Bounds on the Riesz means of mixed Steklov eigenvalues

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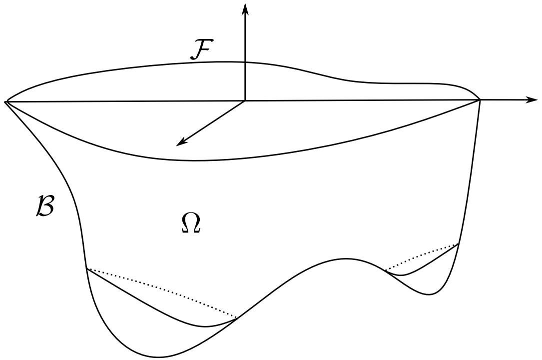
Mixed Steklov Eigenvalue Problem

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz and piecewise smooth boundary $\partial\Omega$. Throughout this talk, we assume that

$$\partial\Omega = \mathcal{F} \cup \mathcal{B}$$

with

$$\mathcal{F} \subset \{x_n = 0\}, \quad \text{and} \quad \mathcal{B} \subset \{x_n < 0\}.$$



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Sloshing Problem:

$$\begin{cases} \Delta f = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}} f = 0, & \text{on } \mathcal{B}, \\ \partial_{x_n} f = \nu f, & \text{on } \mathcal{F}. \end{cases}$$

Sloshing eigenvalues:

$$0 = \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \nearrow \infty$$

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Mixed Steklov-Dirichlet Eigenvalue Problem:

$$\begin{cases} \Delta f = 0, & \text{in } \Omega, \\ f = 0, & \text{on } \mathcal{B}, \\ \partial_{x_n} f = \eta f, & \text{on } \mathcal{F}. \end{cases}$$

Mixed Steklov-Dirichlet Eigenvalues:

$$0 < \eta_1 \leq \eta_2 \leq \eta_3 \leq \cdots \nearrow \infty$$

Dirichlet-to-Neumann map

The eigenvalues of the sloshing problem can be considered as the eigenvalues of the Dirichlet–to–Neumann map

$$\mathcal{D}_N : L^2(\mathcal{F}) \to L^2(\mathcal{F}), \qquad f \mapsto \partial_{\mathbf{n}} \tilde{f},$$

where $\tilde{f}$ is the harmonic extension of f to Ω satisfying the Neumann boundary condition on $\mathcal{B}$.

Similarly, the eigenvalues of the Steklov–Dirichlet problem is equal to the the eigenvalues of the Dirichlet–to–Neumann map

$$\mathcal{D}_D : L^2(\mathcal{F}) \to L^2(\mathcal{F}), \qquad f \mapsto \partial_{\mathbf{n}} \tilde{f},$$

where $\tilde{f}$ is the harmonic extension of f to Ω satisfying the Dirichlet boundary condition on $\mathcal{B}$.

Riesz Means

The Riesz mean $R_\gamma(z)$ of order $\gamma > 0$ is defined as

$$R_\gamma(z) := \sum_j (z - \nu_j)_+^\gamma, \qquad z > 0,$$

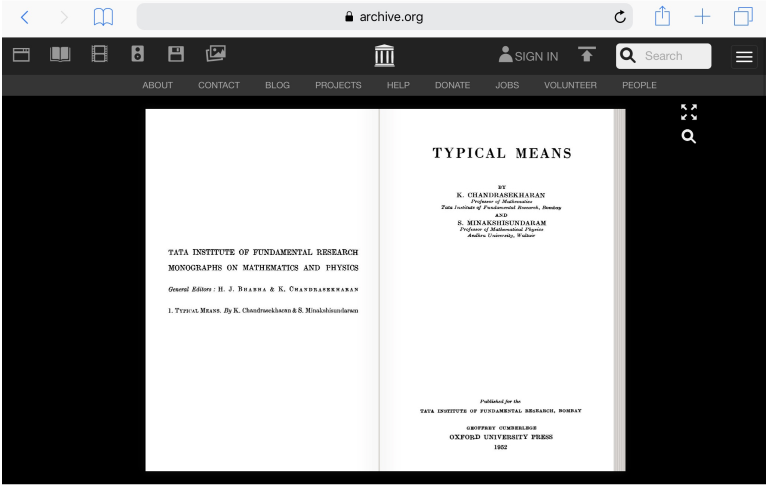
where $(z-\nu)_+ := \max\{0, z-\nu\}$. We may also denote it by $R_\gamma^\Omega(z, \mathcal{D}_N)$ or $R_\gamma^\Omega(z, \mathcal{D}_D)$ to identify the domain and the operator under consideration.

