Stability results for non-autonomous dynamical systems

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The University of Queensland
Australia

Australian Government
Australian Research Council

New Developments in Open Dynamical Systems and Their Applications
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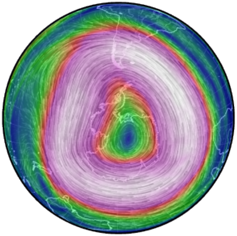
Motivation

- To develop mathematical tools –analytical and numerical– to analyse and understand transport and mixing phenomena in (non-autonomous) dynamical systems.



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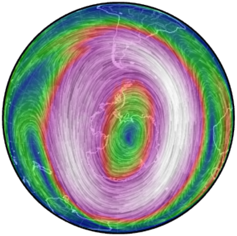

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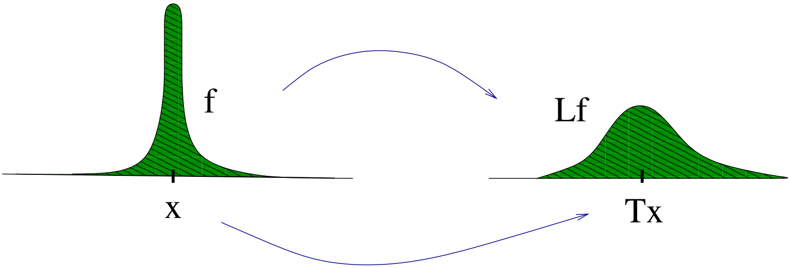
http://earth.nullschool.net

Transfer Operators

- Powerful analytical tool to investigate global properties of dynamical systems, by considering densities, or ensembles of trajectories.

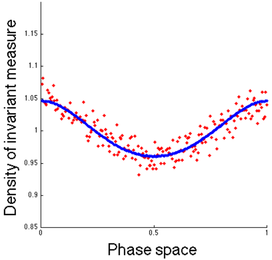


- Linear operators encoding the global dynamics, acting on a linear (Banach, Hilbert) space X,

$$\mathcal{L} : X \to X, \quad \int f \cdot g \circ T \, dm = \int \mathcal{L} f \cdot g \, dm.$$

Transfer Operators

- Very useful for numerical analysis of dynamical systems, e.g. via Markovian models.



Numerical approximations to invariant measure of a dynamical system via transfer operators (blue) and long trajectories (red).

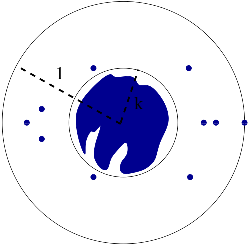
- Ulam discretisation scheme: $\mathcal{P} = \{B_1, \ldots, B_k\}$ partition of the state space into *bins*,

$$\mathbb{E}_{\mathcal{P}}(f) = \sum_{j=1}^{k} \frac{1}{m(B_j)} \Big(\int 1_{B_j} f \; dm \Big) 1_{B_j}.$$

Transfer Operators, Quasi-compactness

- Also useful for the analytical study of transport phenomena in dynamical systems.
- $\mathcal{L}$ is quasi-compact if there exists $0 \leq k < 1$, called *essential spectral radius* of $\mathcal{L}$, such that, outside the disc of radius k:

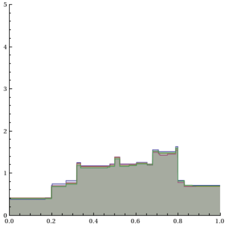
  - The spectrum of $\mathcal{L}$ consists of isolated eigenvalues:
  $$1 = \gamma_1, \ldots, \gamma_m, \quad m \leq \infty,$$
  such that $|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_m| > k$, and
  - Finite-dimensional corresponding generalised eigenspaces:
  $$E_1, \ldots, E_m.$$



