A minimization approach (in De Giorgi's style) to the wave equation on time-dependent domains

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Time-dependent domains

- Several problems in dynamic fracture mechanics lead to the study of the wave equation in time-dependent domains.
- The main difficulty is that at every time t the solution belongs to a different function space V_t contained in an ambient space Hilbert space H independent of t.
- A common situation is $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^d$ and Γ_t is a closed $(d-1)$-dimensional subset of Ω, representing the crack at time t.
- A natural assumption on Γ_t is that it is monotonically increasing with respect to t, thus encoding the fact that, once created, a crack cannot disappear.
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Wave equation in time-dependent domains

- Given $u^0 \in V_0$ $(= H^1(\Omega \setminus \Gamma_0))$ and $u^1 \in H$ $(= L^2(\Omega))$, the Cauchy problem we are interested in is formally written as

$$\begin{cases} u''(t) + Au(t) = 0 & \text{for a.e. } t > 0\,, \\ u(t) \in V_t \ (= H^1(\Omega \setminus \Gamma_t)) & \text{for a.e. } t > 0\,, \\ u(0) = u^0,\ u'(0) = u^1\,, \end{cases}$$

  where $'$ denotes the time derivative and A is a continuous and coercive linear operator mapping V_t into its dual V_t^* ($A = -\Delta$ in the examples).
- Under suitable hypotheses on V_t and A, the existence of a solution has been proven by Larsen and myself through a time-discrete approach, by solving suitable incremental minimum problems and then passing to the limit as the time step tends to zero.
- The purpose of this talk is to show that a solution can be approximated by global minimizers of suitable energy functionals defined as time integrals on $[0, \infty)$.

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General hypotheses on the spaces

- H is a separable Hilbert space and $(V_t)_{t\in[0,\infty)}$ is a family of separable Hilbert spaces with the following properties:
  - (H1) for every $t \in [0,\infty)$ the space V_t is contained and dense in H with continuous embedding;
  - (H2) for every $s, t \in [0,\infty)$, with $s < t$, V_s is a closed subspace of V_t with the induced scalar product; in particular, if $0 \le s < t$ and $v \in V_s$, then we have $\|v\|_{V_s} = \|v\|_{V_t}$.
- The dual of H is identified with H, while for every $t \in [0,\infty)$ the dual of V_t is denoted by V_t^*. Let $\langle \cdot, \cdot \rangle_t$ be the duality product between V_t^* and V_t and let $\|\cdot\|_{V_t^*}$ be the corresponding dual norm. The adjoint of the continuous embedding of V_t into H provides a continuous embedding of H into V_t^* and H is dense in V_t^*.
- On the contrary, for $0 \le s < t$ the adjoint of the continuous embedding of V_s into V_t is not injective from V_t^* into V_s^*, since V_s is not dense in V_t.

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- Let $V_\infty := \bigcup_{t\geq 0} V_t$ and let $a\colon V_\infty \times V_\infty \to \mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(H3) continuity: there exists $M_0 > 0$ such that

$$|a(u,v)| \leq M_0 \|u\|_{V_t} \|v\|_{V_t} \quad \text{for every } t \geq 0 \text{ and } u, v \in V_t;$$

(H4) weak coercivity: there exist $\lambda_0 \geq 0$ and $\nu_0 > 0$ such that

$$a(u,u) + \lambda_0 \|u\|_H^2 \geq \nu_0 \|u\|_{V_t}^2 \quad \text{for every } t \geq 0 \text{ and } u \in V_t\,;$$

(H5) positive semidefiniteness: $a(u,u) \geq 0$ for every $u \in V_\infty$.

- For every $t \geq 0$ let $A_t\colon V_t \to V_t^*$ be the continuous linear operator defined by $\langle A_t u, v\rangle_t := a(u,v)$ for every $u, v \in V_t$. Note that

$$\|A_t u\|_{V_t^*} \leq M_0 \|u\|_{V_t} \quad \text{for every } u \in V_t.$$

- Finally, we set $\mathcal{Q}(u) := a(u,u)$ for every $u \in V_\infty$.

