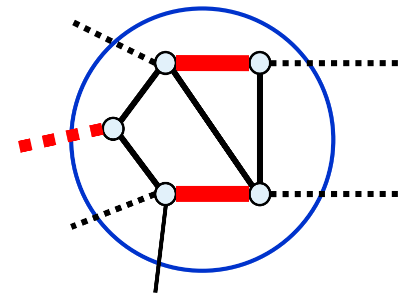
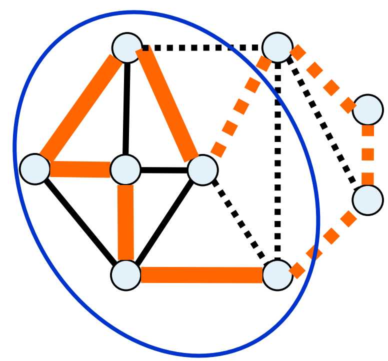
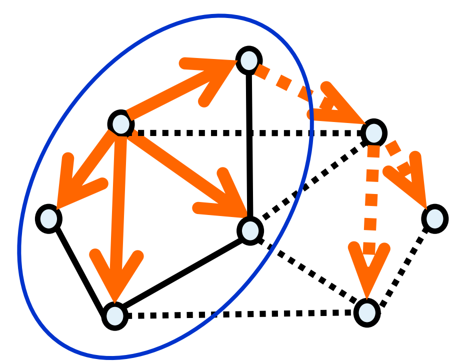
Excluded *t*-factors in Bipartite Graphs:

A Unified Framework for

- Nonbipartite Matchings,
- Restricted 2-matchings, and
- Arborescences







Kenjiro Takazawa

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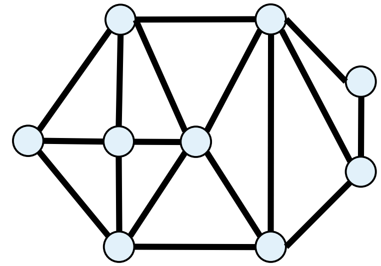
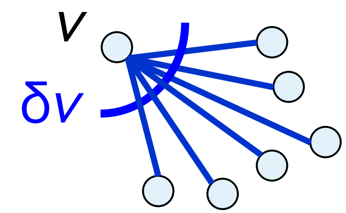
TSP Workshop @ Banff

Sep 24-28, 2018

Matching, 2-matching, and t-matching

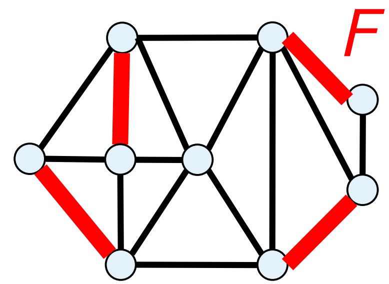
- $G = (V,E)$: Simple, Undirected

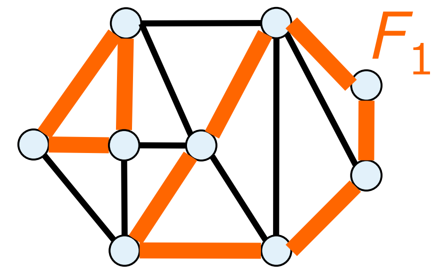


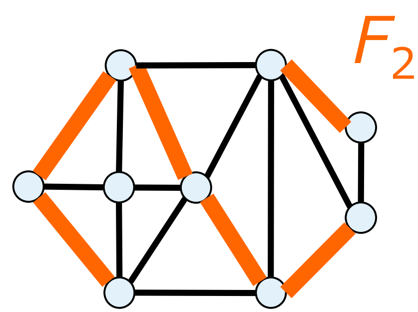


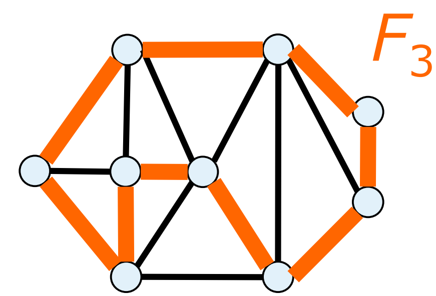
Definition

- $F \subseteq E$ is a **matching**
  $\Leftrightarrow |F \cap \delta v| \leq 1 \quad {}^\forall v \in V$
- $F \subseteq E$ is a **2-matching**
  $\Leftrightarrow |F \cap \delta v| \leq 2 \quad {}^\forall v \in V$
- $F \subseteq E$ is a ***t*-matching**
  $\Leftrightarrow |F \cap \delta v| \leq t \quad {}^\forall v \in V$



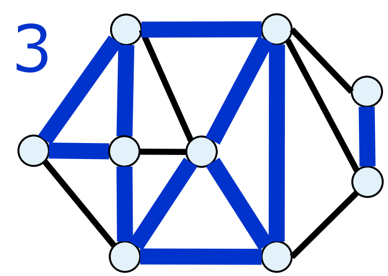






- **Just keep t=1,2 in mind**
- No theoretical difference in ${}^\forall t \in \mathbf{Z}_{>0}$

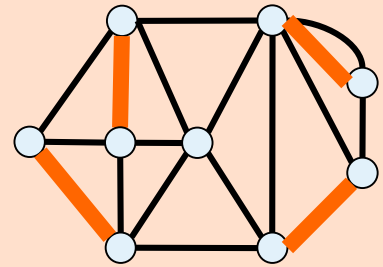
$t = 3$



Our Framework : What are contained ?

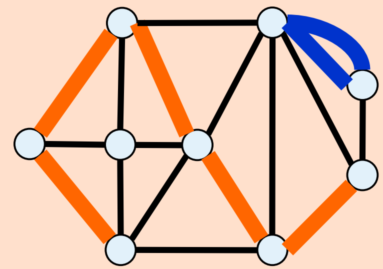
Our Framework

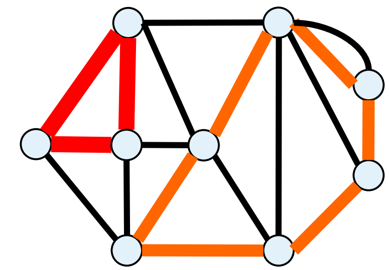
- **Matching**



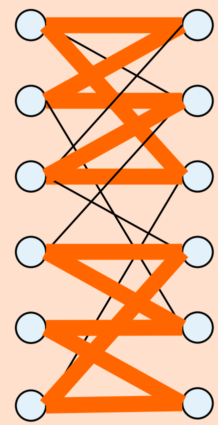
Restriction

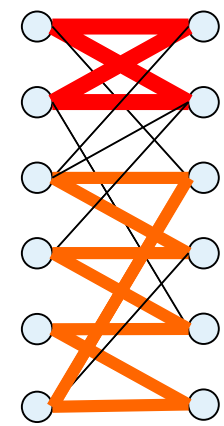
- **Triangle-free 2-matching**
  with edge-multiplicity



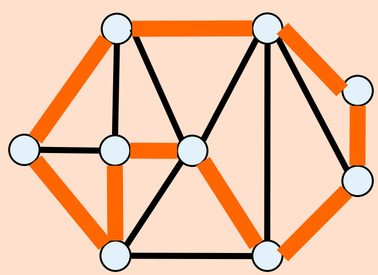


- **Square-free 2-matching in bipartite graph**





- **Hamilton cycle**



Our Result : What did we solve ?

