Random Spanning Trees Effective Resistances & TSP

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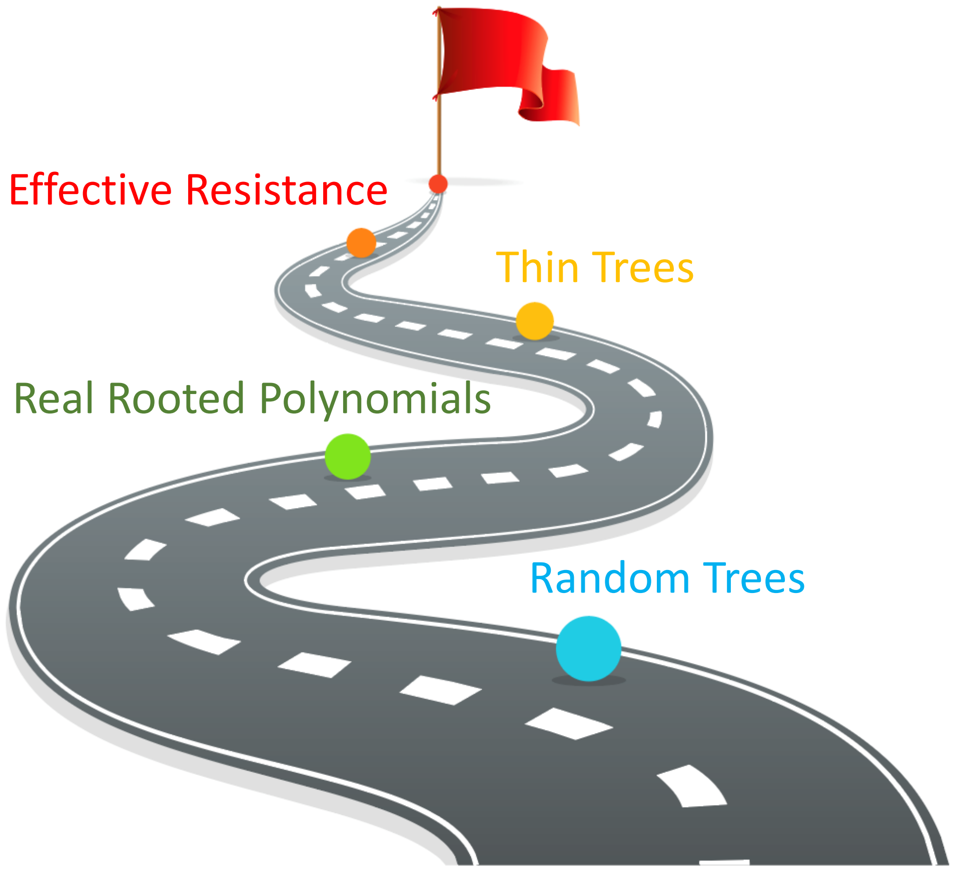
Outline

- TSP & Random Spanning Trees
  - Capacity functions, Max entropy, …
  - Degree distribution of vertices
- ATSP & Random Spanning Trees
  - Thin Trees
  - Spectrally Thin Trees & Effective Resistance
  - Properties of Effective Resistance



TSP Recap

TSP: Choose a connected, Eulerian subgraph of minimum cost.

Approach 1: Choose an Eulerian subgraph then make it connected

How?

Approach 2: Choose a connected subgraph, then make it Eulerian

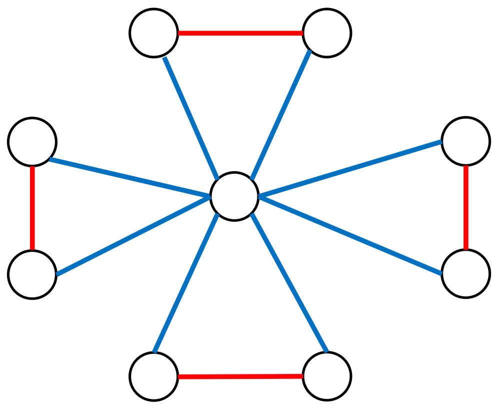
Spanning Trees have beautiful math

Polytime solvable

Christofides Approach

Consider the unit metric space: $c(u, v) = 1$ for all u,v.

Choose a MST & add
min-cost-matching
on odd degree vertices

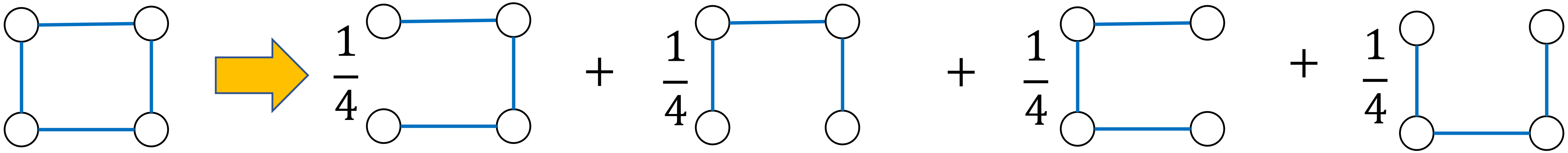


The cost of the matching is n/2 => 3/2 approximation
But, wait, does a typical tree looks like a star?
How many odd degree vertices does it have?

Random Spanning Tree Distributions

Given a graph G=(V,E) with m edges.

Let μ be the uniform distribution over all spanning trees of G.



We can write μ as polynomial,

$$g_\mu(z_{e_1}, \dots, z_{e_m}) = \sum_T \prod_{e \in T} z_e$$

- For any e,

$$z_e \partial_e \log g_\mu(1, \dots, 1) = \frac{z_e \partial_e g_\mu}{g_\mu}(\mathbf{1}) = \frac{\#\{T: e \in T\}}{\#T} = \mathbb{P}_{T\sim\mu}[e \in T]$$

unif dist

- We can evaluate g_μ at any $z \in \mathbb{R}^m$.