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Definition of solution

- Given $T > 0$, we define $\mathcal{W}_T^{0,1} := L^2((0,T);V_T) \cap H^1((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$(u,v)_{\mathcal{W}_T^{0,1}} = (u,v)_{L^2((0,T);V_T)} + (u',v')_{L^2((0,T);H)}.$$

- In order to take into account the constraint $u(t) \in V_t$, we define

$$\mathcal{V}_T^{0,1} := \{u \in \mathcal{W}_T^{0,1} : u(t) \in V_t \text{ for a.e. } t \in (0,T)\},$$

and note that it is a closed subspace of $\mathcal{W}_T^{0,1}$.

- We say that u is a weak solution of the equation

$$u''(t) + A_t u(t) = 0, \quad u(t) \in V_t \qquad \text{for } t \in [0,\infty)$$

if for every $T > 0$ we have $u \in \mathcal{V}_T^{0,1}$ and

$$\int_0^T (u'(t), \psi'(t))_H dt = \int_0^T a(u(t), \psi(t)) dt$$

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Existence and uniqueness

- This definition is different from the one of DM-Larsen 2011, since we use an integration by parts with respect to time. This allows us to avoid the technical problem of the definition of $u''(t)$ as an element of V_t^*, where some difficulties come from the time dependence of the spaces.
- The two definitions turn out to be equivalent (see DM-Toader 2018).
- The existence of a solution with prescribed initial conditions (for $u(0)$ and $u'(0)$) was proved in DM-Larsen 2011. A new proof, which avoids the use of $u''(t)$, in given in DM-Toader 2018.
- The uniqueness of the solution to the Cauchy problem is still open. So far uniqueness has been proved only under very strong additional assumptions on $(V_t)_{t\in[0,\infty)}$, which are satisfied in the case $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$, when the cracks Γ_t are sufficiently regular $(d-1)$-dimensional manifolds and depend regularly on t.

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De Giorgi's approach to the wave equation

- The following conjecture by De Giorgi links the solution of a nonlinear hyperbolic equation to a sequence of minimum problems.

Conjecture (De Giorgi, 1996 in a paper for a celebration of John Nash)

Let $u^0, u^1 \in C_c^\infty(\mathbb{R}^d)$ *and let* $k > 1$ *be an integer; for every* $\varepsilon > 0$ *let* u_ε *be a minimizer if the functional*

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The function spaces for our minimum problems

- Given $T > 0$, we define $\mathcal{W}^{0,2}_T := L^2((0,T);V_T) \cap H^2((0,T);H)$, with the Hilbert space structure induced by the scalar product

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The minimum problems for time dependent domains

- We fix $u^0 \in V_0$, $u^1 \in H$, and a sequence $\{u^1_\varepsilon\} \subset V_0$ such that
$$\|u^1_\varepsilon - u^1\|_H \to 0 \text{ as } \varepsilon \to 0+ \quad \text{and} \quad \varepsilon\|u^1_\varepsilon\|_{V_0} \le C_1,$$
for some constant $C_1 > 0$.
- For every $\varepsilon > 0$ we consider the functional
$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}\Big(\varepsilon^2\|u''(t)\|_H^2 + \mathcal{Q}(u(t))\Big)dt\,.$$
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For every $\varepsilon \in (0,1)$ *the functional* $\mathcal{F}_\varepsilon$ *admits a unique global minimizer* u_ε *in the set* $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$. *Moreover,*

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \bar{C}\varepsilon,$$

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Existence and uniqueness of minimizers

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$$u''(t) + A_t u(t) = 0, \quad u(t) \in V_t \qquad \text{for } t \in [0,\infty)$$

such that $u_{\varepsilon_n} \rightharpoonup u$ *weakly in* $\mathcal{W}^{0,1}_T$ *for every* $T > 0$. *Moreover the following properties hold:*

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If, in addition, $\varepsilon \|u^1_\varepsilon\|_{V_0} \to 0$ *as* $\varepsilon \to 0+$, *then the energy inequality holds:*

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- The previous theorem provides an alternative proof of the existence of a weak solution u of

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- In the case $V_t = H^1(\Omega \setminus \Gamma_t)$, $H = L^2(\Omega)$, and $A_t = -\Delta$, the Euler-Lagrange equation of the minimum problem solved by u_ε formally reads as

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and hence, letting $\varepsilon \to 0$, one *formally* obtains a solution to the wave equation in $\Omega \setminus \Gamma_t$.