Our Framework

- Matching

- Matroid
- Arborescence

- Triangle-free 2-matching with edge-multiplicity
- Square-free 2-matching in bipartite graph

P

NP-hard

- Hamilton cycle

Our Result

➢ *Min-max theorem*
➢ *LP with dual integrality*
➢ *Combinatorial algorithm*

- **Path-matching** [Cunningham, Geelen '97]
- **Even factor** [Cunningham, Geelen '01]

- **$K_{t,t}$-free t-matching** [Frank '03]

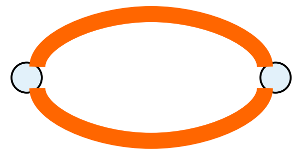
- **2-matching covering 3,4-edge cuts** [Kaiser, Škrekovski '04,08] [Boyd, Iwata, **T.** '13]

Contents

Triangle-free 2-matching

Definition (Triangle-free 2-matching)

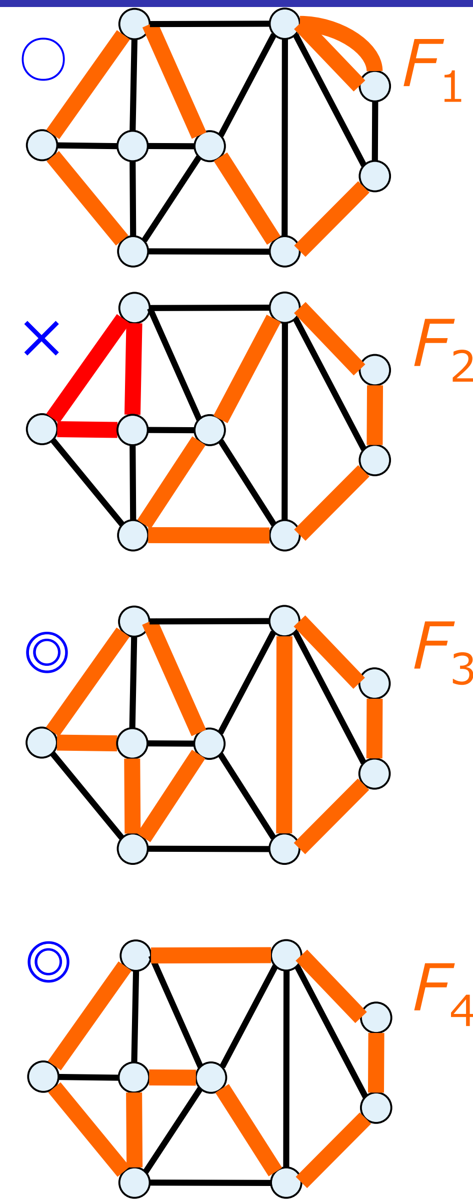
- 2-matching $\boldsymbol{x} \in \{0,1,2\}^E$ is **Triangle-free**
  $\Leftrightarrow$ Excluding cycles of length 3

➢ Allowing multiplicity 2: 

Theorem [Cornuéjols & Pulleyblank '80]

- Max. $\sum x(e)$: **P**
- Max. $\sum w(e)x(e)$: **P**
  - ➢ *Min-max theorem*
  - ➢ *LP with dual integrality*
  - ➢ *Combinatorial algorithm*

- **No multiplicity** allowed:
  - ➢ Max. $|F|$: Algorithm [Hartvigsen '84]
  - ➢ Max. $w(F)$: **Open**
    - ➢ *Discrete convexity* [Kobayashi '14]

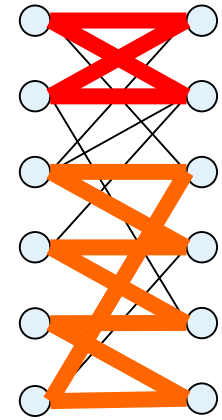


Square-free 2-matching in bipartite graph

Definition (Square-free 2-matching)

- 2-matching $F \subseteq E$ is **Square-free**
  $\Leftrightarrow$ Excluding cycles of length 4

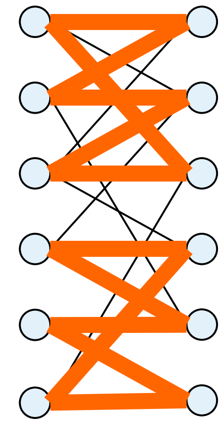
×



Previous work for bipartite graphs

- Max. $|F|$: **P**
  - *Min-max theorem* [Z. Király '99, Frank '03]
  - *Combinatorial algorithm* [Hartvigsen '06; Pap '07]
  - *Canonical decomposition* [**T.** '15]
- Max. $w(F)$: **NP-hard** [Z. Király '99]
  - **P** under *a certain assumption on* ***w*** (☞p. 19)
    - ✓ *LP with dual integrality* [Makai '07]
    - ✓ *Combinatorial algorithm* [**T.** '09]

◎

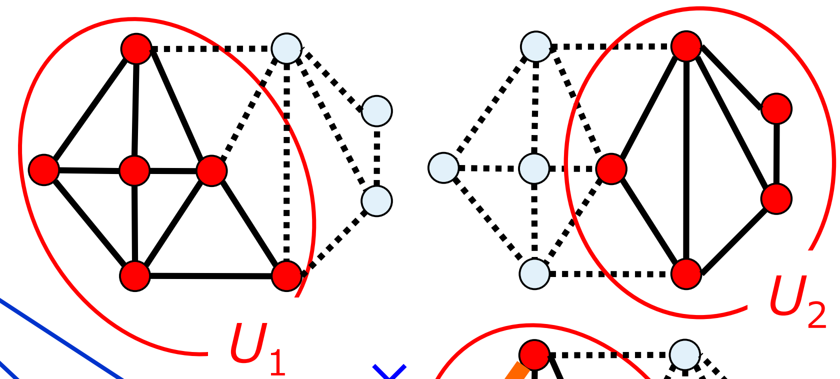


- Max. $|F|$ in **nonbipartite** graphs: **Open**
  - *Discrete convexity* [Kobayashi, Szabó, **T.** '12]

Contents

Our Framework: $\mathcal{U}$-feasible t-matching

- $\mathcal{U} \subseteq 2^V$: **Vertex subset family**
- Each $U \in \mathcal{U}$ has a t-factor



