| < \xi$.
- Wasting arc length: choose $(\partial_2 v)^2$ to cancel out (on average) $\omega^2\left(\xi^2 - x_1^2\right)$ in $|x_1| < \xi$. Choose $\partial_2 u_2$ to cancel out oscillations around average.

Outline

The main result

Theorem (Kohn, O.)

There exists constants E_0, C, C' such that

$$E_0 + Ch^{4/3} \leq \min_{\boldsymbol{u},v} E^{(h)}(\boldsymbol{u},v) \leq E_0 + C'h^{4/3}.$$

The minimum is over $\boldsymbol{u} \in W^{1,2}(\Omega,\mathbb{R}^2)$, $v \in W^{2,2}(\Omega,\mathbb{R}^2)$ vanishing on $x_2 = 0$ and l.

Two parts of the proof:

- The lower bound requires an argument for any $\boldsymbol{u}$ and v.
- The upper bound is an ansatz (a choice of $\boldsymbol{u}$ and v).

The leading-order energy E_0

Main point: the zones under vertical tension always contribute energy E_0, and making $\boldsymbol{u}, v$ nonzero can only increase the energy.

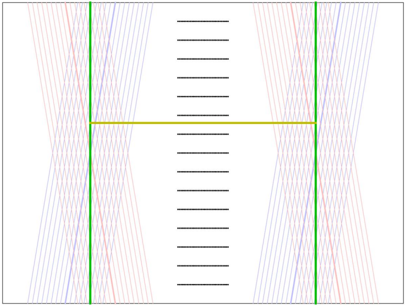
$$\begin{aligned} E^{(h)}(\boldsymbol{u}, v) &= \int_{-1/2}^{1/2}\int_0^l |\mathbf{M}|^2 + h^2|\mathbf{B}|^2 \, dx_2 \, dx_1 \\ &\geq \int_{-1/2}^{1/2}\left(\int_0^l m_{22}\, dx_2\right)_+^2 dx_1 \\ &\geq \frac{1}{2}\int_\xi^{1/2} \omega^4 (x_1^2 - \xi^2)^2 \, dx_1 = E_0 \end{aligned}$$

Remark: We are minimizing the relaxed problem.

An outline of the lower bound

We assume that $E^{(h)}(\boldsymbol{u}, v) < E_0 + \varepsilon$ and find a contradiction if ε is too small. This proof has two main steps:

1. The outer edges contain rigid lines: displacements are small.
2. Horizontal lines are stretched if the wrinkles have large amplitude, but bending resistance keeps the amplitude from being too small.



Sources: Strauss, Proc. Sympos. Pure Math. 1973; Bella and Kohn, Comm. Pure Applied Math 2014.

Proof of the lower bound (1.1)

$$E^{(h)} - E_0 = \int_\Omega \left|\mathbf{M}^{(\mathrm{ex})}\right|^2 + h^2|\mathbf{B}|^2 + \frac{1}{2}\omega^2\left(x_1^2 - \xi^2\right)_+ (\partial_2 v)^2 < \varepsilon$$

where $\mathbf{M}^{(\mathrm{ex})}$ is the excess strain:

$$\mathbf{M}^{(\mathrm{ex})} = \mathbf{M} - \frac{1}{2}\omega^2\left(x_1^2 - \xi^2\right)_+ \boldsymbol{e}^{(2)} \otimes \boldsymbol{e}^{(2)}$$

1. Tension in the vertical direction: for any $R > \xi$, $\|\partial_2 v\|_{L^2(|x_1|>R)} \lesssim \varepsilon^{1/2}$.
2. Small displacement: $\|v\|_{L^2(|x_1|>R)L^\infty(x_2)} \lesssim \varepsilon^{1/2}$.
3. There exist $\xi'_{\mathrm{left}} < R$ and $\xi'_{\mathrm{right}} > R$ such that $\|v(\xi', x_2)\|_{L^\infty(x_2\in[0,l])} \lesssim \varepsilon^{1/2}$

Next: Control ∇v in the outer zones.

Proof of the lower bound (1.2)

$$\mathrm{M}^{(\mathrm{ex})} = \mathsf{e}(\boldsymbol{u}) + \frac{1}{2}\nabla v \otimes \nabla v + \omega\, \mathsf{sym}\left(\nabla(x_1 v) \otimes \boldsymbol{e}^{(2)}\right) + \dots$$

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- **An observation:** Tension in direction $\boldsymbol{a}$ (unit vector) gives control on $\langle \boldsymbol{a}, \mathrm{M}^{(\mathrm{ex})}\boldsymbol{a}\rangle$, which gives control on $\langle \boldsymbol{a}, \mathsf{e}(\boldsymbol{u})\boldsymbol{a}\rangle$.
- **A Problem:** We have vertical, but not horizontal, tension ($\boldsymbol{a} = \boldsymbol{e}^{(2)}$). We have control on $u_2(\xi', x_2)$, but not $u_1(\xi', x_2)$.
- **The resolution:** Use tension in two diagonal directions $\boldsymbol{a}^{\pm}$.