Transfer Operators, Spectral Properties

- It is now known that for a rich class of transformations T (including piecewise smooth expanding/hyperbolic maps) and appropriate X, $\mathcal{L}$ is quasi-compact. Furthermore,

$$1 = \gamma_1 \text{ simple} \quad \leftrightsquigarrow \quad \text{Ergodic system;}$$
$$f_1 \in E_1 \quad \leftrightsquigarrow \quad \text{Density of physical invariant measure.}$$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j x) \quad =: \quad \lim_{n\to\infty} \frac{1}{n} S_n g(x) = \int g f_1 dm, m \text{ a.e. } x \in I.$$



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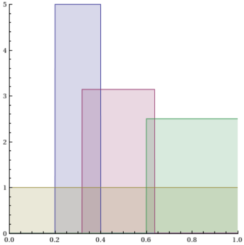

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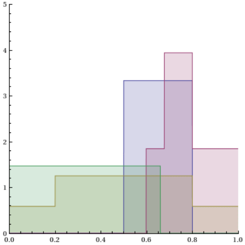

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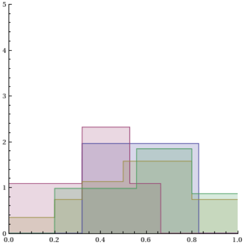

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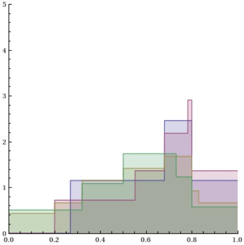

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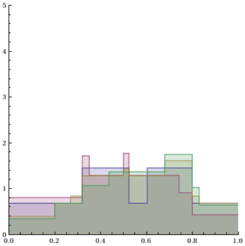

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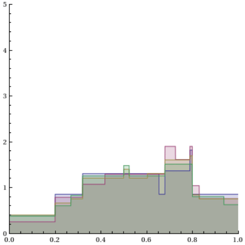

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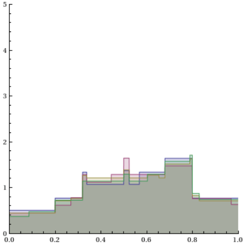

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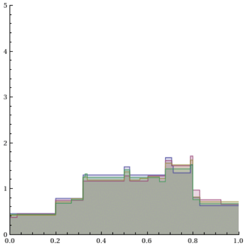

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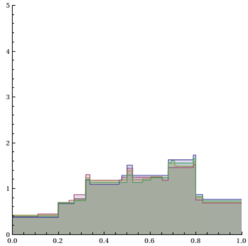

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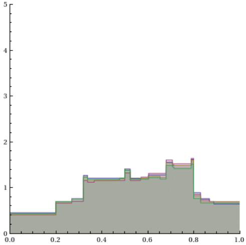

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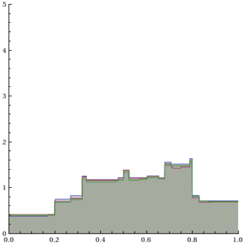
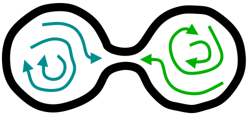
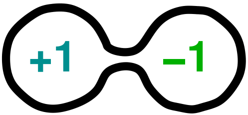


Transfer Operators, Spectral Properties

Dellnitz, Deuflhard, Junge and collaborators in the 1990's suggested the connection

$$f_2 \in E_2 \quad \leftrightsquigarrow \quad \text{Almost-invariant sets.}$$





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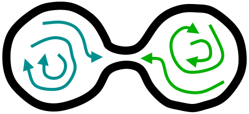

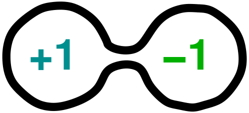

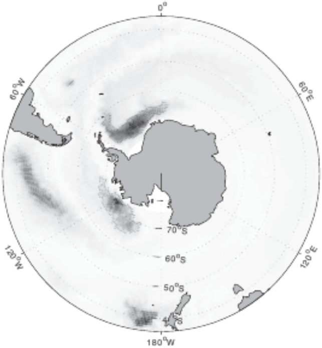

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Non-Autonomous Dynamical Systems: Introduction