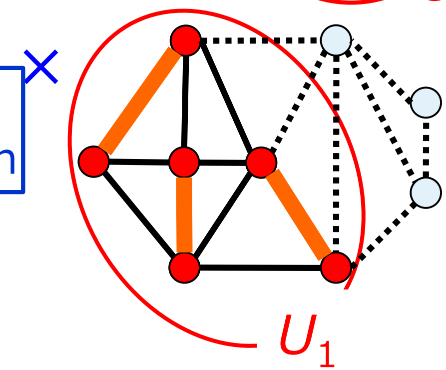
Definition

t-matching $F \subseteq E$ is **$\mathcal{U}$-feasible**

$\Leftrightarrow |F[U]| \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor \ {}^{\forall} U \in \mathcal{U}$

$\Leftrightarrow$ Excluding t-factors in $G[U]$ ${}^{\forall} U \in \mathcal{U}$

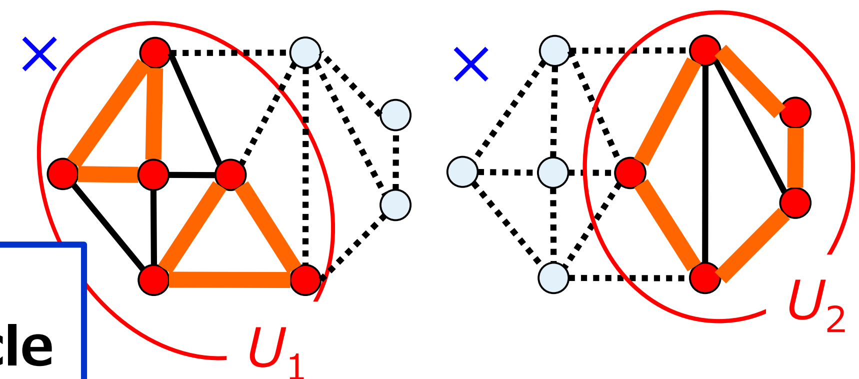
Answer to Michel Goemans' question



- $t=1$: $|F[U]| \le \left\lfloor \frac{|U|-1}{2} \right\rfloor = \begin{cases} \frac{|U|}{2} - 1 & (|U|\text{: even}) \\ \cancel{\frac{|U|-1}{2} \quad (|U|\text{: odd})} \end{cases}$

- $t=2$: $|F[U]| \le \left\lfloor \frac{2|U|-1}{2} \right\rfloor = |U| - 1$

[**T.** '17]



➢ **$\mathcal{U}=2^V \setminus \{\emptyset, V\}$**
➔ $\mathcal{U}$-feasible 2-factor=**Hamilton cycle**

Our Result

Our assumption

- G: **Bipartite**
- $^\forall U \in \mathcal{U}$ is **"factor-critical"** (☞ p. 14)

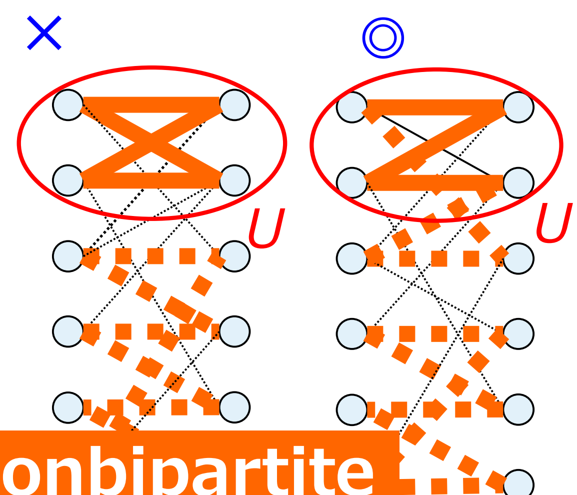
Our result

- *Min-max theorem*
- *Combinatorial algorithm*

Weighted (Assumption on w)

- *LP with dual integrality*
- *Combinatorial algorithm*

- Strong assumption…? **NO!!**
  - **Square-free 2-matching**
    - $\mathcal{U} = \{U : U \subseteq V, |U|=4\}$
    - $t=2$
  - **$K_{t,t}$-free t-matching**
  - **Nonbipartite matching**
  - **Triangle-free 2-matching**
  - **Path-matching / Even factor**
  - **Arboresence**



Nonbipartite
(☞Next slides)

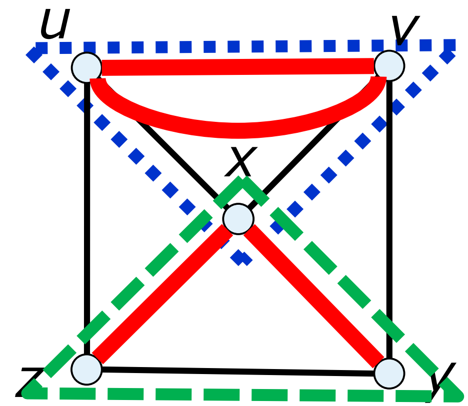
Special Case: Triangle-free 2-matching

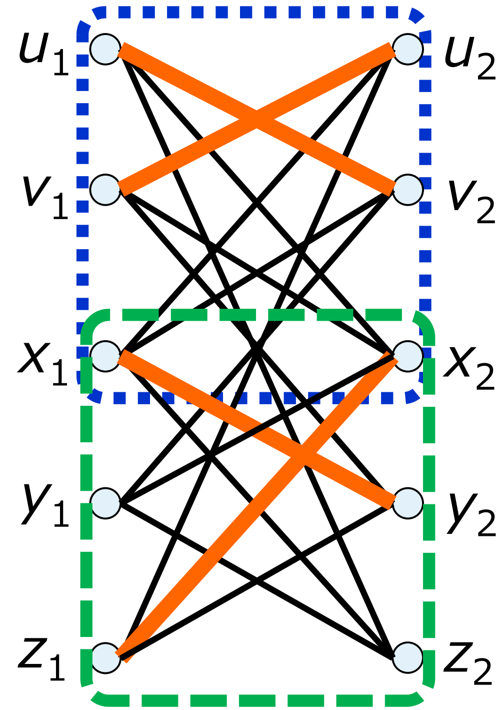
- $G=(V,E)$: **Nonbipartite graph**

- $G'=(V',E')$: **Bipartite graph**
  - $V' = V_1 \cup V_2$
  - $E' = \{u_1v_2,\ v_1u_2 : uv \in E\}$
- $t = 1$
- $\mathcal{U} = \{U_1 \cup U_2 : U \subseteq V,\ |U|=3\}$

Proposition

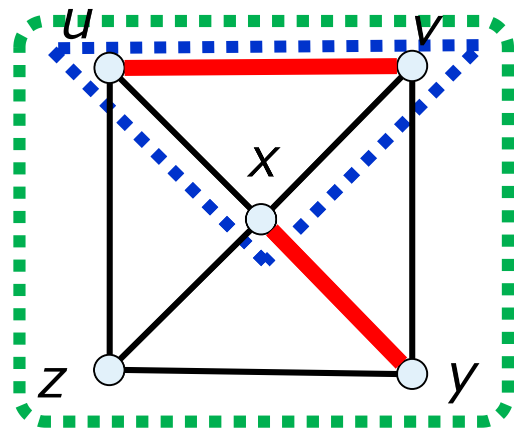
Triangle-free 2-matching in G
$\rightleftarrows$ $\mathcal{U}$-feasible 1-matching in G'