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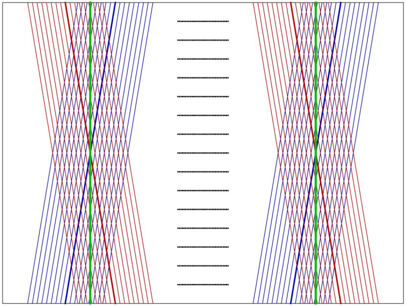
Proof of the lower bound (1.3)

Take vectors $\boldsymbol{a}^{\pm}$, sets $\Omega^{\pm}$ as shown. $\Omega^0 = \Omega^+ \cap \Omega^-$.

Goal: Show that $\boldsymbol{u}$ is small on the green lines.



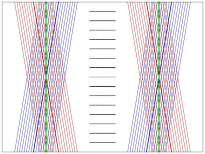
- Blue: Lines parallel to $\boldsymbol{a}^+$ shading region Ω^+.
- Red: Lines parallel to $\boldsymbol{a}^-$ shading region Ω^-.
- Green: Lines $x_1 = \xi'$.

Proof of the lower bound (1.4)

1. Integrate along diagonal lines:

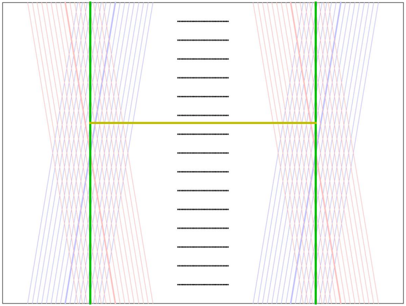
$$\begin{aligned} \varepsilon &\gtrsim \int_{\Omega^\pm} \left|\mathbf{M}^{(\mathrm{ex})}\right|^2 \mathrm{d}\boldsymbol{x} \gtrsim \int_{\Omega^\pm} \langle \boldsymbol{a}^\pm, \mathbf{M}^{(\mathrm{ex})} \boldsymbol{a}^\pm \rangle^2 \, \mathrm{d}\boldsymbol{x} \\ &\gtrsim \left(\int_{\Omega^\pm} \left\langle \boldsymbol{a}^\pm, \left[\frac{1}{2} \nabla v \otimes \nabla v - \begin{pmatrix} 0 & \omega v \\ \omega v & 0 \end{pmatrix} \right] \boldsymbol{a}^\pm \right\rangle \mathrm{d}\boldsymbol{x} \right)^2 \end{aligned}$$

2. Conclude that $\|\nabla v\|_{L^2(\Omega^0)} \lesssim \varepsilon^{1/4}$.
3. Triangle Inequality: $\|\mathrm{e}(\boldsymbol{u})\|_{L^1(\Omega^0)} \lesssim \varepsilon^{1/2}$.
4. Another diagonal line argument: $\left| \int_0^l u_1(\xi', x_2) \, \mathrm{d}x_2 \right| \lesssim \varepsilon^{1/2}$.



Lower bound part 2: picture

Goal: Show that v is small along the gold line (the wrinkles have small amplitude).



- Green: Lines $x_1 = \xi'$. Displacements are small.
- Gold: Line across wrinkles. Cannot be stretched much.

Proof of the lower bound (2.1)

We control the horizontal stretching across the wrinkles to show that v cannot be too large. First: Jensen's Inequality on the $(1,1)$ membrane term. Let $\Omega' = \{|x_1| < \xi'\}$.

$$\begin{aligned} \varepsilon &\geq \int_\Omega \frac{1}{2}(m_{11}^{(\mathrm{ex})})^2 \geq \frac{1}{2}\left(\int_{\Omega'} \partial_1 u_1 + \frac{1}{2}\partial_1 v^2\right)^2 \\ &\gtrsim \left(\int_{\Omega'} \partial_1 v^2\right)^2 - \left|\int_{\Omega'} \partial_1 u_1\right|^2 \end{aligned}$$

so $\|\partial_1 v\|_{L^2(\Omega')} \lesssim \varepsilon^{1/4}$, and therefore $\|v\|_{L^2(\Omega')} \lesssim \varepsilon^{1/4}$.

Proof of the lower bound (2.2)

The membrane term prefers that v be small. The bending term prefers to have $\partial_{22} v$ small:

$$h^2 \int_\Omega (\partial_{22} v)^2 \leq \varepsilon$$

By interpolation, the slopes must be small:

$$\|\partial_2 v\|_{L^2(\Omega')} \leq \|v\|_{L^2(\Omega')}^{1/2} \|\partial_{22} v\|_{L^2(\Omega')}^{1/2} \lesssim \left(\varepsilon^{3/4} h^{-1}\right)^{1/2}$$

Proof of the lower bound (2.3)

$$m_{22} = \partial_2 u_2 + \omega x_1 \partial_2 v + \frac{1}{2}\left((\partial_2 v)^2 - \omega^2\left(\xi^2 - x_1^2\right)\right)$$

We now have a contradiction: the wrinkles must waste an $O(1)$ amount of arclength.