- The evolution rule,

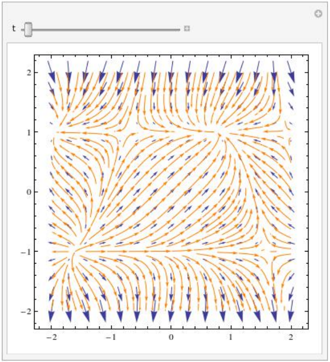
$$T_\omega : D \to D, \quad \omega \in \Omega,$$

is dictated by an external driving system $\sigma : \Omega \to \Omega$.

- Analogy:

autonomous	$\leftrightsquigarrow$	picture
non-autonomous	$\leftrightsquigarrow$	movie

- Also known as:
  - Skew products, cocycles
  - Forced, time-dependent, and random dynamical systems (RDS).



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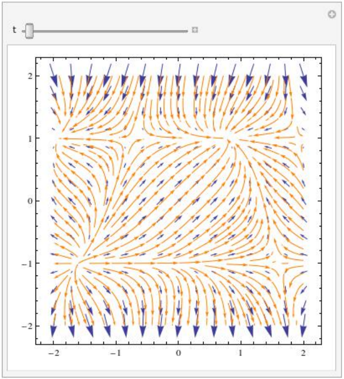


The Driving System

$$\sigma : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$$

- Invertible;
- Probability preserving:

$$\mathbb{P}(\sigma^{-1}E) = \mathbb{P}(E) \text{ for all measurable } E \subset \Omega;$$

- Ergodic:

$$E = \sigma^{-1}(E) \Rightarrow \mathbb{P}(E) = 0 \text{ or } \mathbb{P}(E) = 1.$$

♦ Examples

- Autonomous system:

$$\Omega = \{\omega_0\},\ \mathbb{P} = \delta_{\omega_0},\ \sigma = \text{Id}.$$

- Deterministic forcing:

$$\Omega = S^1,\ \mathbb{P} = \text{Leb},\ \sigma(\omega) = \omega + \alpha \pmod 1, \alpha \notin \mathbb{Q}.$$

- Stationary noise:

$$\Omega = [-\epsilon, \epsilon]^{\mathbb{Z}},\ \mathbb{P} = \text{product of uniform measures},\ \sigma = \text{shift}.$$

Non-Autonomous Systems

- External driving system

$$\sigma : \Omega \to \Omega,$$

measure preserving transformation of $(\Omega, \mathcal{F}, \mathbb{P})$.

- Several, possibly uncountably many, evolution rules

$$T_\omega : D \to D, \quad \omega \in \Omega.$$

- Associated transfer operators,

$$\mathcal{L}_\omega \in L(X), \quad \omega \in \Omega.$$

- Random dynamical system,

$$\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L}).$$

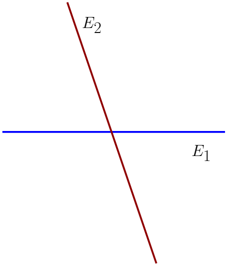
$$\mathcal{L}(\omega, n) = \mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

Multiplicative Ergodic Theorems: Introduction

Spectral type decompositions for non-autonomous dynamical systems. (Into non-linear time-varying modes, in order of decay rate.)

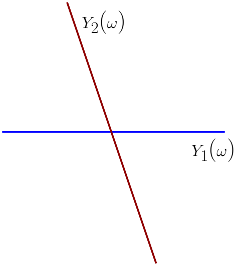
Autonomous

- $\mathcal{L}$ quasi-compact operator
- γ_i isolated eigenvalues
- E_i (generalised) eigenspaces



Non-autonomous

- $\mathcal{R}$ quasi-compact RDS
- λ_i Lyapunov exponents
- $Y_i(\omega)$ Oseledets spaces