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Main estimates

Theorem (Main estimates)

There exists a constant $C > 0$ *such that for every* $\varepsilon \in (0,1)$ *the minimizer* u_ε *of* $\mathcal{F}_\varepsilon$ *in* $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$ *satisfies the estimates:*

$$\int_t^{t+\tau} \mathcal{Q}(u_\varepsilon(s))ds \leq C\,\tau \qquad \textit{for every } t \geq 0 \textit{ and } \tau \geq \varepsilon\,,$$

$$\|u_\varepsilon(t)\|_H^2 \leq C(1+t^2) \qquad \textit{for every } t \geq 0,$$

$$\|u'_\varepsilon(t)\|_H \leq C \qquad \textit{for every } t \geq 0.$$

- The proof of these estimates follows the lines of Serra-Tilli 2016 with an important change, described in the next slide.

- A step is obtained by using an inner variation $u_\varepsilon(\varphi_\delta(t))$ for a suitable function $\varphi_\delta\colon [0,\infty) \to [0,\infty)$. Since in our case we have to require that $u_\varepsilon(\varphi_\delta(t)) \in V_t$ for a.e. $t > 0$, by the monotonicity of $t \mapsto V_t$ this variation is admissible only if $\varphi_\delta(t) \le t$ for a.e. $t > 0$.
- By the technical definition of φ_δ, this inequality leads to the constraint $\delta > 0$. Therefore the standard comparison between the functional on $u_\varepsilon(\varphi_\delta(t))$ and on the minimizer $u_\varepsilon(t)$, in the limit as $\delta \to 0+$, gives only an inequality, instead of the equality proven in Serra-Tilli 2016. This inequality, however, turns out to be enough to obtain the estimates of the theorem with minor changes.

Main change in the proof of the estimates

- A step is obtained by using an inner variation $u_\varepsilon(\varphi_\delta(t))$ for a suitable function $\varphi_\delta\colon [0,\infty) \to [0,\infty)$. Since in our case we have to require that $u_\varepsilon(\varphi_\delta(t)) \in V_t$ for a.e. $t > 0$, by the monotonicity of $t \mapsto V_t$ this variation is admissible only if $\varphi_\delta(t) \leq t$ for a.e. $t > 0$.
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Proof of the main theorem (first part)

- By the estimates for every $T > 0$ the sequence $\{u_{\varepsilon_n}\}$ is equibounded in $\mathcal{W}^{0,1}_T$ and hence there exist a subsequence, not relabeled, and a function $u \in \mathcal{W}^{0,1}_T$ such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $\mathcal{W}^{0,1}_T$. Moreover, since $\{u_{\varepsilon_n}\} \subset \mathcal{V}^{0,2}_T \subset \mathcal{V}^{0,1}_T$ and $\mathcal{V}^{0,1}_T$ is a closed subspace of $\mathcal{W}^{0,1}_T$, we have that $u \in \mathcal{V}^{0,1}_T$.
- For every $T > 0$ the Euler equation satisfied by u_{ε_n} and an integration by parts lead to

$$\int_0^T (u'_{\varepsilon_n}(t), \varepsilon_n^2 \psi'''(t) + 2\varepsilon_n \psi''(t) + \psi'(t))_H dt = \int_0^T a(u_{\varepsilon_n}(t), \psi(t)) dt.$$

for every $\psi \in C^\infty_c((0,T); V_T)$ with $\psi(t) \in V_t$ for every $t \in (0,T)$.
- To prove the previous result we have to approximate an arbitrary test function ψ satisfying the constraint $\psi(t) \in V_t$ for a.e. $t > 0$ by sums of functions of the form $\varphi(t)v$ with $v \in V_s$ and $\varphi \in C^2(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset [s, \infty)$, which still satisfy the constraint.

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Proof of the main theorem (conclusion)

- For every $T > 0$ we can pass to the limit as $n \to \infty$ in the equality
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and we obtain
$$\int_0^T (u'(t), \psi'(t))_H dt = \int_0^T a(u(t), \psi(t))dt$$
for every $\psi \in C_c^\infty((0,T); V_T)$ with $\psi(t) \in V_t$ for every $t \in (0,T)$.
- An easy approximation argument shows that the same equality is satisfied for every $\psi \in \mathcal{V}_T^{0,1}$ with $\psi(0) = \psi(T) = 0$.
- Therefore u is a weak solution of the equation
$$u''(t) + A_t u(t) = 0, \quad u(t) \in V_t \qquad \text{for } t \in [0, \infty).$$
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THANK YOU FOR YOUR ATTENTION!

AND MANY THANKS TO THE ORGANIZERS!