A Linear Algebraic View

For a pair of vertices u,v, let $b_{u,v} = 1_u - 1_v$

$$b_{u,v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \\ \leftarrow u \\ \\ \\ \leftarrow v \\ \\ \\ \\ \end{matrix}$$

For an edge $e = (u, v)$ let $L_{u,v} = b_{u,v} b_{u,v}^T$

Also, for $S \subseteq E$, let $L_S = \sum_{e \in S} L_e$

It follows that $\det\left(\cancel{\frac{\mathbf{1}\mathbf{1}^T}{n}} + L_T\right) = 1$ for all trees T.

Write the (n-1)-dimensional space orthogonal to **1** in $\mathbb{R}^{n-1}$

Then,

Cauchy-Binet identity

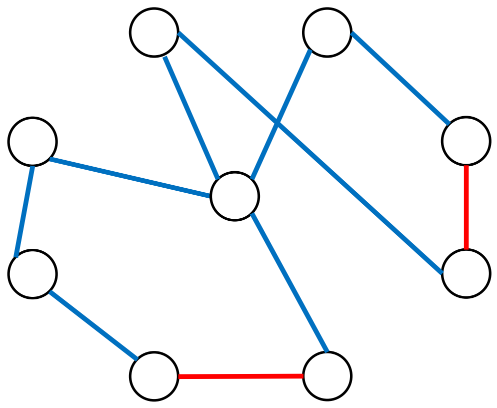
$$g_\mu(z) = \sum_T \det(L_T) \prod_{e \in T} z_e = \det\left(\sum_e z_e L_e\right)$$

So, g_μ is computable and log-concave & we can generate random trees.

Back to TSP on Unit Metric

Choose a uniformly random spanning (cost n)

Add min-cost matching.



What is $\mathbb{E}$[#even-deg / #deg-2]?

- $\exists$ A bijection between spanning trees of K_n & seq of n-2 numbers in $\{1, \ldots, n\}$
- For any v, $\mathbb{P}[d_T(v) = 2] = \frac{n(n-1)^{n-3}}{n^{n-2}} = \left(1 - \frac{1}{n}\right)^{n-3} \approx \frac{1}{e}$.

So, $\mathbb{E}[c(\text{matching})] \leq \frac{n}{2}\left(1 - \frac{1}{e}\right) < \frac{4n}{3}$.

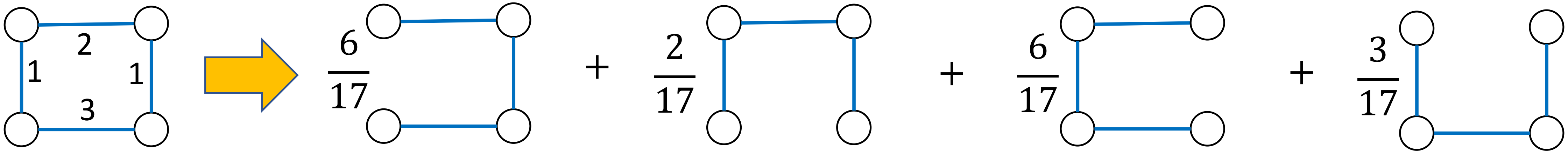
Can we extend this approach to any metric?

1. How to bound c(T)?
2. How to bound $\mathbb{P}[d_T(v) = 2]$?
3. How to bound $\mathbb{E}[c(\text{matching})]$?

First Challenge: Choosing the Right Distribution

A uniformly random spanning tree can have a cost >> OPT-TSP

Let’s allow weighted-unif distributions:



Let x be OPT of LP.

$$
\begin{aligned}
\min \quad & \sum_e x_e c(e) \\
\text{s.t.,} \quad & x(\delta(S)) \geq 2 \quad \forall S \subseteq V \\
& x(\delta(v)) = 2 \quad \forall v \in V
\end{aligned}
$$

Choose a weights s.t., for all e: $\mathbb{P}_{T\sim\mu}[e \in T] \approx x_e \left(1 - \frac{1}{\mathrm{n}}\right) =: y_e$

Because, then $\mathbb{E}[c(T)] \approx \sum_e x_e c(e) \leq \mathrm{OPT}.$

x OPT of LP

$y = \left(1 - \frac{1}{n}\right) x.$

Idea: Choose $z > 0$ s.t., $g_\mu(z)$ gives the right marginals.

i.e., choose z s.t.,

$$\forall e: \quad z_e \partial_e \log g_T = y_e \quad \Leftrightarrow \quad \forall e: \; \partial_e \log g_\mu = \frac{y_e}{z_e}$$

Reverse engineering, z must be

$$\inf_{z>0} \log \frac{g_{\mu_T}(z)}{\prod_e z_e^{y_e}} = \inf_z \log g_\mu(z) - \sum_e y_e \log z_e$$