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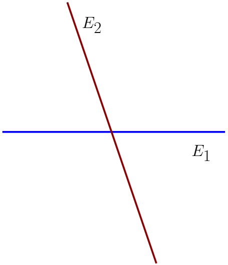

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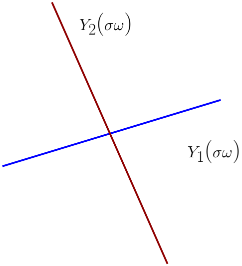

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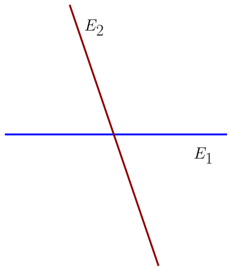

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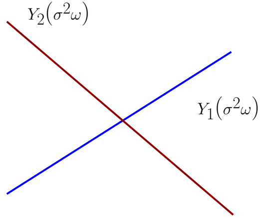

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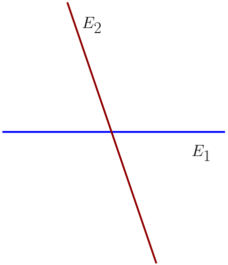

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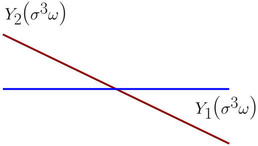

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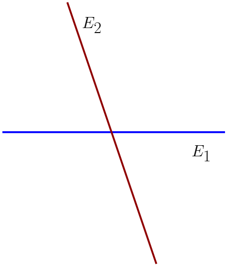

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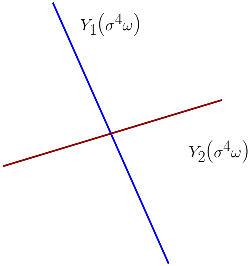

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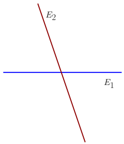

$$\mathcal{L}e_i = \gamma_i e_i$$

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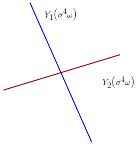


$$\mathcal{L}_\omega(Y_i(\omega)) = Y_i(\sigma\omega)$$

$$\frac{1}{n}\log \|\mathcal{L}_\omega^{(n)} y_i(\omega)\| \to \lambda_i$$

Multiplicative Ergodic Theorems: History

Oseledets splittings:

- For invertible (injective) operators:
  - Oseledets '68, Raghunathan '79 (matrices);
  - Ruelle '79 (Hilbert spaces);
  - Mañé '83, Thieullen '87, Lian–Lu '10, Blumenthal '16 (Banach spaces).

  (In the non-invertible case, the above show existence of Oseledets filtration.)
- For semi-invertible operators: (σ invertible)
  - Froyland–Lloyd–Quas '10 (matrices);
  - Froyland–Lloyd–Quas '13 (restricted type of operators);
  - GT–Quas '14, '15 (separable Banach spaces).

Multiplicative Ergodic Theorem: Setting

- Let $(X, \|\cdot\|)$ be a Banach space with separable dual.
- Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a random dynamical system with ergodic and invertible base σ.
- Integrability: $\log^+ \|\mathcal{L}(\omega)\| \in L^1(\mathbb{P})$.
- Strong measurability: For each $f \in X$, $\omega \mapsto \mathcal{L}_\omega f$ is measurable.
- Quasi-compactness: $\lambda^* > \kappa^*$.

$$\lambda^*(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|$$

maximal Lyapunov exponent (analog of the *spectral radius*);

$$\kappa^*(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log \mathrm{ic}(\mathcal{L}_\omega^{(n)})$$

index of compactness (analog of the *essential spectral radius*)

$$\mathrm{ic}(\mathcal{L}) := \inf \left\{ r > 0 : \begin{array}{l} \mathcal{L}(B_X) \text{ can be covered with} \\ \text{finitely many balls of radius } r \end{array} \right\}.$$

Multiplicative Ergodic Theorem

Theorem (Semi-invertible Oseledets theorem [GT-Quas '14])

