Compact, Provably-Good LP Relaxations for Orienteering ~~and RVRP~~

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with

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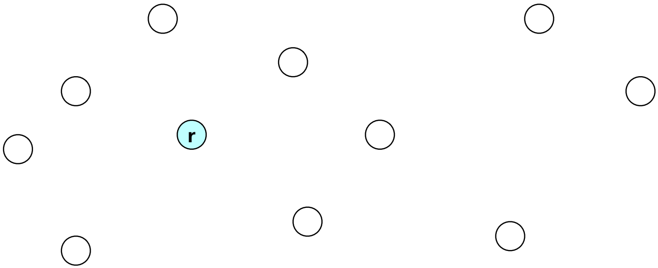
BIRS TSP Workshop - 2018

Orienteering

Starting from **Corbet Hall**, visit as many sights in Banff during the Wednesday break.



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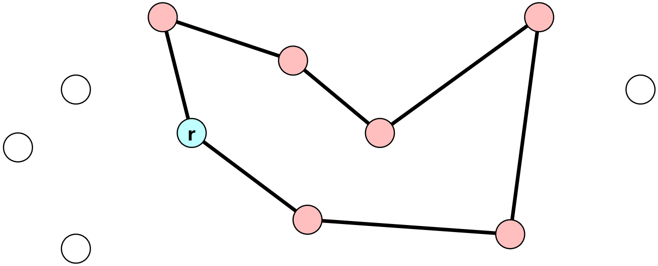
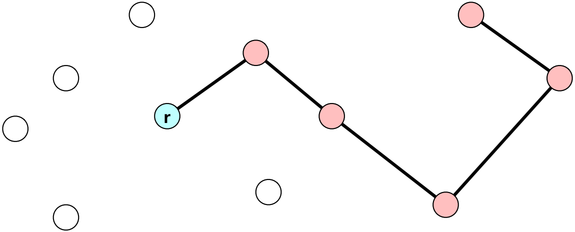


Rooted Orienteering

Given metric distances $d()$ over points $V \cup \{r\}$ where:

- V - clients
- r - depot

Each $v \in V$ has a reward $\rho(v) \geq 0$. Also have a distance bound $D \geq 0$.



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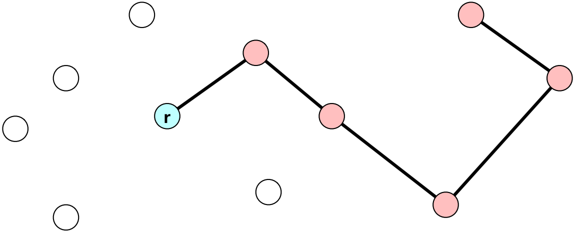

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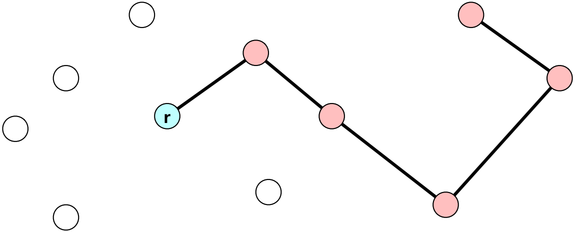

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If an end vertex t is also specified (could be $t = r$), we call this **Point-to-Point Orienteering**.

A Brief History

- First, a 4-approximation for rooted orienteering [Blum et al, 1994].
- Then, a 3-approximation for Point-to-Point Orienteering. [Bansal et al, 2004].
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Briefly, the asymmetric version is also studied.

- An $O(\log^2 OPT)$-approximation. [Chekuri, Korula, and Pal, 2007].
- An $O(\rho \cdot \log n)$-approximation: $\rho =$ ATSP integrality gap. [Nagarajan and Ravi, 2007].
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Notice: The improved integrality gap bound for ATSP led to an improved approximation for a different problem!

Specific Results

Poly-size LP relaxations with the following integrality gap bounds.

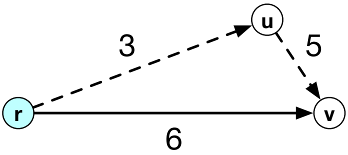
- **Rooted Orienteering**: 3
- **Point-to-Point Orienteering**: 6
- **RVRP**: A natural relaxation with a gap of 27, an unnatural relaxation with a gap of 15.
  This beats a 28.86-approximation that used a large configuration LP [F. and Swamy, 2014].

The Regret Metric

We shift focus to a new metric called the **regret metric**.

$$d^{reg}(u, v) := d(r, u) + d(u, v) - d(r, v).$$

How much longer is $r \to u \to v$ than $r \to v$ directly?