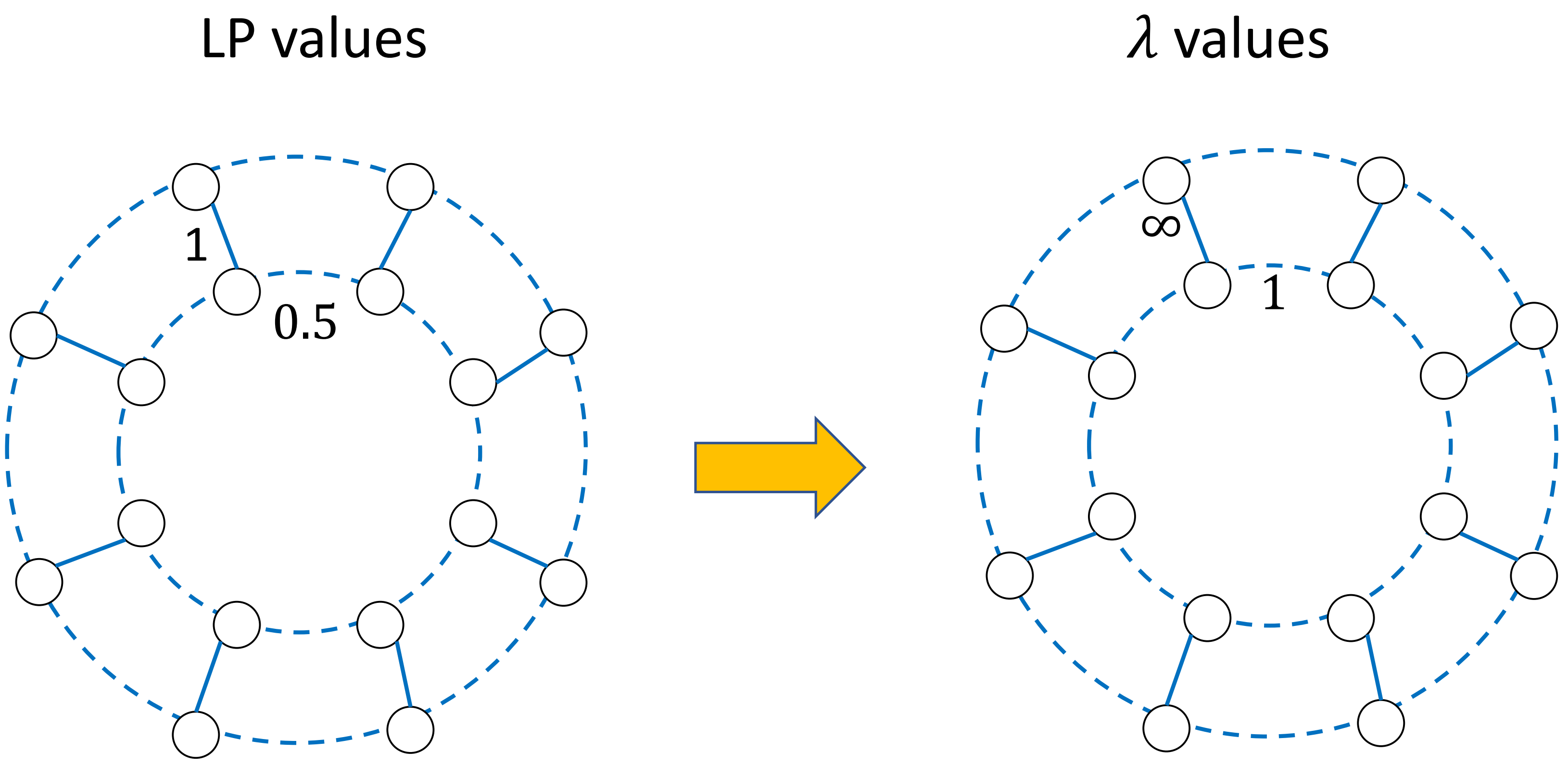
- Dual of max-entropy CP
- A.k.a., Gurvits' capacity fn
- Works for any set system

If y is not in the spanning tree polytope, then OPT=$-\infty$.

Otherwise, let $\lambda = z$ and let $g_{\lambda * \mu}(z_{e_1}, \dots, z_{e_m}) = g_\mu(\lambda_{e_1} z_{e_1}, \dots, \lambda_{e_m} z_{e_m})$.

$$\mathbb{P}_{T \sim \lambda * \mu}(e \in T) = y_e \Rightarrow \mathbb{E}_{T \sim \lambda * \mu}[c(T)] = \mathrm{OPT}.$$

Example



Second Challenge: Degree Distributions

Let $S \subseteq E$. We claim, for $T \sim \lambda * \mu$, $|T \cap S|$ is very strongly concentrated.

Fact 1: $\sum_k \mathbb{P}[|T \cap S| = k]t^k$ is a real rooted polynomial (RR)

Fact 2: If $\sum_k a_k t^k$ is RR $\exists$ Bernoullies $B_1, \dots, B_n$ s.t.,

$$\forall k\colon\ \mathbb{P}[B_1 + \cdots + B_n = k] \propto a_k$$

Fact 3: If $\sum_i a_i t^i$ is RR, then $a_1, a_2, \dots, a_n$ is a log-concave seq, i.e.,

$$\forall i\colon a_k^2 \geq a_{k-1} \cdot a_{k+1}$$

[Anari-Liu-O-Vinzant'18]: Log-concavity holds for any matroid

Fact 1: $\sum_k \mathbb{P}[|T \cap S| = k]t^k$ is RR

Recall $g_{\lambda * \mu}(z) = \det(\sum_e \lambda_e z_e L_e)$. So,

$A \succcurlyeq 0$ $B \succcurlyeq 0$

$$\sum_k \mathbb{P}[|T \cap S| = k]t^k \propto \det\left(t \sum_{e \in S} \lambda_e L_e + \sum_{e \notin S} \lambda_e L_e \right)$$

We can rewrite

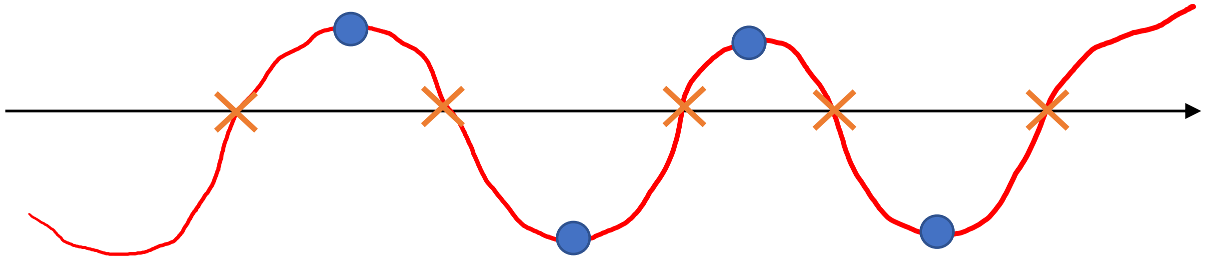
$$\det(tA + B) = \det(\mathrm{A})\det(tI + A^{-1/2} B A^{-1/2})$$