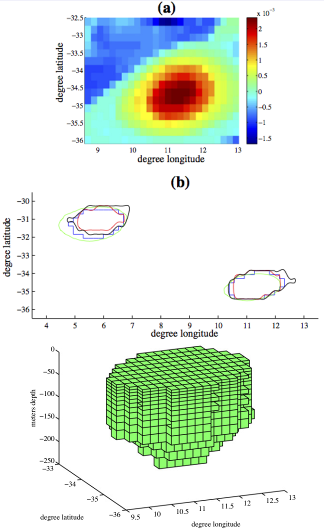
$\mathcal{R}$ has an Oseledets splitting:
There are at most countably many exceptional Lyapunov exponents, $\lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa^*$; and there exists a unique measurable and equivariant splitting of X,

$$X = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega), \text{ defined for } \mathbb{P} \text{ a.e. } \omega \in \Omega,$$

with $V(\omega)$ closed and $Y_j(\omega)$ finite dimensional, such that:

- *For every $v \in Y_j(\omega) \setminus \{0\}$, $\lim_{n\to\infty} n^{-1} \log \|\mathcal{L}_\omega^{(n)} v\| = \lambda_j$.*
- *For every $v \in V(\omega)$, $\lim_{n\to\infty} n^{-1} \log \|\mathcal{L}_\omega^{(n)} v\| \leq \kappa^*$.*

Approximation and Identification of Coherent Structures



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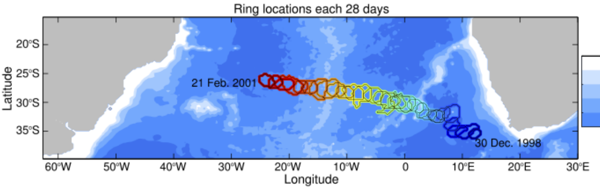
- The Oseledets spaces $Y_j(\omega)$ can be approximated using a *singular value decomposition* (SVD) type construction. [Froyland–Santitisadeekorn–Monahan '10, GT–Quas '15]



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Stability?

Question

How does spectral data from transfer operators (Lyapunov exponents, Oseledets splitting) change when the dynamical system is perturbed?

- Relevant perturbations:
  - Model errors.
  - Noise.
  - Numerical approximations: Ulam and Fourier-based methods.
- Early work, autonomous setting:
  - Keller–Liverani '99:
    Stability of spectral data for quasi-compact operators (isolated eigenvalues and corresponding eigenspaces).

Stability for non-autonomous systems

- Setting: Perturbations
  - Initial system:

$$\mathcal{R} = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L}).$$

  - Perturbations:

$$\mathcal{R}_k = (\Omega, \mathbb{P}, \sigma, X, \mathcal{L}_k), \quad \mathcal{L}_k \text{ `close to' } \mathcal{L}.$$

- Previous positive stability results, closest to our setting:
  - Ledrappier–Young '91, Ochs '99;
  - Baladi–Kondah–Schmitt '96, Bogenschütz '00.
- Warning! Negative stability results:
  - Bochi '02, Bochi–Viana '05.

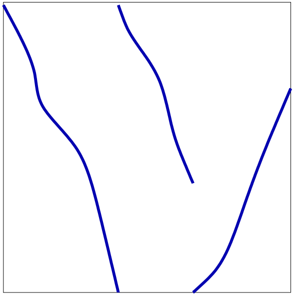
(I)
Stability of random absolutely continuous invariant measures for piecewise expanding interval maps

Setting: Lasota–Yorke Maps

- Let LY be the set of non-singular, finite-branched, piecewise monotonic and piecewise smooth interval maps,

$$T : I \to I.$$

- For each $T \in LY$,
  - $\mu(T) := \operatorname{essinf}_{x \in I} |T'(x)|$
  - $N(T) :=$ number of branches of T