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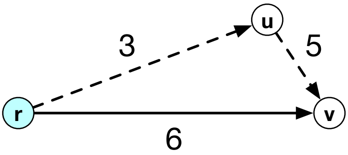

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Key Properties:

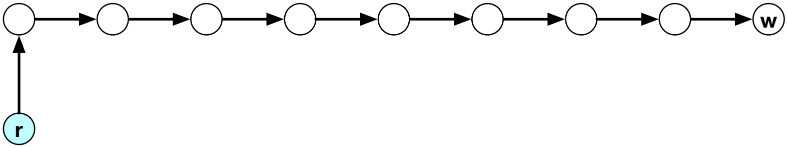
- For any $v \in V$, $d^{reg}(r, v) = 0$.
- For any $r \to v$ path P, $d^{reg}(P) = d(P) - d(r, v)$.

Pruning w.r.t. Regret

Before presenting the LP, we briefly discuss a slightly weaker goal.

Observe a rooted $r - w$ path P is a feasible orienteering solution iff $d^{reg}(P) \leq D - d(r, v)$.

Now suppose P is an $r - w$ path with $d^{reg}(P) \leq \alpha \cdot (D - d(r, w))$.

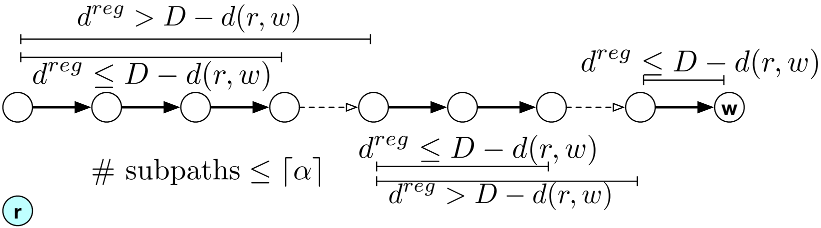


Claim

If w has maximum distance from r among all clients, we can *chop* P to a feasible solution with value $\geq \rho(P)/\lceil \alpha \rceil$.

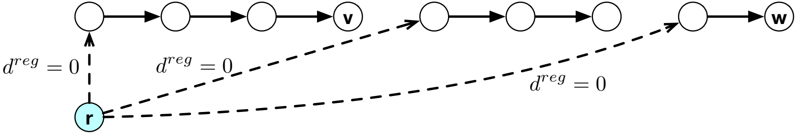
Pruning w.r.t. Regret

First, break $P - \{r\}$ into $\lceil \alpha \rceil$ subpaths, each having d^{reg}-distance $\leq D - d(r, w)$.



Pruning w.r.t. Regret

Make each subpath a rooted path by prepending r. Recall $d^{reg}(r, x) = 0$ for all $x \in V$.



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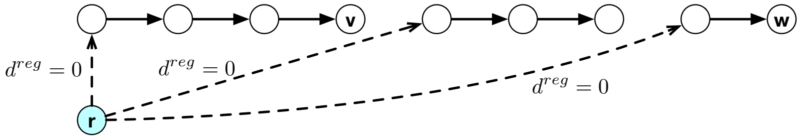

Any of these r-rooted subpaths P' ending at, say, v has length

$$d^{reg}(P') + d(r, v) \leq (D - d(r, w)) + d(r, v) \leq D.$$

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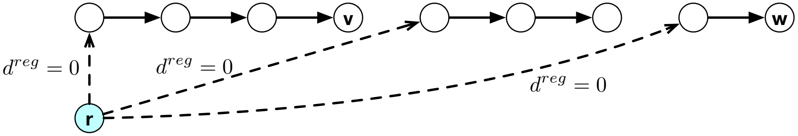

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$$d^{reg}(P') + d(r, v) \leq (D - d(r, w)) + d(r, v) \leq D.$$

So the most profitable path has value $\geq \rho(P)/\lceil \alpha \rceil$.

The LP

Before forming the LP, guess the node w on the optimum path that is furthest from r and discard farther nodes.

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We deal with a **bidirected-cut** relaxation of the problem.

- z_v - indicating we visit v.
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\text{s.t.}: & x(\delta^{in}(v)) & \geq & x(\delta^{out}(v)) & v \in V & \text{(preflow)} \\
& x(\delta^{in}(S)) & \geq & z_v & v \in S \subseteq V & \text{(clients reachable)} \\
& x(\delta^{out}(r)) & = & 1 & & \text{(one path)} \\
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Notes: Can "fold" the guess into the LP to avoid guessing. *i.e.* (x^w, z^w) variables. Can make poly-size using flow variables.