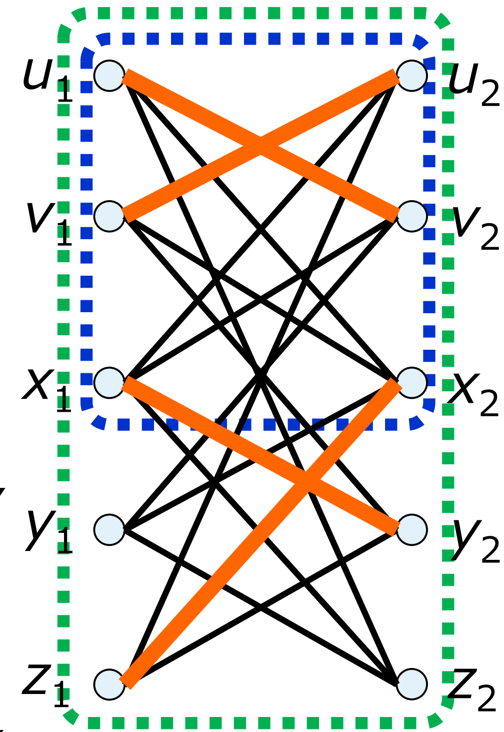


Special Case: Nonbipartite Matching

- $G=(V,E)$: **Nonbipartite graph**



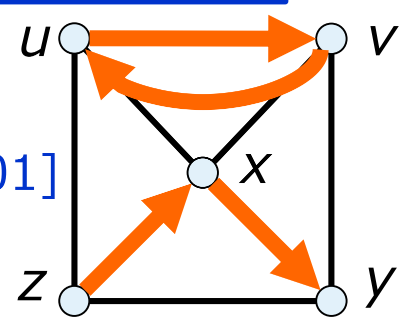
- $G'=(V',E')$: **Bipartite graph**
  - $V' = V_1 \cup V_2$
  - $E' = \{u_1v_2,\ v_1u_2 : uv \in E\}$
- $t = 1$
- $\mathcal{U} = \{U_1 \cup U_2 : U \subseteq V,\ |U| \text{ is odd}\}$



Proposition

$2 \cdot |\text{max matching in } G|$
$= |\text{max } \mathcal{U}\text{-feasible 1-matching in } G'|$

Dipaths and even dicycles
= Even factor [Cunningham, Geelen '01]



Special Case: Arborescence

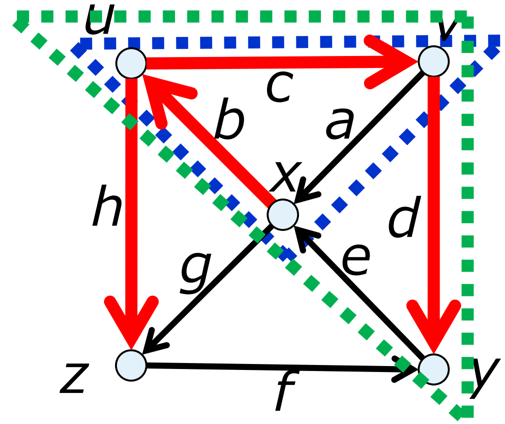
- $D=(V,E)$: (Nonbipartite) **Digraph**

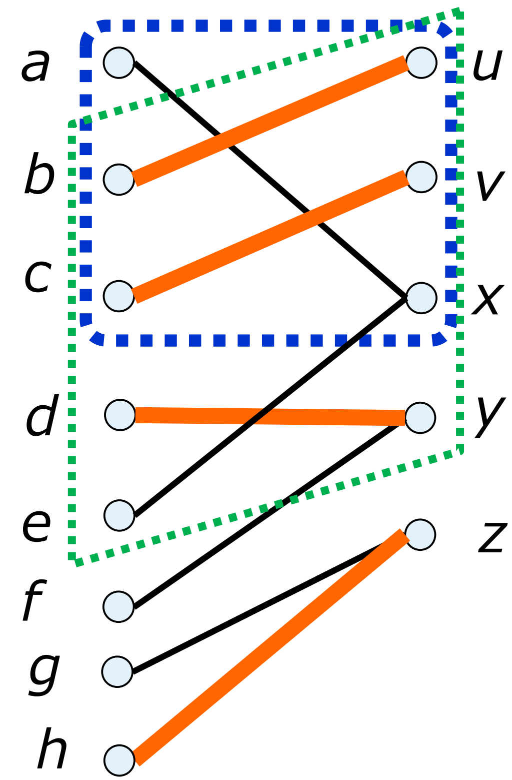
⇩

- $G'=(V',E')$: **Bipartite graph**
  - ➢ $V' = A \cup V$
  - ➢ $E' = \{av \mid v\text{: head of } a \text{ in } D\}$
- $t = 1$
- $\mathcal{U} = \{A(C) \cup V(C) \mid C\text{: cycle in } D\}$

Proposition

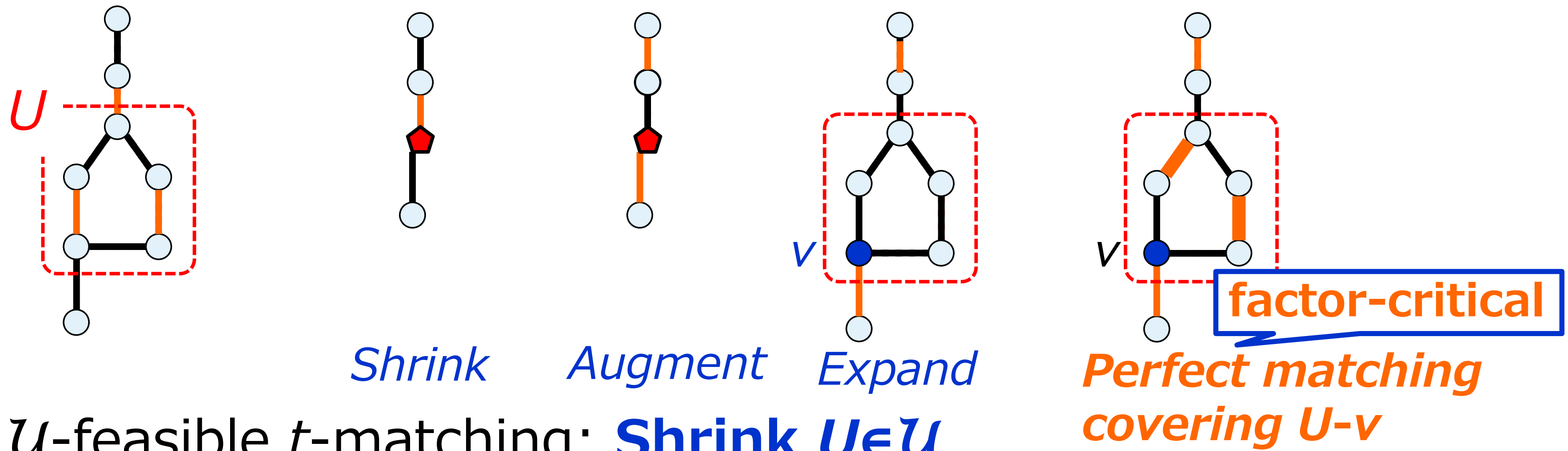
Arborescence in D
$\rightleftarrows$ $\mathcal{U}$-feasible 1-matching in G'