Setting: Random Lasota–Yorke Maps

- $\sigma : \Omega \circlearrowleft$ ergodic, invertible $\mathbb{P}$-preserving transformation.
- A good random Lasota–Yorke map $\mathcal{T}$ is a function

$$\begin{aligned} \mathcal{T} : \Omega &\to LY, \\ \omega &\mapsto T_\omega, \text{ such that} \end{aligned}$$

  - $(\omega, x) \mapsto T_\omega(x)$ is measurable.
  - Expansion: $\lim_{K\to\infty} \int_\Omega \log \min(\mu(T_\omega), K) d\mathbb{P} > 0$.
  - Number of branches: $\log^+(N(T_\omega)/\mu(T_\omega)) \in L^1(\mathbb{P})$.
  - Distortion: $\log^+(\mathrm{var}(1/|T'_\omega|)) \in L^1(\mathbb{P})$.

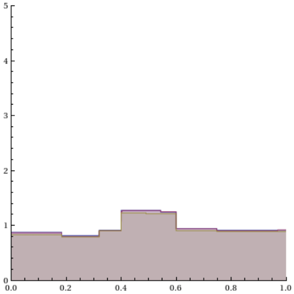
Random Lasota–Yorke Maps: Existence of Random acims

Definition

A ***random acim*** *for* $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, BV, \mathcal{L})$ *is a non-negative measurable function* $F : \Omega \times I \to \mathbb{R}$*, with* $f_\omega := F(\omega, \cdot) \in BV$*, such that* $\|f_\omega\|_1 = 1$ *and for every* $\omega \in \Omega$*,* $\mathcal{L}_\omega f_\omega = f_{\sigma\omega}$*.*

Theorem (Buzzi '99)

Let $\mathcal{R}$ *be a good random Lasota–Yorke map. Then,* $\mathcal{R}$ *has at least one and at most finitely many random acims.*



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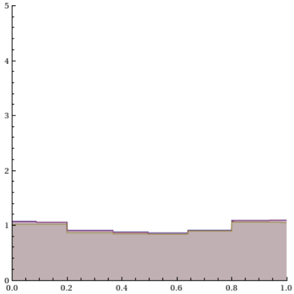

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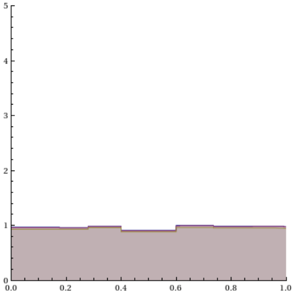

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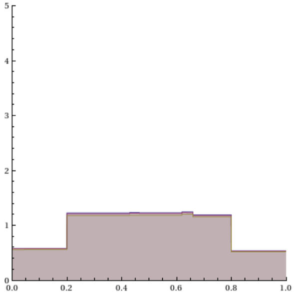

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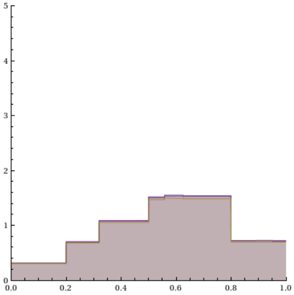


Perturbations: the Ulam Scheme

Ulam discretisations

$$\mathcal{L}_{k,\omega} = \mathbb{E}_k \circ \mathcal{L}_\omega$$

$\mathbb{E}_k$ is the conditional expectation with respect to the uniform partition of I into k intervals $\mathcal{P}_k = \{B_1, \dots, B_k\}$,

$$\mathbb{E}_k(f) = \sum_{j=1}^{k} \frac{1}{m(B_j)} \Big(\int 1_{B_j} \, f \; dm \Big) 1_{B_j},$$

- Very effective numerical approximation scheme.

Perturbations: Convolutions

Convolutions

$$\mathcal{L}_{k,\omega} f(x) = Q_k * \mathcal{L}_\omega f(x) = \int Q_k(y) \mathcal{L}_\omega f(x-y) dy$$

$\{Q_k\}_{k\in\mathbb{N}}$ are densities on $\mathbb{S}^1$, with $Q_k \to \delta_0$ weakly.