But this is the characteristic polynomial of $A^{-1/2} B A^{-1/2}$

This is a symmetric matrix so it has real eigenvalues and the roots of characteristic polynomial is real

Fact 3: $\sum_{k=0}^{n} a_k t^k$ is RR=> $a_k^2 \geq a_{k-1} \cdot a_{k+1}$

Cl 1: If $f(t)$ is RR then, so is $f'(t)$



Cl 2: If $f(t)$ is RR, then so is $t^n f\left(\frac{1}{t}\right)$

If a is a root of $f\left(\frac{1}{t}\right)$, then $\frac{1}{a}$ is a root of f.

Let $f(t) = \sum_k a_k t^k$

- $g(t) = f^{(k-1)}(t)$ is RR
- $h(t) = t^{n-k+1} g\left(\frac{1}{t}\right)$ is RR
- $\ell(t) = h^{(n-k-1)}(t)$ is RR

But, $\ell(t) \propto \frac{a_{k-1}}{\binom{n}{k-1}} t^2 + \frac{2a_k}{\binom{n}{k}} t + \frac{a_{k+1}}{\binom{n}{k+1}}$, so,

$$\frac{a_k^2}{\binom{n}{k}} \geq \frac{a_{k-1}^2}{\binom{n}{k-1}} \cdot \frac{a_{k+1}^2}{\binom{n}{k+1}}$$

Parity of Degree Cuts

Lem: $\mathbb{P}_{T\sim\lambda*\mu}[d_T(v) = 2] \geq \frac{1}{e}$.

- $\mathbb{E}[d_T(v)] = \sum_{e\sim v} x_e \approx 2$.
- By log-concavity

$$\mathbb{P}[d_T(v) = 2]^2 \geq \mathbb{P}[d_T(v) = 1] \cdot \mathbb{P}[d_T(v) = 3].$$

This refutes the case where $\mathbb{P}[d_T(v) = 1] = \mathbb{P}[d_T(v) = 3] = \frac{1}{2}$.

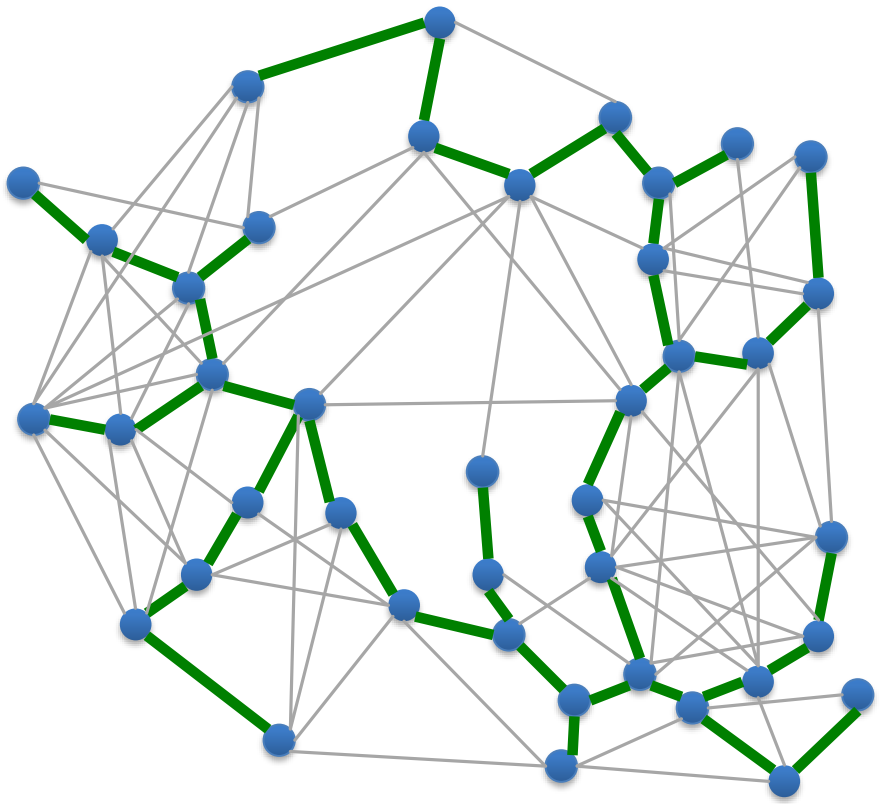
More generally, log concavity implies sub-Gaussian tail

Are we done??

Third Challenge: Bounding Cost of Matching

Let $T \sim \lambda * \mu$. What is the expected cost matching on odd degree vertices of T?

T locally looks like a Hamiltonian path with a constant probability.



Thm: O-Saberi-Singh'11

~~Conj~~: For every graphic metric, the expected cost of M for $T \sim \lambda * \mu$ is at most $OPT\left(\frac{1}{2} - \epsilon\right)$ when x is not near integral.

Summary

Let x be the LP solution

- Set

$$\lambda = \operatorname*{argmin}_z \log \frac{g_\mu(z)}{\prod_e z_e^{x_e(1-1/n)}}$$

Gives, $\mathbb{P}_{T\sim\lambda*\mu}[e] \approx x_e$.

- Then, for any $S \subseteq E$: $|S \cap T|$ is log-concave, unimodal, concentrated, … for $T \sim \lambda * \mu$

- Local Hamiltonian properties => $\mathbb{E}[c(\text{matching})] \leq c(x)\left(\frac{1}{2} - \epsilon\right)$

  if we have cut metric and x is not near integral

- Is $\mathbb{E}_{T\sim\lambda*\mu}[c(\text{matching})] < \left(\frac{1}{2} - \epsilon\right)\text{OPT}$?