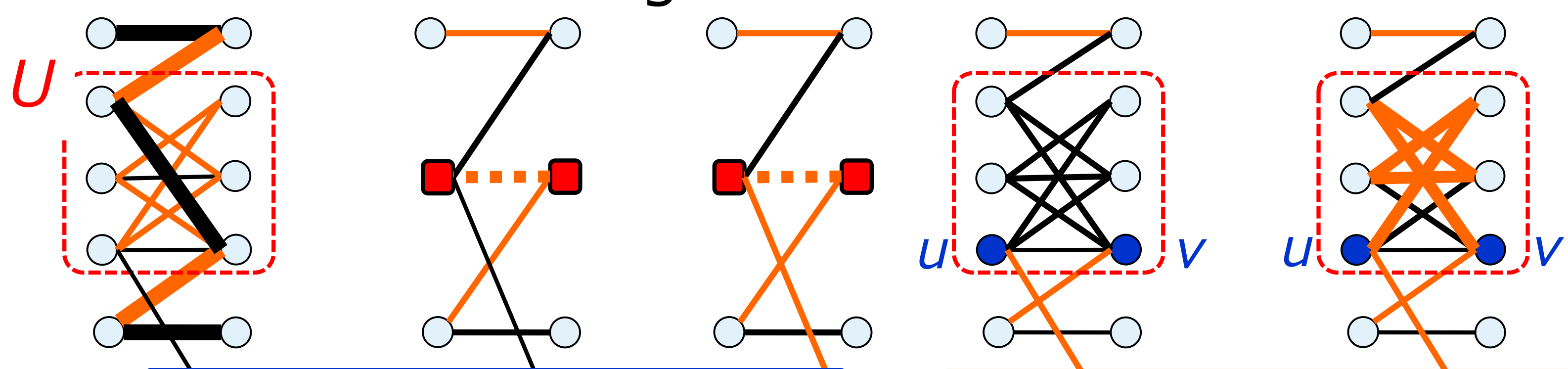


Algorithm

- Nonbipartite matching: **Shrink odd cycles** [Edmonds ’65]



- $\mathcal{U}$-feasible t-matching: **Shrink $U \in \mathcal{U}$**



Assumption on $(G,\mathcal{U},t)$

$U \in \mathcal{U}$ is “**factor-critical**”

***t*-matching s.t.**

- **u, v : Degree 1 (=t-1)**
- **U-{u,v} : Degree 2 (=t)**

More about Shrinking/Expanding $U \in \mathcal{U}$

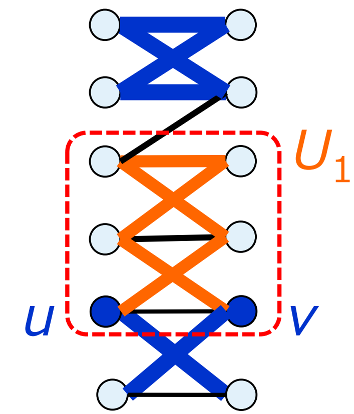
Definition [Factor-criticality]

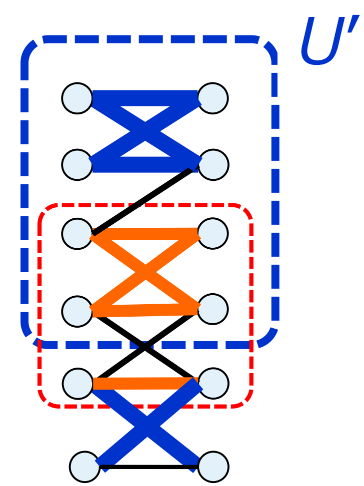
$(G, \mathcal{U}, t)$ is **factor-critical** if:
$^\forall U_1, \ldots, U_k \in \mathcal{U}$,
$^\forall$feasible edge set $\boldsymbol{F} \subseteq E$ in $G/(U_1 \cup \cdots \cup U_k)$
$^\exists \boldsymbol{F_i} \subseteq E[U_i]$ for each $i = 1, \ldots, k$, s.t.

- $|F_i| = \left\lfloor \frac{t|U_i| - 1}{2} \right\rfloor$
- $F \cup F_i$ is a t-matching

Definition [Feasibiity]

A t-matching $\boldsymbol{F} \subseteq E$ in $G/(U_1 \cup \cdots \cup U_k)$ is **feasible** if $^\exists F_i \subseteq E[U_i]$ s.t.
$\boldsymbol{F} \cup F_1 \cup \cdots \cup F_k$ is a $\mathcal{U}$-feasible t-matching





- {**Testing feasibility** / **Finding $\boldsymbol{F_i}$** } depend on $(G, \mathcal{U}, t)$, typically done in O(1) or O(n)

Min-max Theorem

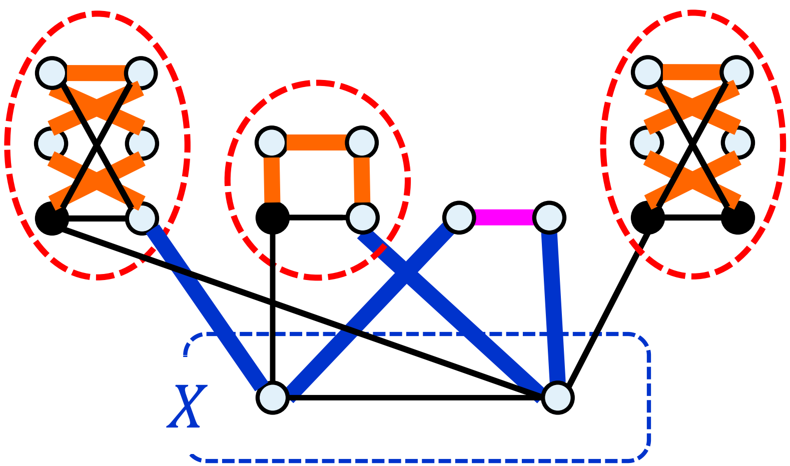
Theorem

- G: Bipartite
- $(G,\mathcal{U},t)$ is factor-critical

- Nonbipartite matching
- Triangle-free 2-matching
- Square-free 2-matching
- Even factor
- Arborescence
- $K_{t,t}$-free t-matching