When $\gamma \to 0$, we get the counting function

$$N(z) := \sum_{\nu_j < z} 1 = \sup\{k : \nu_k < z\}$$

and by convention we denote $R_0(z) := N(z)$.

https://archive.org/details/typicalmeans032098mbp



Typical Means by Chandrasekharan and Minakshisundaram (1952)

Asymptotics

Sandgren 1955

$$\boxed{N(z) \sim \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathcal{F}| z^{n-1}, \qquad z \nearrow \infty,}$$

where ω_{n-1} is the volume of a unit ball in $\mathbb{R}^{n-1}$, and $|\mathcal{F}|$ denote the $(n-1)$-Euclidean volume of $\mathcal{F}$.

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Using the following identity

$$R_\gamma(z) = \gamma \int_0^\infty (z-t)_+^{\gamma-1} R_0(t)dt = \gamma \int_0^z (z-t)^{\gamma-1} R_0(t)dt,$$

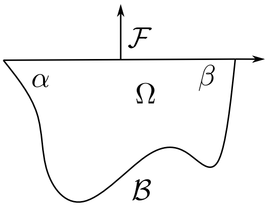
we immediately get

$$\boxed{R_\gamma(z) \sim C_{n,\gamma} |\mathcal{F}| z^{n+\gamma-1}, \qquad z \nearrow \infty,}$$

where

$$C_{n,\gamma} := \frac{1}{(4\pi)^{\frac{n-1}{2}}} \frac{\Gamma(\gamma+1)\Gamma(n)}{\Gamma(\frac{n+1}{2})\Gamma(n+\gamma)}.$$

Two-term asymptotic in dimension 2



Levitin, Parnovski, Polterovich, Sher (2017)

Let Ω be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega = \mathcal{F} \cup \mathcal{B}$, where $\mathcal{F}$ is connected. Let $\alpha, \beta \in (0, \frac{\pi}{2})$ be the interior angles between $\mathcal{F}$ and $\mathcal{B}$. Then the following asymptotic expansion holds.

Sloshing eigenvalues

$$\nu_k|\mathcal{F}| = \pi k - \frac{\pi}{2} - \frac{\pi^2}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + o(1), \qquad k \nearrow \infty.$$

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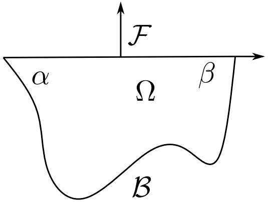

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Steklov-Dirichlet eigenvalues

$$\eta_k |\mathcal{F}| = \pi k - \frac{\pi}{2} + \frac{\pi^2}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + o(1), \qquad k \nearrow \infty.$$

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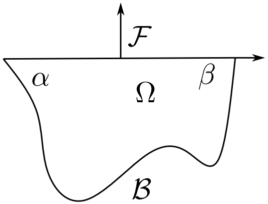

Asymptotic of Riesz means (Ferrulli, Lagacé - 2018)

Let Ω be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega = \mathcal{F} \cup \mathcal{B}$, where $\mathcal{F}$ is connected. Let $\alpha, \beta \in (0, \frac{\pi}{2})$ be the interior angles between $\mathcal{F}$ and $\mathcal{B}$. Then the following asymptotic expansion holds.

$$R_1^\Omega(z, \mathcal{D}_N) = \frac{1}{2\pi}|\mathcal{F}|z^2 + \frac{\pi}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) z + o(z).$$