- How about half-integral instances, better than 3/2?

- A reduction from general TSP to half-integral or 1/C-integral for some $C = O(1)$?

- TSP for matroids?

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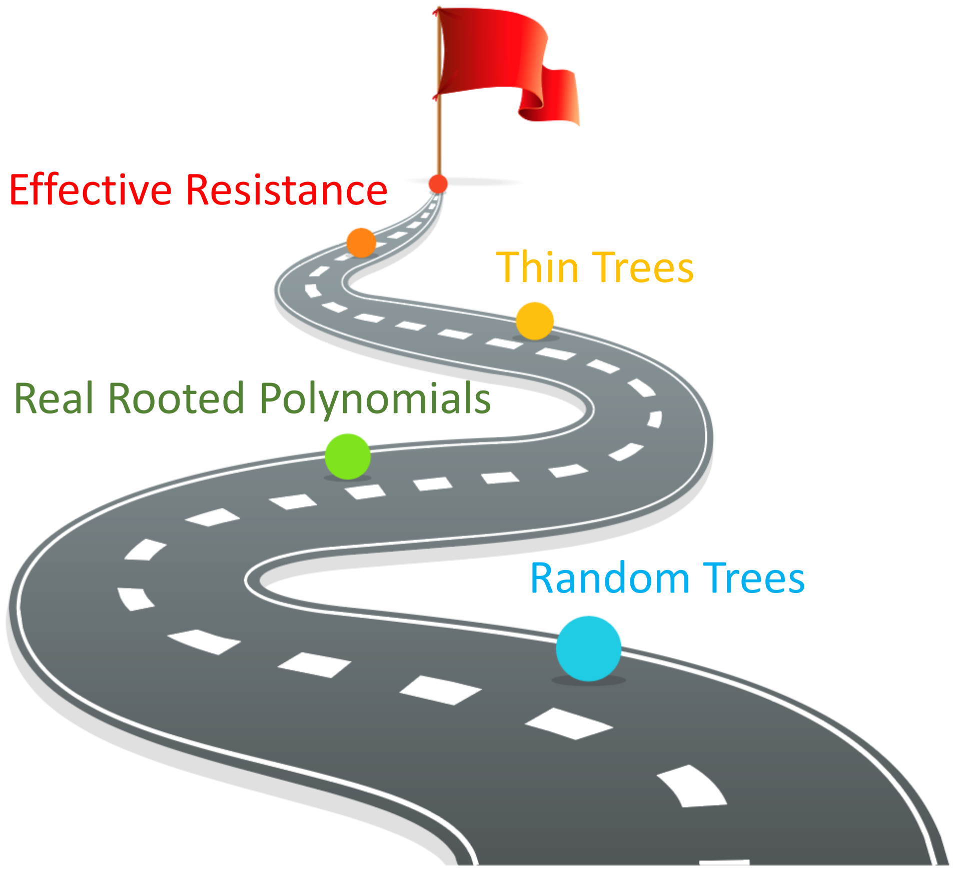


ATSP

Find a connected, Eulerian subgraph of a directed graph with minimum cost.

Follow a similar approach:

Choose a random spanning tree then make it Eulerian

min cost flow problem

What is $\mathbb{E}[c(\text{flow})]$?

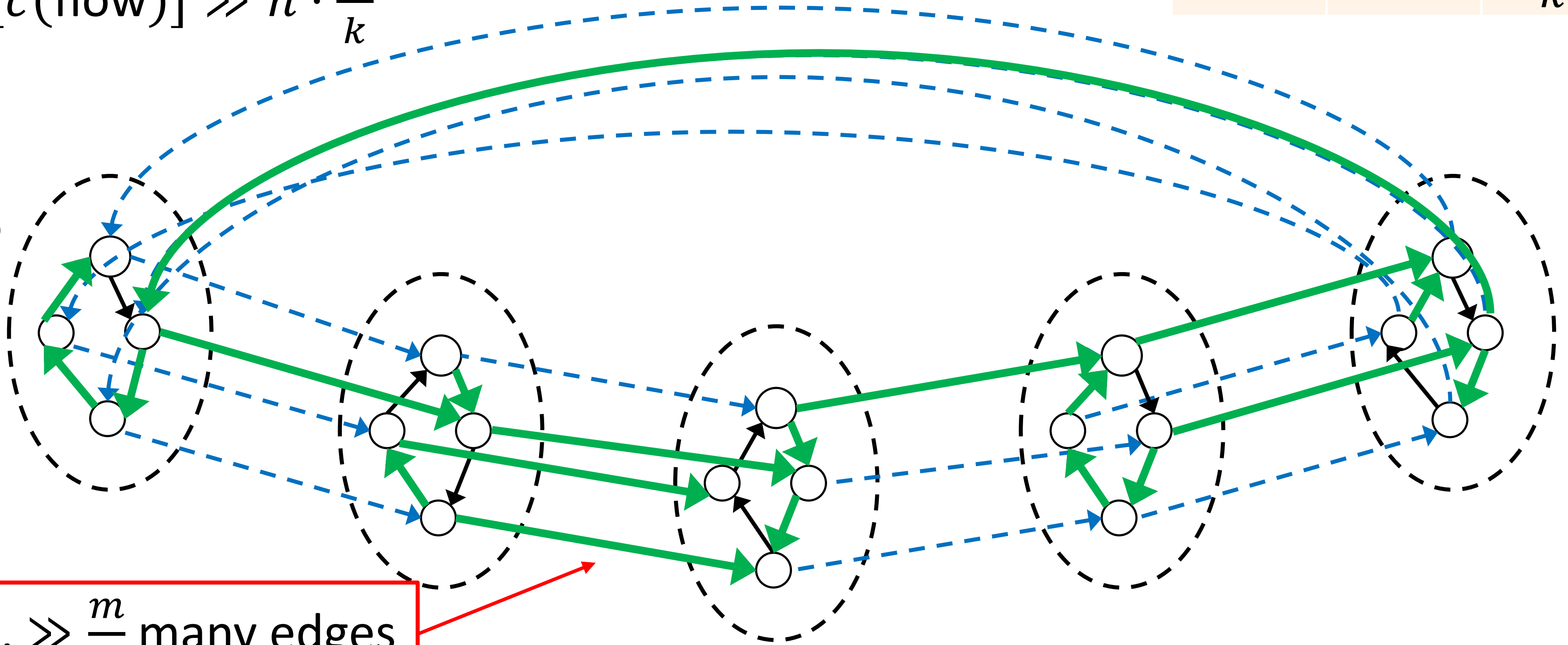
Example: Local mistakes => Blow-up costs