A Decomposition Theorem

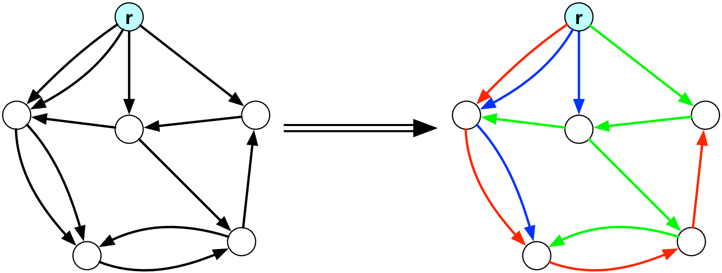
Let $D = (V + r, A)$ be a multi-digraph satisfying preflow conditions at each $v \in V$:

$$|\delta^{in}(v)| \geq |\delta^{out}(v)|.$$

Let λ_v be the $r - v$ edge connectivity.

Theorem (Bang-Jensen, Frank, and Jackson, 1995)

For any $K > 0$, there are K arc-disjoint r-branchings where each vertex v lies on $\min\{K, \lambda_v\}$ *branchings.*

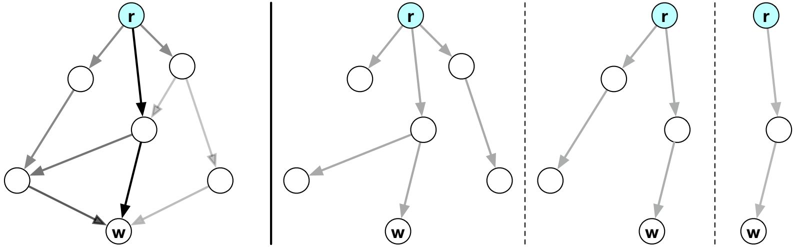


The fractional version:

Theorem

The preflow x dominates a convex combination of r-branchings where each $v \in V$ lies on a z_v-weight of these branchings.

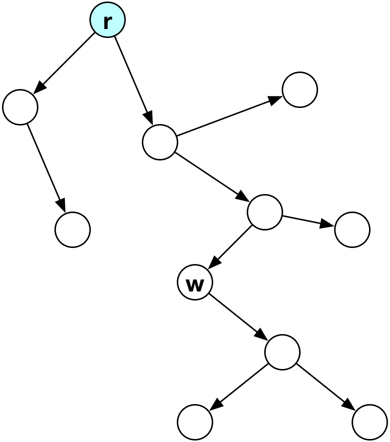
Note, w lies on each branching.



Can be found in poly-time [Post and Swamy, 2015].

The Rounding Algorithm

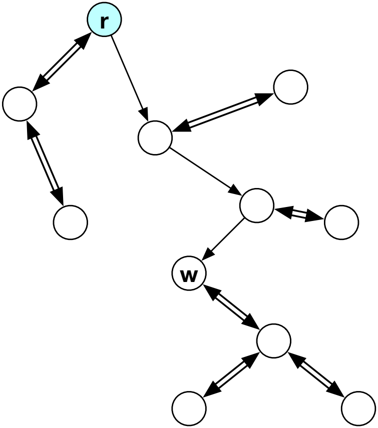
Sample a random branching B in the decomposition.



The expected $d()$-cost of B is $\leq D$.

The Rounding Algorithm

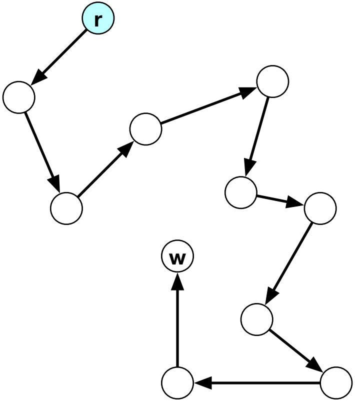
Double edges not on the $r - w$ path.



The expected $d()$-cost is $\leq D + (D - d(r, w))$.

The Rounding Algorithm

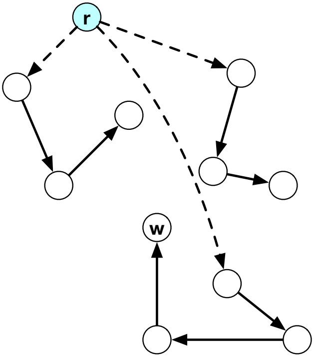
Of course, shortcut the resulting Eulerian walk to an $r - w$ path.



The expected $d()$-cost of these paths is still $\leq D + (D - d(r, w))$.