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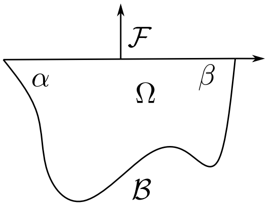

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Asymptotically Sharp Bounds for Riesz Means

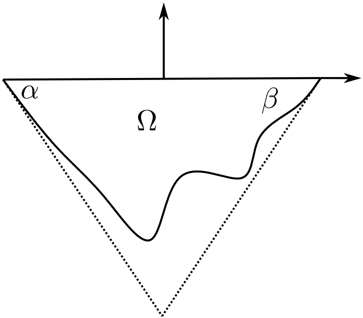
Main results

Theorem (H. , Laptev)

Let $\Omega \subset \mathbb{R}^2$ be a subset of a triangular domain, as shown in the picture, and the interior angles $\alpha, \beta \in (0, \frac{\pi}{2})$ of Ω coincide with the ones for the triangle. Then

$$R_1(z, \mathcal{D}_N) \geq \frac{1}{2\pi}|\mathcal{F}|z^2 + \frac{1}{2\pi}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right)\left(z - \frac{1 - e^{-2hz}}{2h}\right),$$

where $h = |\mathcal{F}|\frac{\tan(\alpha)\tan(\beta)}{\tan(\alpha)+\tan(\beta)}$ is the hight of the triangle.



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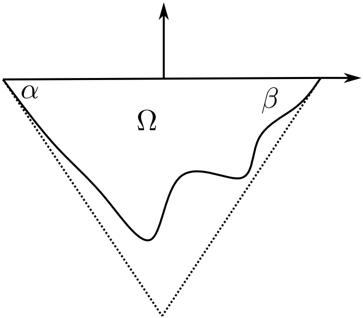

Main results

Theorem (H. , Laptev)

Let $\Omega \subset \mathbb{R}^2$ *with* $\partial\Omega = \mathcal{F} \cup \mathcal{B}$ *be a subset of* $\mathcal{F} \times (-\infty, 0)$. *Then*

$$R_1^{\Omega}(z, \mathcal{D}_N) \geq \frac{1}{2\pi}|\mathcal{F}|z^2 + \frac{1}{2}z,$$

and

$$R_1^{\Omega}(z, \mathcal{D}_D) \leq \frac{1}{2\pi}|\mathcal{F}|z^2 - \frac{1}{2}z + \frac{\pi}{2|\mathcal{F}|}.$$

Domain Monotonicity for Mixed Steklov Eigenvalues

Theorem (Banuelos, Kulczycki, Polterovich, Siudeja - 2010)

Let $\Omega \subset \tilde{\Omega}$ *be subdomains of* $\mathbb{R}^n$ *with* $\partial\Omega = \mathcal{F} \cup \mathcal{B}$ *and* $\partial\tilde{\Omega} = \mathcal{F} \cup \tilde{\mathcal{B}}$*. Then the following inequality holds.*

$$\nu_k(\Omega) \leq \nu_k(\tilde{\Omega}), \qquad \forall k \geq 1.$$

In particular,

$$R^{\Omega}(z, \mathcal{D}_N) = \sum_j (z - \nu_j(\Omega))_+ \geq \sum_j (z - \nu_j(\tilde{\Omega}))_+ = R^{\tilde{\Omega}}(z, \mathcal{D}_N).$$

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Main Results

Theorem (H. , Laptev)

Let Ω be a bounded domain of $\mathbb{R}^n$ and $\partial\Omega = \mathcal{F} \cup \mathcal{B}$. Then

$$R(z, \mathcal{D}_N) \geq C_{n,1}|\mathcal{F}|z^n - c(n) \int_0^z \int_{\mathcal{B}} \langle \mathbf{n}, e_n \rangle e^{2x_n r} r^{n-1} \, ds dr,$$

where $c(n) = \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}}$.