➔ $\max\{|F| : F \text{ is a } \mathcal{U}\text{-feasible } t\text{-matching}\}$

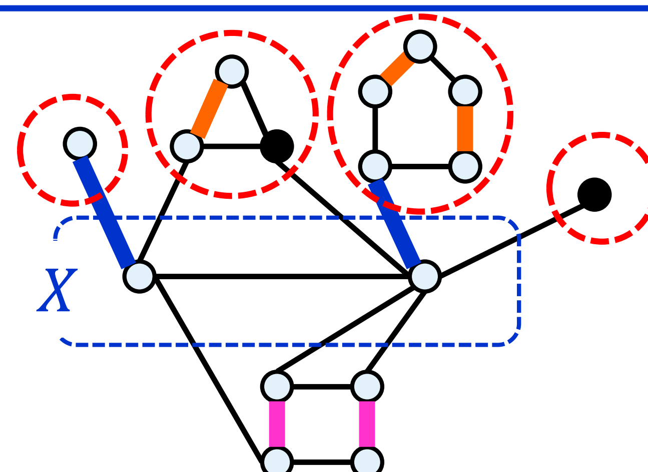
$$=\min\{\, t|X| + |E[C_{V-X}]| + \sum_{U\in\mathcal{U}(V-X)} \left\lfloor \frac{t|U|-1}{2} \right\rfloor \,\}$$



Theorem [Tutte '47, Berge '58]

$\max\{|M| : M \text{ is a matching}\}$

$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(X) : X \subseteq V\}$



News to Cycle Covers

Theorem [Karp, Ravi ’17][van Zuylen ’18+] etc.

A **cubic bipartite graph** has a **square-free 2-factor (cycle cover excluding C_4)**

Theorem [T. 17]

In a ***d*-regular bipartite graph** ($d \geq 4$),
a **2-factor (cycle cover)** excluding C_4 and
C_6 with ≥ 2 chords exists and can be found in $O(n^2 m)$ time

- **First positive result for $C_{\leq 6}$-free 2-matching**

Corollary

In a ***d*-regular bipartite graph** ($d \geq 4$), if $^\forall C_6$ has ≥ 2 chords,
a **2-factor (cycle cover)** with $\leq$ ***n*/8 cycles** exists and
can be found in $O(n^2 m)$ time

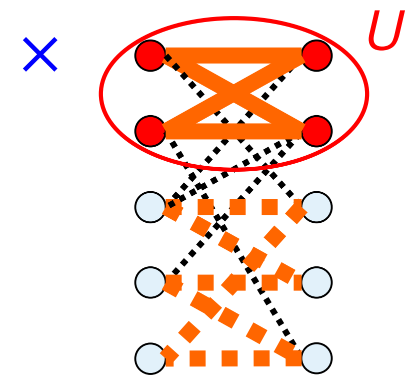
Contents

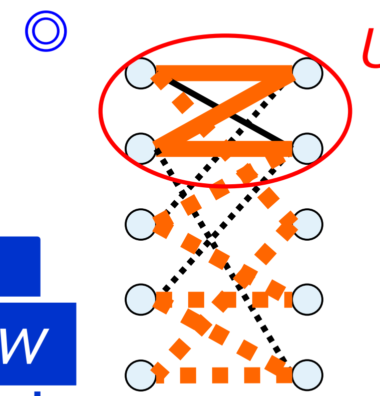
LP for Square-free 2-matching

Max weight square-free 2-matching

Maximize	$\Sigma_{e\in E}\, w(e)\, x(e)$	
subject to	$\Sigma_{e\in \delta v}\, x(e) \le 2$	$(v\in V)$
	$\boldsymbol{\Sigma_{e\in E[U]}\, x(e) \le 3}$	$(U\subseteq V,\ \lvert U\rvert=4)$
	$\boldsymbol{0 \le x(e) \le 1}$	$(e\in$

$$\left\lfloor \frac{t|U|-1}{2} \right\rfloor$$



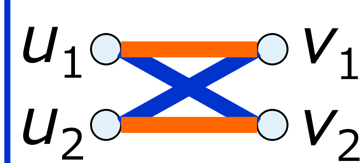


Theorem [Makai '07, T. '09]

Assumption on w

- G: Bipartite
- w is **vertex-induced on $^\forall$square U**
  i.e., $w(u_1v_1)+w(u_2v_2) = w(u_1v_2)+w(u_2v_1)$

➔ ***This LP** has an **integral opt solution***
The dual LP** has an **integral opt solution



Our Result: LP for $\mathcal{U}$-feasible t-matching

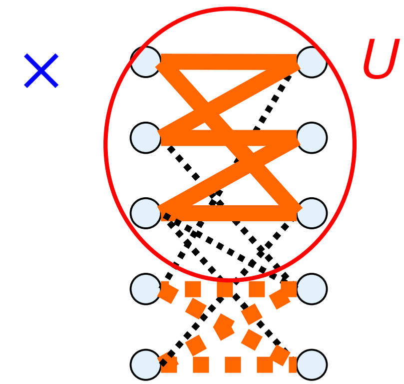
Max weight $\mathcal{U}$-feasible t-matching

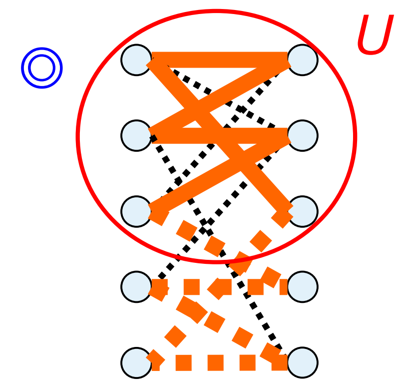
Maximize $\quad \sum_{e \in E} w(e)\, x(e)$

subject to $\quad \sum_{e \in \delta v} x(e) \le t \qquad (v \in V)$

$$\sum_{e \in E[U]} x(e) \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor \qquad (U \in \mathcal{U})$$

$$x(e) \ge 0 \qquad (e \in E)$$





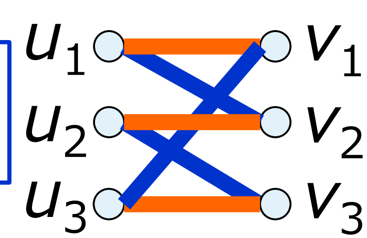
Theorem

Proved by our Primal-Dual Algorithm

- G: Bipartite
- $(G, \mathcal{U}, t)$ is factor-critical
- w is **vertex-induced on $\forall U \in \mathcal{U}$**
  i.e., in $G[U]$, the weights of perfect matchings are identical

Can be weakened [Thanks, Michel !]