	c(e)	x_e
→	0	$1-\frac{1}{k}$
⇢	1	$\frac{1}{k}$

Set $k = \frac{\log m}{m}$,
n blocks for $n \gg m$

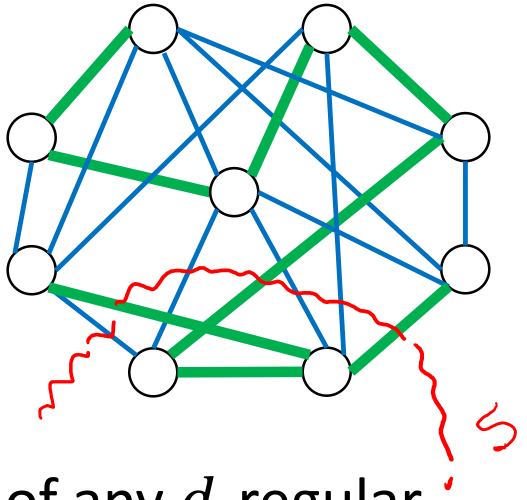
So, $\mathbb{E}[c(\text{flow})] \gg n \cdot \frac{m}{k}$

m-vertices
In each blob



w.h.p. $\gg \frac{m}{k}$ many edges

Thin Trees



Given $G = (V, E)$, a Tree T is α-thin w.r.t. G if for all $S \subseteq V$,

$$|T(S, \bar{S})| \leq \alpha |\delta(S)|$$

What is the thinness of this tree?

Example: Any bounded degree spanning tree is an $O\left(\frac{1}{d}\right)$-thin tree of any d-regular expander graph.

Finding $\frac{f(n)}{k}$-thin tree In k-edge conn graph

→

O(f(n))-approximation for ATSP

In Pursuit of Thin Trees

Thm [Asadpour-Goemans-Madry-O-Saberi'10]: Given a k-edge-connected graph, choose λ s.t., for all e, $\mathbb{P}_{T\sim\lambda*\mu}[e] \leq \frac{2}{k}$. Then,

$$\mathbb{P}_{T\sim\lambda*\mu}\left[T \text{ is } O\left(\frac{\log n}{k\cdot\log\log n}\right)\text{–thin}\right] \geq 1 - O\left(\frac{1}{n}\right)$$

Pf Idea: For every $S \subseteq V$: $|T \cap \delta(S)|$ is strongly concentrated around $\frac{2|\delta(S)|}{k}$ by the log-concavity property.

The proof follows by a careful Chernoff bound union bound.

Revised Plan

Prove that for a small α

$$\mathbb{P}_{T\sim\mu}[T \text{ is } \alpha\text{-thin}] > 0$$

Use probabilistic method? Lovasz Local lemma?

- Exponentially many bad events, i.e., cuts
- A lot of interactions between cuts

Use spectral approach?

$$\mathbb{P}_{T\sim\mu}[T \text{ is } \alpha\text{-spectrally-thin}] > 0$$

Spectrally thin trees

For a graph G, T is α-spectrally thin if

- For all $x \in \mathbb{R}^V$, $x^T L_T x \leq \alpha x^T L_G x$
- $L_T \preccurlyeq \alpha L_G$
- $L_G^{-1/2} L_T L_G^{-1/2} \preccurlyeq \alpha I$

Implies α-thinness letting $x = \mathbf{1}_S$

Easy to check in polytime

Exercise: Find an $O\left(\frac{1}{\log n}\right)$-spectrally-thin-tree in $\log n$-dim hypercube

Spectrally thin trees & Effective Resistance

Suppose T is α-spectrally thin, i.e., $L_T \preccurlyeq \alpha L_G$.

Then, for every $e \in T$ we must have: $L_e \preccurlyeq L_T \preccurlyeq \alpha L_G$

Or,

Rank 1

$$L_G^{-1/2} L_e L_G^{-1/2} \preccurlyeq \alpha I$$

So,

$b_e b_e^T$

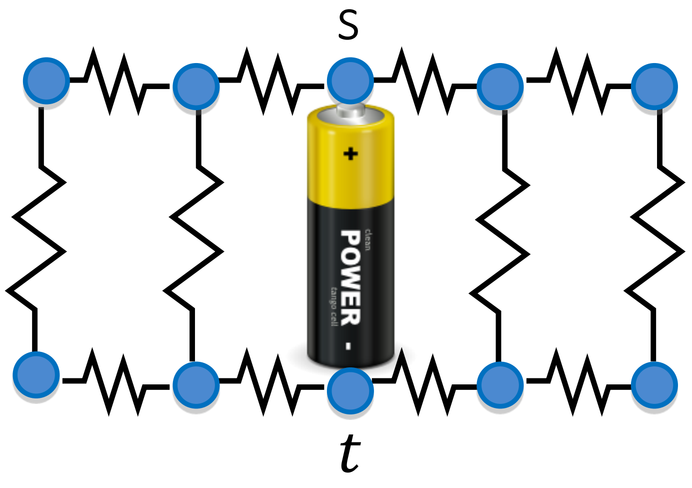
$$\mathrm{Tr}\left(L_G^{-1/2} L_e L_G^{-1/2}\right) \leq \alpha$$

