Strongly Polynomial Algorithms for Some Parametric Global Minimum Cut Problems

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Outline

1. Global Min Cut
   - Non-Parametric
   - Parametric
   - The Parametric Problems

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4. Conclusion

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- Can compute a global min cut in $O(mn + n^2 \log n)$ deterministic time (Stoer-Wagner = SW, Nagamochi-Ibaraki = NI), or $\tilde{O}(n^2)$ randomized time (Karger-Stein = KS), or $\tilde{O}(m)$ randomized time (Karger = K).

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- There are only $O(n^{\lfloor 2\alpha \rfloor})$ α-approximate min cuts; when $\alpha < \frac{4}{3}$ they can all be computed in $O(n^4)$ deterministic time (NI), or $\tilde{O}(n^{\lfloor 2\alpha \rfloor}) = \tilde{O}(n^2)$ randomized time (KS).

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- A vitally important subproblem in separating TSP facets.

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  - It will turn out to further highlight how the small number of α-approximate solutions leads to more efficient algorithms.

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- We also want to see if one is harder than the other.

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- However, we'd still like to do better than generic Megiddo.

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Non-param GMC	SW $O(mn + n^2 \log n)$	K $\tilde{O}(m)$ (KS $\tilde{O}(n^2)$)
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Megiddo $d = 1$	SW $O(n^5 \log n)$	KS $O(n^2 \log^5 n)$
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Summary of running times so far.

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Big gap between non-parametric and computing all of $Z(\mu)$ running times, even for $d = 1$

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Much smaller gap between non-parametric and Megiddo running times (compare to $Z(\mu)$ times in blue); for $d = 1$, KS gap is just logs. Note that using Megiddo to solve P_{NB} is just general Megiddo with d set to 1.

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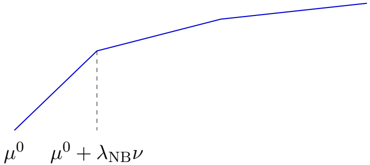
P_{NB} ($\sim d = 1$)	???	???
$P_{\max}$ ($\sim$ gen'l d)	???	???

Hoped-for results in this paper in red. Compare to non-param lower bounds in green, various upper bounds in blue.

Reducing P_{NB} to $P_{\max}$ with $d = 1$

- P_{NB} wants us to compute λ_{NB}:



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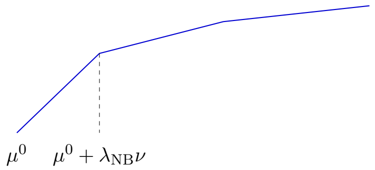

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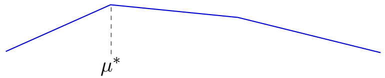

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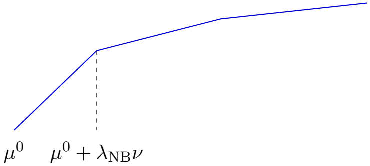

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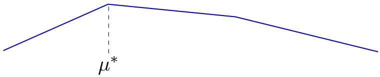

- Thus P_{NB} cannot be any harder than $P_{\max}$ for $d = 1$, though it could be easier.

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Using Stoer-Wagner to Solve P_{NB}

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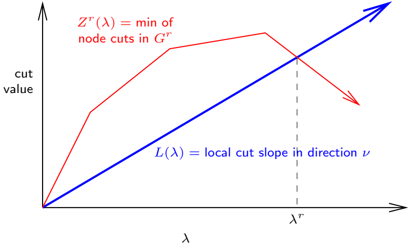

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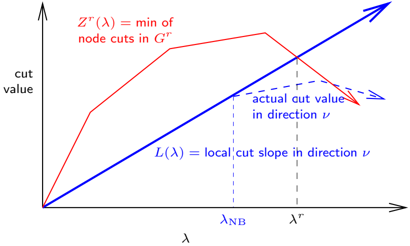

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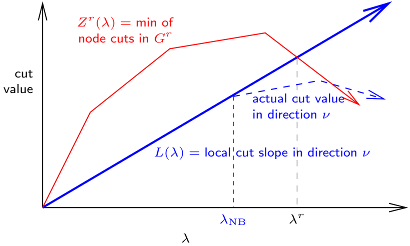

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P_{NB} ($\sim d = 1$)	SW $O(mn + n^2 \log n)$	???
$P_{\max}$ ($\sim$ gen'l d)	???	???

Here we saved a lot w.r.t. Megiddo, and matched the non-parametric lower bound.

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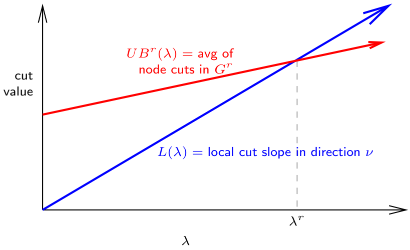

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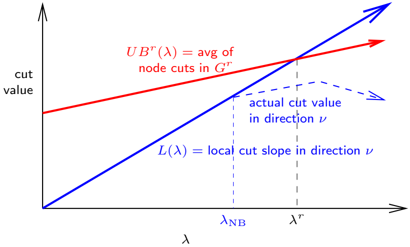

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- Thus using the KS framework is correct, and runs in same $\tilde{O}(n^2)$ time as KS.
- There is a minor technical point about how to implement the random edge contractions: Here the parametric costs interfere with the KS matrix update technique, but we can replace the static matrices with separate matrices for $\vec{c}^0$ and $\vec{c}^1$ to achieve the same effect.