➔ ***This LP** has an **integral opt solution***
The dual LP** has an **integral opt solution



Our Result: LP for $\mathcal{U}$-feasible t-matching

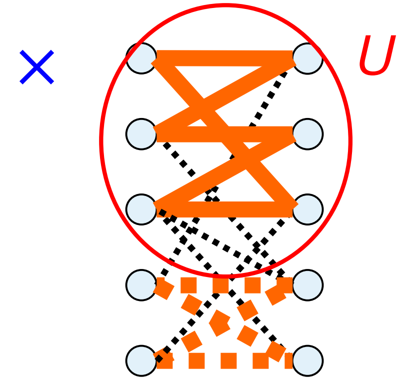
Max weight $\mathcal{U}$-feasible t-matching

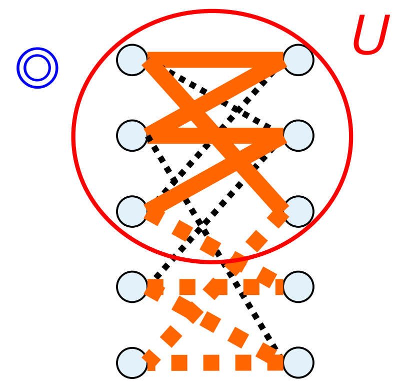
Maximize $\quad \sum_{e\in E} w(e)\, x(e)$

subject to $\quad \sum_{e\in\delta v} x(e) \le t \qquad (v\in V)$

$$\boldsymbol{\sum_{e\in E[U]} x(e)} \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor \qquad (U\in\mathcal{U})$$

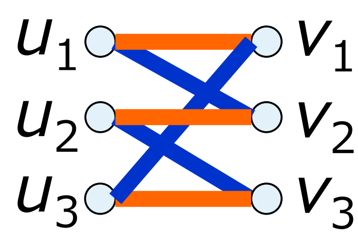
$$\boldsymbol{x(e) \ge 0} \qquad (e\in E)$$





Special cases

- **Subtour Elimination**
  - $t=2 \rightarrow \left\lfloor \frac{t|U|-1}{2} \right\rfloor = |U| - 1$
- **Blossom Constraint**
  - $t=1,\ |U|=2\cdot(\text{odd}) \rightarrow \left\lfloor \frac{t|U|-1}{2} \right\rfloor = \left\lfloor \frac{|U|-1}{2} \right\rfloor$
- **Arborescence Polytope**



Branching (Arborescence) Polytope

Max weight arborecence

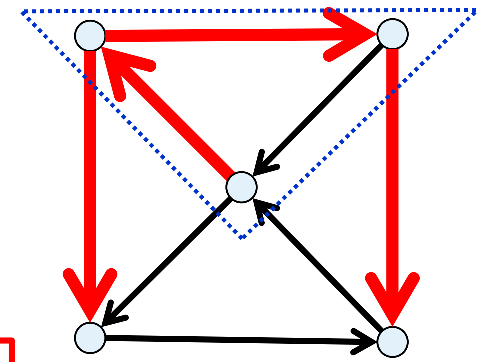
Maximimize $\quad \sum_{a\in A} w(a)\, x(a)$

subject to $\quad \sum_{a\in\delta^-(v)} x(a) \le 1 \qquad (v\in V)$

$\qquad\qquad \boldsymbol{\sum_{a\in A[U]} x(a) \le |U| - 1} \qquad (U\subseteq V)$

$\qquad\qquad \boldsymbol{x(a)\ge 0} \qquad (a\in A)$

$\left\lfloor \frac{|A(C)\cup V(C)|-1}{2} \right\rfloor$ (C: cycle)

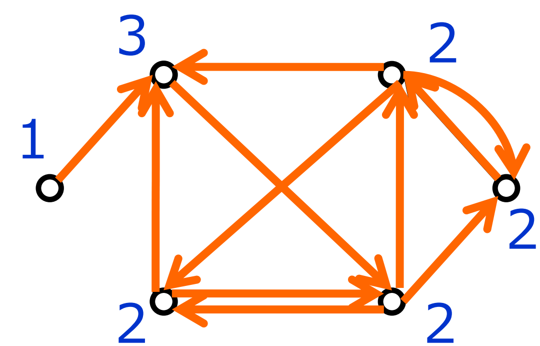


Theorem [Edmonds '70]

*The above linear system is **TDI***

b-branching [Kakimura, Kamiyama, T '18]

- Degree bound 1 ➔ $b(v)$
- Graphic matroid $|U|$ - 1
  ➔ Sparsity matroid $b(U)$ - 1



Subtour Elimination for TSP

IP for TSP [Dantzig, Fulkerson, Johnson '54]

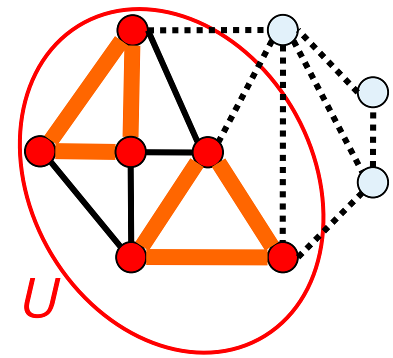
Minimize $\quad \sum_{e\in E} w(e)\, x(e)$

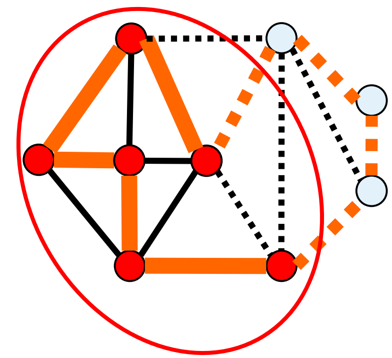
subject to $\quad \sum_{e\in \delta v} x(e) = 2 \qquad (v\in V)$

$\boldsymbol{\sum_{e\in E[U]} x(e)} \le \boldsymbol{|U| - 1} \qquad (U\subseteq V)$

$\boldsymbol{x(e)\in\{0,1\}} \qquad (e\in E)$

$\left\lfloor \frac{2|U|-1}{2} \right\rfloor$





Conjecture [Goemans '95] etc.

w is metric ➔ **Integrality gap** $\le \frac{4}{3}$

i.e., OPT(IP) $\le \frac{4}{3}$ OPT(LP)

Blossom Const for Nonbipartite Matching

Max. weight matching

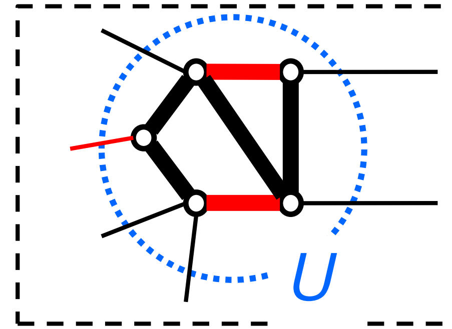
Maximize $\quad \Sigma_{e\in E}\, w(e)\, x(e)$