Recall $L_e = b_e b_e^T$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$b_e^T L_G^{-1} b_e \leq \alpha$$

Reff(e) – Effective resistance of e

Properties of Effective Resistance



- $\mathrm{Reff}(e) = \mathbb{P}_{T\sim\mu}[e \in T]$

- Reff(s,t)=$\max_x \frac{(x_s - x_t)^2}{\sum_{u\sim v}(x_u - x_v)^2}$

- For all s, t there are at least $\frac{1}{\mathrm{Reff}(s,t)}$ many edge disjoint paths from s to t.

Claim: For any e, $\mathbb{P}_{T\sim\mu}[e \in T] = \mathrm{Reff}(e)$

$$\mathbb{P}_{T\sim\mu}[e \in T] = z_e \partial_e \log g_\mu(\mathbf{1}) = \frac{z_e \partial_e g_\mu}{g_\mu}(\mathbf{1})$$

$$= \frac{z_e \partial_e \det\left(\sum_f z_f L_f\right)}{\det(\sum_f z_f L_f)}(\mathbf{1})$$

$$= \frac{\partial_\mathrm{x} \det(L_G + x L_e)\,|_{x=0}}{\det(L_G)}$$

$$= \frac{\det(L_G)\,\partial_x \det\left(I + x L_G^{-1/2} L_e L_G^{-1/2}\right)|_{x=0}}{\det(L_G)}$$

$$= \mathrm{Tr}\left(L_G^{-1/2} L_e L_G^{-1/2}\right) = b_e L_G^{-1} b_e$$

Claim: For any pair of vertices s, t,

$$b_{s,t} L_G^{-1} b_{s,t} = \max_x \frac{(x_s - x_t)^2}{\sum_{u\sim v}(x_u - x_v)^2}$$

Pf: Since $L_G^{-1/2} L_{s,t} L_G^{-1/2}$ is rank 1,

$$b_{s,t}^T L_G^{-1} b_{s,t} = \max_y \frac{y^T L_G^{-1/2} L_{s,t} L_G^{-1/2} y}{y^T y}$$

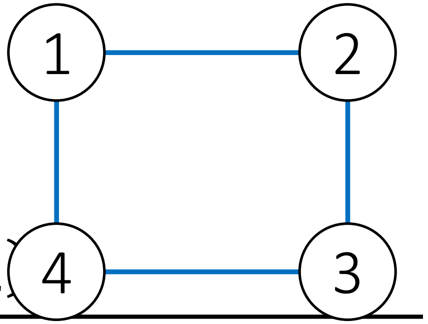
$$= \max_x \frac{x^T L_{s,t} x}{x^T L_G x}$$

$$= \max_x \frac{(x_s - x_t)^2}{\sum_{u\sim v}(x_u - x_v)^2}$$

Example

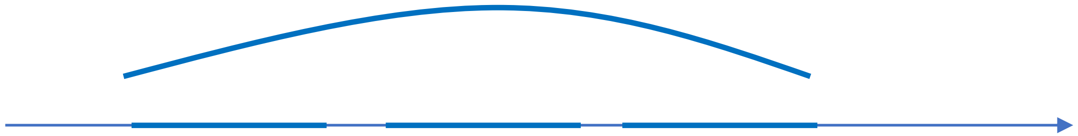
What is Reff(1,4)?

$$\mathbb{P}[(1,4) \in T] = \frac{3}{4}$$



$$\max_x \frac{(x_1 - x_4)^2}{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2} = \frac{3^2}{1^2 + 1^2 + 1^2 + 3^2} = \frac{3}{4}$$

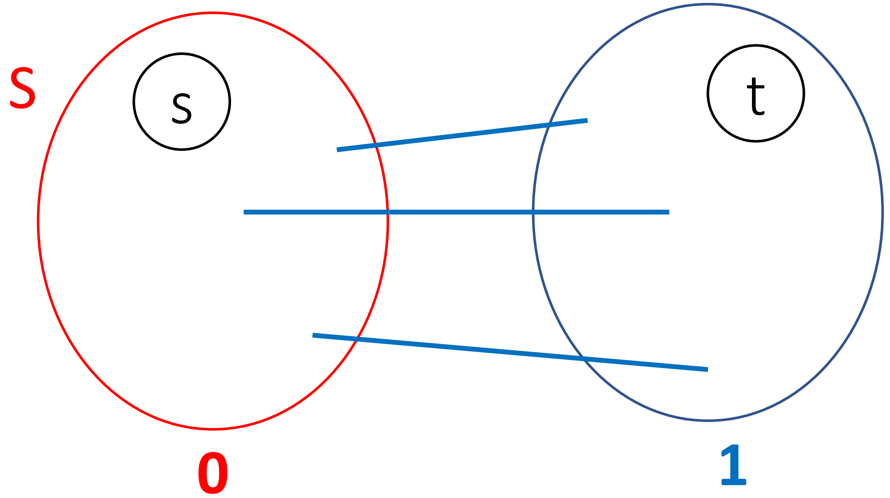
$+ (x_1 - x_4)^2$



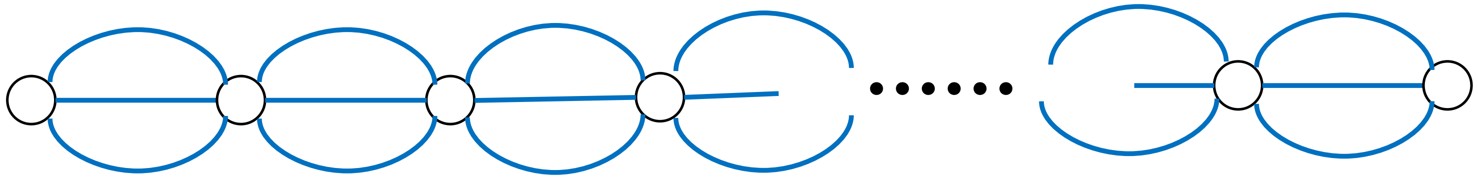
Claim: For all s, t there are at least $\frac{1}{\text{Reff}(s,t)}$ many edge disjoint paths from s to t.