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Here we saved only log factors w.r.t. Megiddo, but that's all the gap we had to work with; our ideas don't seem to extend to Karger's improvement.

Outline

1. Global Min Cut
   - Non-Parametric
   - Parametric
   - The Parametric Problems
2. Faster Algorithms for P_{NB}
   - Deterministic
   - Randomized
3. Faster Algorithms for $P_{\max}$
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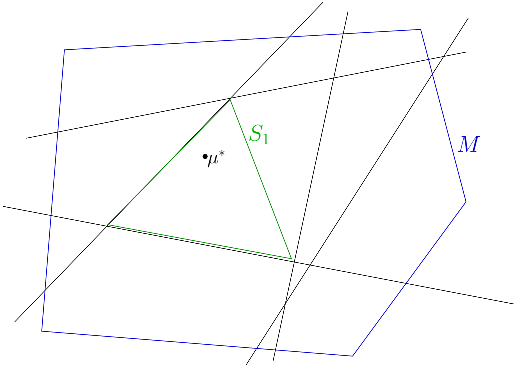
  - an unknown target (think μ^*).
- Then the task is to find a simplex in a cell of $\mathcal{H} \cap P$ containing μ^*.

Weak Duality between GMC and Max Spanning Tree

- Define $\mathcal{H}_1$ as the set of $O(m^2)$ hyperplanes where $c_\mu(e) = c_\mu(e')$ and run PLA for $(\mathcal{H}_1, M, \mu^*)$ to get simplex S_1.



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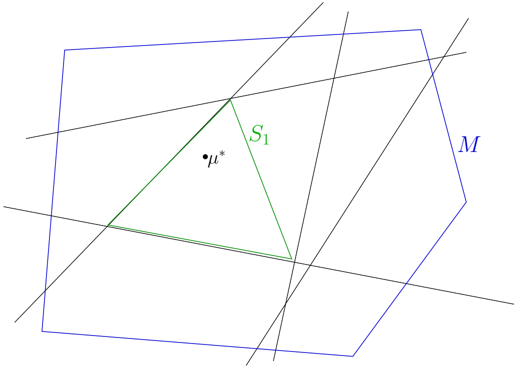

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- Now we need to use PLA a second time to further narrow in on μ^* so we can get the cuts inducing it via α-approximate cuts.

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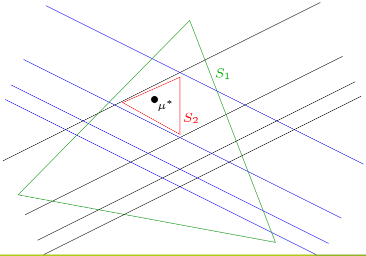
- Define $\mathcal{H}_2$ as the $O(m \log n)$ hyperplanes where $c_\mu(e) = g_i(\bar{e}, \mu)$, $\forall$ $e \in E$, $i = 1, \dots, p$, and set $S_2 = \mathrm{PLA}(\mathcal{H}_2, S_1, \mu^*)$:



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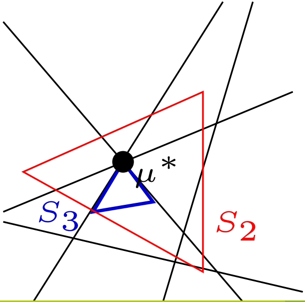
- Instead, define $\mathcal{H}_3$ as the $O(n^4)$ hyperplanes where $c_\mu(C) = c_\mu(C')$ for C, $C' \in \mathcal{C}$ and set $S_3 = \mathrm{PLA}(\mathcal{H}_3, S_2, \mu^*)$.



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We saved a lot compared to Megiddo, but even for $d = 1$ still much slower than our deterministic P_{NB} algorithm, suggesting that $P_{\max}$ for $d = 1$ is strictly harder than P_{NB}.

Summary of Running Times

Problem	Deterministic	Randomized
Non-param GMC	SW $O(mn + n^2 \log n)$	K $\tilde{O}(m)$ (KS $\tilde{O}(n^2)$)
All $\alpha < \frac{4}{3}$-approx	NI $O(n^4)$	KS $\tilde{O}(n^2)$
Megiddo $d = 1$	SW $O(n^5 \log n)$	KS $O(n^2 \log^5 n)$
Megiddo gen'l d	SW $O(n^{2d+3} \log^d n)$	KS $O(n^2 \log^{4d+1} n)$
$Z(\mu)$ $d = 1$	$O(mn^4 \log n + n^5 \log^2 n)$	$O(n^4 \log n)$ K
$Z(\mu)$ gen'l d	(big) AMMQ	$O(n^{2d+2} \log n)$ K
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Notice that running time for our $P_{\max}$ algorithm is just log factors more than for computing all $\bar{\alpha}$-approximate min cuts.

Solving $P_{\max}$ Randomly

- So far we don't know how to do this ...

Final Summary of Running Times

Problem	Deterministic	Randomized
Non-param GMC	SW $O(mn + n^2 \log n)$	K $\tilde{O}(m)$ (KS $\tilde{O}(n^2)$)
All $\alpha < \frac{4}{3}$-approx	NI $O(n^4)$	KS $\tilde{O}(n^2)$
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New results in this paper in red. Compare to non-param lower bounds in green, various upper bounds in blue.

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  - Can we use Karger's ideas to further speed up P_{NB} to $\tilde{O}(m)$?
  - There should be a faster, specialized, randomized algorithm for $P_{\max}$.

Any questions?

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Comments?