Equivalently: The expected $d^{reg}()$-cost is $\leq 2 \cdot (D - d(r, w))$.

Chop into rooted paths with $d^{reg}()$-distance $\leq D - d(r, w)$.
i.e. Feasible orienteering solutions!



If the original path P had regret $\alpha_P \cdot (D - d(r, w))$, this creates $\leq \lceil \alpha_P \rceil \leq \alpha_P + 1$ paths.

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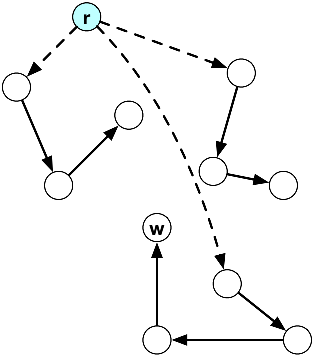

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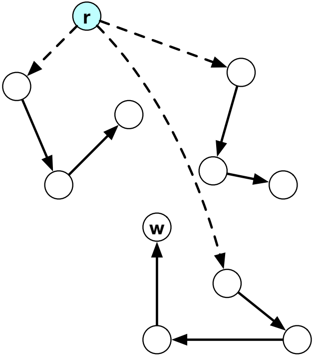

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Some subpath created this way has value $\geq OPT_{LP}/3$.

Algorithm Summary

1. Guess the furthest node w.
2. Solve the LP.
3. Decompose (x, z) into branchings.
4. For each branching:
   - Double edges not on the $r - w$ path.
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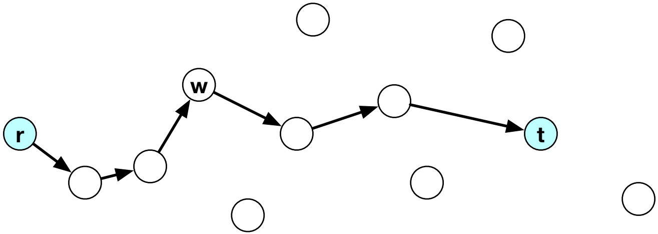
Comment

Without the guess, the gap is very bad. Even if we just guess the furthest distance but not the node itself!

Point-to-Point Orienteering

Suppose we want an $r - t$ path of bounded length.



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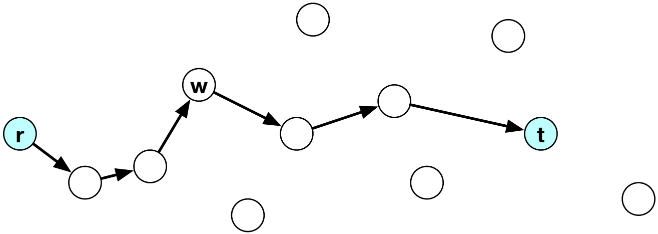

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LP: one unit of $r - w$ flow x^L and one unit of $w - t$ flow x^R.

Also z_v^L and z_v^R variables indicating if v is visited before w or after w, respectively.

To round it, the x^L-flow is a preflow from r with cost at most $D - d(w,t)$, so do as before.

This produces a path ending at some v with length

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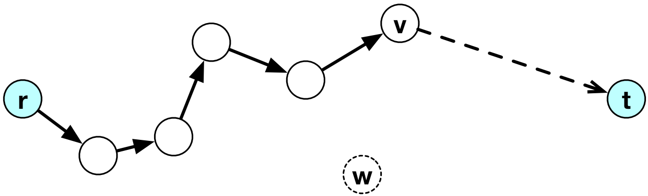

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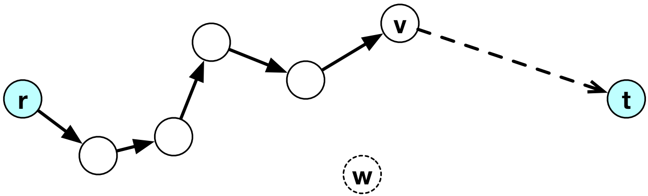

Extending from v to an $r - t$ path yields a path with distance

$$D + [d(r, v) + d(v, t)] - [d(r, w) + d(w, t)] \leq D$$

with at least $1/3$ the value of z^L.

Similarly, the reverse of x^R is a preflow out of t so we can get a feasible solution with at least $1/3$ the value of z^R.

The best solution overall has value $\geq OPT_{LP}/6$.

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Using Lagrangian relaxation, one can find a bipoint "solution": two branchings spanning the guess w with "average" cost $\leq D$ and average profit $\geq OPT$.

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All done!

Thank You