Fix any cut $(S, \bar{S})$ where $s \in S$ and $t \notin S$. We show $|\delta(S)| \geq \frac{1}{\text{Reff}(s,t)}$

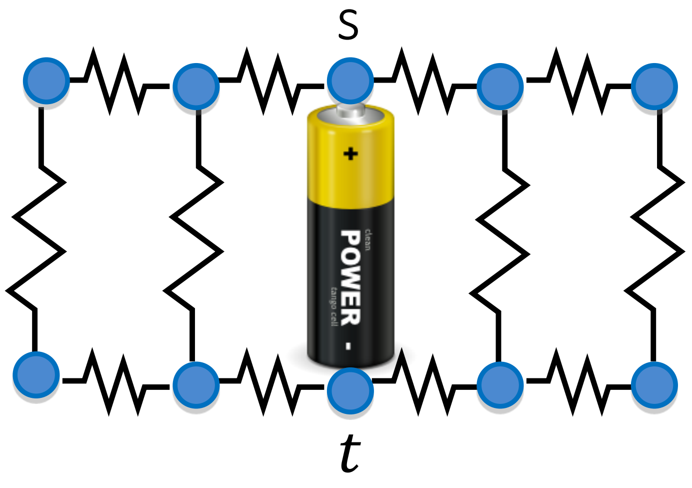
Then,
$$(x_s - x_t)^2 = 1$$
$$\sum_{u \sim v} (x_u - x_v)^2 = |\delta(S)|$$
$$\text{Reff}(s,t) \geq \frac{1}{|\delta(S)|}$$



The converse is false because the paths may be long



Properties of Effective Resistance



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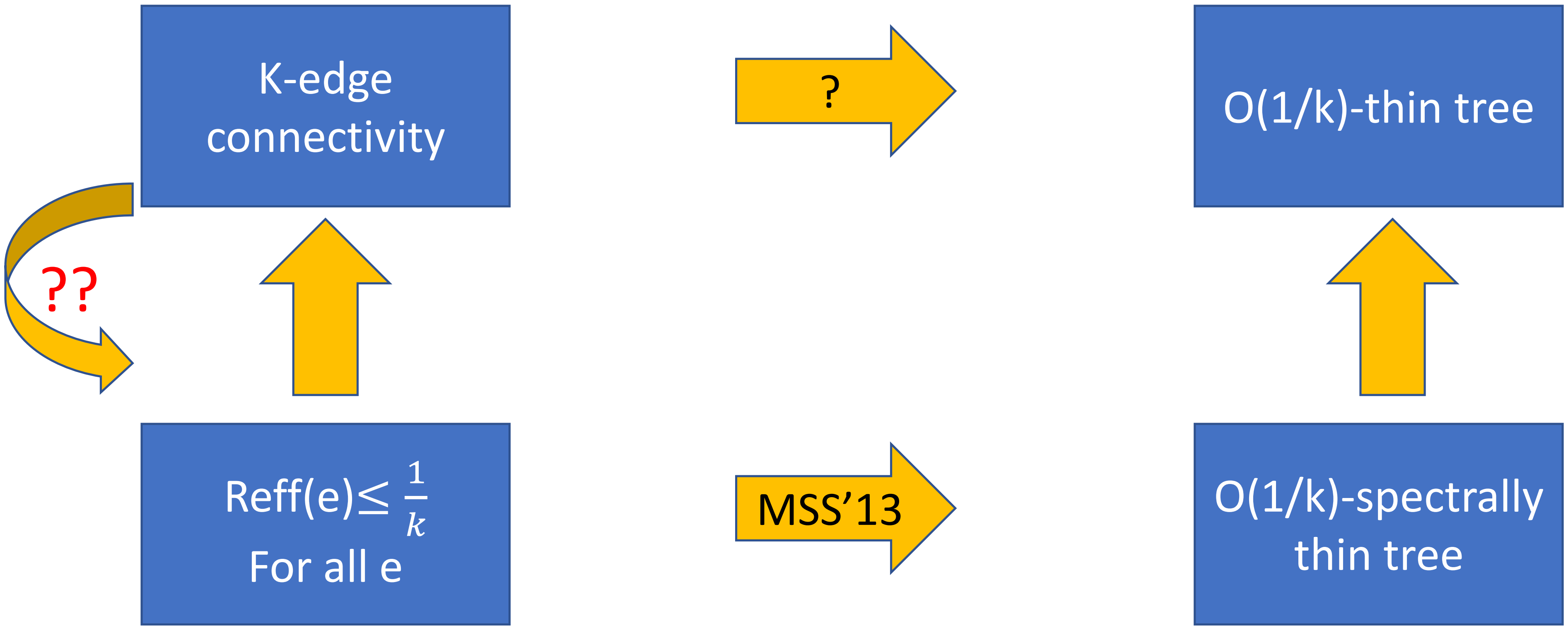
Summary / Plan



- O(1/k)-thin forest of linear size in k-edge-connected graphs

- Weak thin-tree conj:

There is k_0 s.t., every k_0-edge connected graph has a 0.99-thin tree

- Strong thin-tree conj:

There is $C > 0$ s.t., every k-edge connected graph has an $O\left(\frac{1}{k}\right)$-thin tree