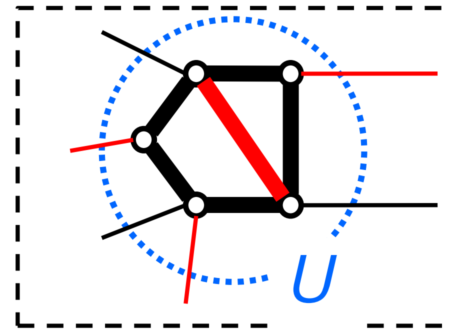
subject to $\quad \Sigma_{e\in \delta v}\, x(e) \leq 1 \qquad (v \in V)$

$$\boldsymbol{\Sigma_{e\in E[U]}\, x(e)} \leq \frac{|U|-1}{2} \qquad (U \subseteq V,\ |U| \text{ is odd})$$

$x(e) \geq 0 \qquad E)$

$$\left\lfloor \frac{|U|-1}{2} \right\rfloor$$





Theorem [Cunningham, Marsh '78]

- *The above linear system is **TDI***

Triangle-free Const for 2-matching

Max weight triangle-free 2-matching

Maximimize $\quad \sum_{e\in E} w(e)\, x(e)$

subject to $\quad \sum_{e\in\delta v} x(e) \le 2 \qquad (v\in V)$

$\qquad \boldsymbol{\sum_{e\in E[U]} x(e)} \le \mathbf{2} \qquad (U\subseteq V,\ |U|=3)$

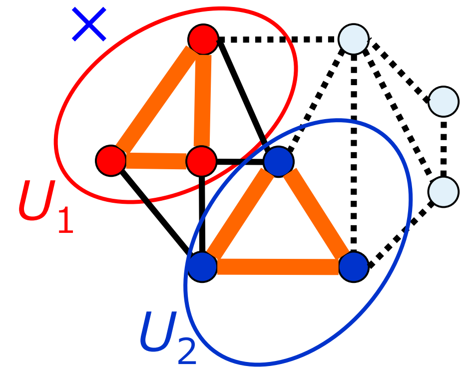
$\qquad \boldsymbol{x(e)\ge 0} \qquad (e\in E)$

$\left\lfloor \dfrac{2|U|-1}{2} \right\rfloor$

Theorem [Cornuéjols & Pulleyblank '80]

*This LP has an **integer optimal solution***

×



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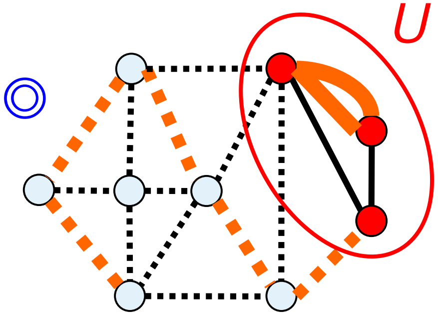
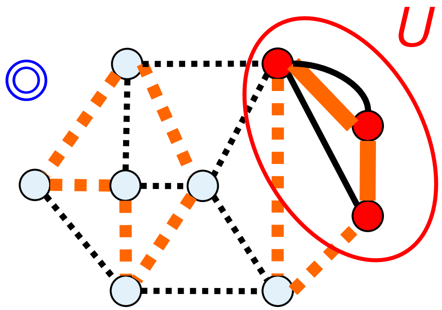


◎



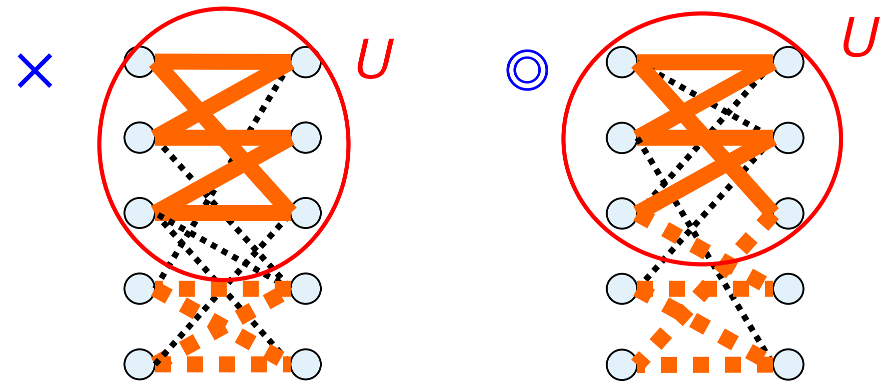
Contents

Summary

Our Framework

- ➢ **$\mathcal{U}$-feasible t-matching:**

$$|F[U]| \leq \left\lfloor \frac{t|U|-1}{2} \right\rfloor \quad {}^{\forall}U \in \mathcal{U}$$



Special Cases

- ➢ **Nonbipartite matching**
- ➢ **Triangle-free 2-matching** with edge multiplicity
- ➢ **Even factor**
- ➢ **Arborescence**
- ➢ **Square-free 2-matching**
- ➢ **$K_{t,t}$-free t-matching**
- ➢ 2-matchings covering edge cuts
- ➢ Hamilton cycles

Solved:

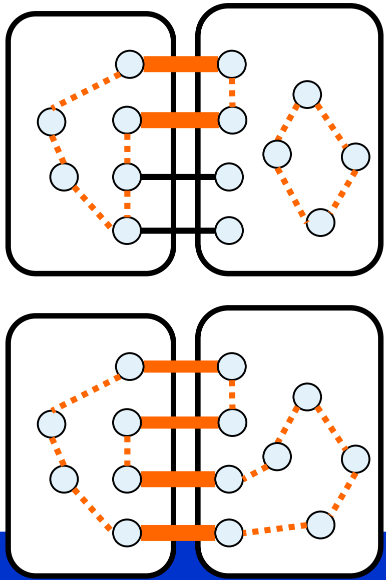
- G: Bipartite
- $(G, \mathcal{U}, t)$ is factor-critical
- w is vertex-induced on ${}^{\forall}U \in \mathcal{U}$

- ➢ ***Min-max theorem***
- ➢ ***LP with dual integrality***
- ➢ ***Combinatorial algorithm***

END of slides

Supplements below

2-factor Covering Edge Cuts



Definition (A–covering 2-factor)

2-factor F is ***A*-covering** ($A \subseteq \mathbb{Z}$)

$\overset{\text{def}}{\Leftrightarrow}$ F intersects every k-edge cut $^{\forall}k \in A$

- **Hamilton cycle** = $\mathbb{Z}$-covering 2-factor

Previous work

- 2-edge connected cubic graph:
  - **{3,4}-covering 2-factor** exists [Kaiser, Škrekovski '08], and can be found in $O(n^3)$ time [Boyd, Iwata, **T.** '13]
  - Min-weight **{3}-covering 2-factor** in $O(n^3)$ time [BI**T.** '13]
- Graphs w/o **{4,5}-covering 2-factor** [Čada, Chiba, Ozeki, Vrána, Yoshimoto '13]

- **Application**: Approximation of min. 2-edge connected subgraph [BI**T.** '13]