$$\begin{aligned} \varepsilon^{1/2} &\gtrsim \int_{\Omega'} \left|\partial_2 v^2 - \omega^2(\xi^2 - x_1^2)_+\right| \\ &\geq \int_{\Omega'} \left|\omega^2(\xi^2 - x_1^2)_+\right| - \int_{\Omega'} \left|\partial_2 v^2\right| \end{aligned}$$

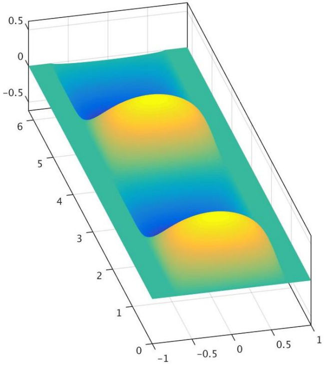
This gives a contradiction if $\varepsilon < Ch^{4/3}$ for some C.

The main point: this proves a lower bound for the energy. Along the way we showed inequalities about any low energy state.

Ansatz sketch: first attempt

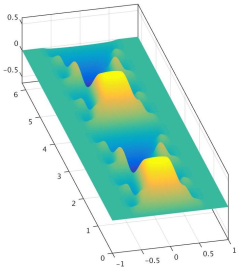
$$m_{22} = \partial_2 u_2 + \omega x_1 \partial_2 v + \frac{1}{2}\left((\partial_2 v)^2 - \omega^2\left(\xi^2 - x_1^2\right)\right)$$

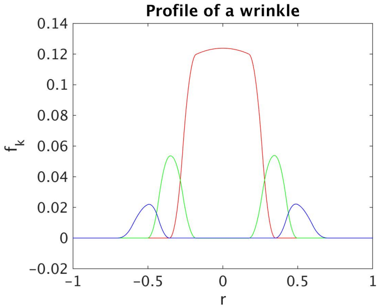
- **The basic idea:** Wrinkling can waste arclength to avoid compression. The lower bound suggests the wavelength.
- **A natural first attempt:** $v(x_1, x_2) = \lambda f(x_1) \sin\left(\frac{x_2}{\lambda}\right)$ where is λ the wavelength and $f(x_1)$ controls the amplitude.
- **Choosing $\boldsymbol{u}$:** pick $\boldsymbol{u}$ to cancel out the two highest-order membrane terms m_{11} and m_{12}.
- **The problem:** The optimal $f(x_1) = \omega\sqrt{2(\xi^2 - x_1^2)_+}$ is not $W^{2,2}$, which gives infinite energy.



Ansatz sketch: refinement

- **Idea:** We have two parameters to play with: the wavelength and the amplitude. Varying both with x_1 allows us to make f less singular.
- **The old ansatz (reminder):** $v(x_1, x_2) = \lambda f(x_1) \sin\left(\frac{x_2}{\lambda}\right)$
- **Idea:** $v(x_1, x_2) = \lambda(x_1) f(x_1) \sin\left(\frac{x_2}{\lambda(x_1)}\right)$
- **The new ansatz:** $v(x_1, x_2) = \sum_{k=0}^{N} \lambda_k f_k(x_1) \sin\left(\frac{x_2}{\lambda_k}\right)$

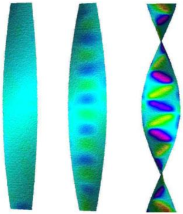




Do we expect refinement?

There is **no physical evidence** for refinement of wrinkles. However, they do change shape.

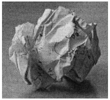
- This is a small-h theory. In regimes studied, $h \approx 0.05$. The difference between $O(h)$ energy (no refinement) and $O(h^{4/3})$ energy (refinement) is not too large. Prefactors might be more significant.
- The ansatz should not be taken too seriously. We needed to change the frequency from place to place. We took only two non-zero frequencies at each x_1 for convenience.

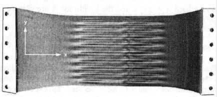


Source: Chopin Kudrolli PRL 2013.

Other morphologies for the twisted ribbon

1. The creased ribbon resembles crumpling due to confinement. The triangular facets are highly regular.





2. The stretched ribbon resembles the Cerda-Mahadevan experiment [PRL 2003].

Some open questions about twisted ribbons

- Ground state: are wrinkles horizontal with refinement (as in ansatz), diagonal (suggested by experiments) or something else?
- Creases are probably found if tension is low ($\frac{1}{2} - \xi \ll 1$). Similar results with two small parameters (thickness and tension)? Phase transitions?
- Nonlinear version: solving the relaxed problem (finding E_0 and identifying the wrinkled zone) seems hard.

Summary (twisted ribbons)

- We proved a lower bound for our energy and found a matching ansatz.
- The energy scales as $E_0 + Ch^{4/3}$, which indicates that some zone is stretched (E_0) and that there is microstructure ($h^{4/3}$).
- The lower bound does not identify the shape of the wrinkles, or tell us if there are multiple length scales.
- In proving the lower bound, we showed that low energy states are rigid near the edges and wrinkle in the center.
- The ansatz uses a cascade of wrinkles.

Thanks for your attention!