Main Results

Theorem (H. , Laptev)

Assume that Ω *(with* $\partial\Omega = \mathcal{F} \cup \mathcal{B}$*) is a subset of* $\mathcal{F} \times (-\infty, 0)$*. Then*

$$R^{\Omega}(z, \mathcal{D}_N) \geq C_{n,1}|\mathcal{F}|z^n + \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \frac{|\mathcal{F}|}{(2h_\Omega)^n} \left(\Gamma(n) - \Gamma(n, 2h_\Omega z)\right),$$

where h_Ω *is the depth of* Ω *and*

$$\Gamma(n, x) := (n-1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$$

is the incomplete Γ*-function. (Notice that* $\Gamma(n) - \Gamma(n, x) > 0$ *for every* $x > 0$*, and every* $n \in \mathbb{N}$*.)*

Main Results

Theorem (H. , Laptev)

Let Ω be a bounded domain in $\mathbb{R}^n$ and subset of an infinite cylinder $\mathcal{F} \times [-\infty, 0]$, where $\partial\Omega = \mathcal{F} \cup \mathcal{B}$. Here $\mathcal{F}$ is the free part of the boundary. Then for every $z > 0$ we have

$$R^{\Omega}(z, \mathcal{D}_D) = \sum_j (z - \eta_j)_+ \leq C_{n,1} |\mathcal{F}| z^n.$$

Averaged Variational Principle

Theorem (El Soufi, Harrell, Stubbe - 2015)

Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\mathcal{F})$ consisting of the eigenfunctions associated with $\{\nu_j\}_{j=1}^{\infty}$ and let $f_\xi \in \mathcal{H}(\Omega)$ be a family of harmonic functions where ξ varies over a measure space $(\mathfrak{M}, \mu)$. Let $\mathfrak{M}_0$ be a measurable subset of $\mathfrak{M}$. Then for any $z > 0$ we have

$$\sum_j (z - \nu_j)_+ \int_{\mathfrak{M}} \left| \int_{\mathcal{F}} \varphi_j \bar{f}_\xi ds \right|^2 d\mu \geq z \int_{\mathfrak{M}_0} \int_{\mathcal{F}} |f_\xi|^2 ds\, d\mu - \int_{\mathfrak{M}_0} \mathrm{Re} \int_{\partial\Omega} \frac{\partial f_\xi}{\partial \mathbf{n}} \bar{f}_\xi \, ds\, d\mu.$$

Sketch of the Proof for Sloshing Porblem

We choose a suitable family of harmonic test functions.

$$f_{\xi'}(x) = e^{ix'\xi' + x_n|\xi'|}$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$. In the previous theorem take $\mathfrak{M} = \mathbb{R}^{n-1}$ and $\mathfrak{M}_0 = \{|\xi'| \leq z\}$.

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$$\sum_j (z - \nu_j)_+ \int_{\mathbb{R}^{n-1}} |\hat{\varphi}_j(\xi')|^2 \, d\xi' \geq |\mathcal{F}| \int_{|\xi'| \leq z} (z - |\xi'|) \, d\xi' - \int_{|\xi'| \leq z} \int_{\mathcal{B}} \langle \mathbf{n}, e_n \rangle |\xi'| e^{2x_n|\xi'|} \, ds d\xi',$$

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$$\sum_j (z-\nu_j)_+ \int_{\mathbb{R}^{n-1}} |\hat{\varphi}_j(\xi')|^2 \, d\xi' \ge |\mathcal{F}| \int_{|\xi'|\le z} (z - |\xi'|) \, d\xi' - \int_{|\xi'|\le z} \int_{\mathcal{B}} \langle \mathbf{n}, e_n \rangle |\xi'| e^{2x_n|\xi'|} \, ds d\xi',$$

where $\hat{\varphi}_j(\xi') = \int_{\mathcal{F}} e^{ix'\xi'} \varphi_j(x') ds$. Therefore,

$$R(z, \mathcal{D}_N) = \sum_j (z-\nu_j)_+ \ge \frac{\omega_{n-1}}{n(2\pi)^{n-1}} |\mathcal{F}| z^n - \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \int_0^z \int_{\mathcal{B}} \langle \mathbf{n}, e_n \rangle e^{2x_n r} r^{n-1} \, ds dr.$$

Thank You For Your Attention!