- Uniform densities: Model of iid noise (on average)

$$Q_k = \frac{1}{2\epsilon_k} \mathbb{1}_{[-\epsilon_k, \epsilon_k]}.$$

- Fejér kernels: Cesàro average of partial sums of Fourier series

$$Q_k(x) = \frac{\sin(\pi k x)^2}{k \sin(\pi x)^2}.$$

Stability Theorem Application: Static Perturbations

♦ Static perturbations

Each T_ω is perturbed to a nearby map $T_{k,\omega}$,
$\mathcal{L}_{k,\omega}$ is the transfer operator of $T_{k,\omega}$.

- Modelling errors
- Model iid additive noise:
  $\Xi = [-1,1]^{\mathbb{Z}}$, equipped with the product of uniform measures,
  s left shift on Ξ.
  Set $\bar{\Omega} = \Omega \times \Xi$, $\bar{\sigma} = \sigma \times s$ and for $(\omega, \xi) \in \bar{\Omega}$,

$$T_{k,(\omega,\xi)}(x) = T_\omega(x) + \epsilon_k \xi_0.$$

Stability Theorem for Random Acims

Theorem (Froyland–GT–Quas '14 & Froyland–GT–Murray '17)

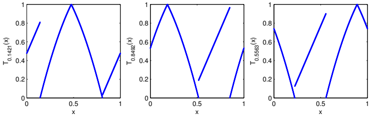
- *Let $\mathcal{R}$ be a covering good random Lasota–Yorke map.*
- *Let $\{\mathcal{R}_k\}$ be either*
  - *The sequence of Ulam discretisations, corresponding to uniform partitions $\mathcal{P}_k$ $(^*)$, or*
  - *A sequence of random perturbations by convolution with Q_k, with $Q_k \to \delta_0$ weakly.*
  - *A sequence of static perturbations of size $\epsilon_k \to 0$.*

Then, for each sufficiently large k, $\mathcal{R}_k$ has a unique random acim.
Let $\{F_k\}_{k\in\mathbb{N}}$ be the sequence of random acims for $\mathcal{R}_k$.
Then, $\lim_{k\to\infty} F_k = F$ fibrewise in $|\cdot|_1$.
(That is, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lim_{k\to\infty} |f_\omega - f_{k,\omega}|_1 = 0$.)

Comments on the Proof

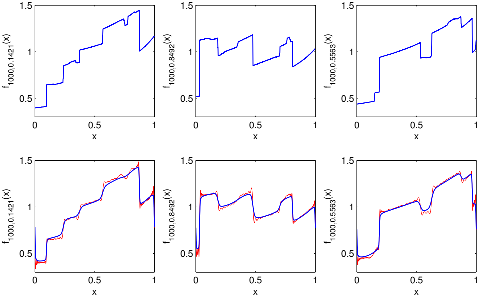
- Convergence is established in a strong sense.
- Previous stability results deal with small perturbations of an autonomous expanding system.
  (Baladi, Kondah, Schmidt, Bogenschütz)
- The proof combines ergodic theoretical tools with *classical* functional analysis tools for autonomous systems (Buzzi, Blank, Keller, Liverani), including quantitative control on the skeleton of *(random) periodic turning points*.

Stability: Numerical Example



- $\sigma : \mathbb{S}^1 \circlearrowleft$ be a rigid rotation by angle $\alpha = 1/\sqrt{2}$

$$T_\omega(x) = \begin{cases} 3(x-\omega) - 2.9(x-\omega)(x-\omega-\frac{1}{3}), & \omega \le x < \omega + \frac{1}{3}; \\ -3(x-\omega) + 1 - 2.9(x-\omega-\frac{1}{3})(x-\omega-\frac{2}{3}), & \omega + \frac{1}{3} \le x < \omega + \frac{2}{3}; \\ \frac{7}{3}(x-\omega-\frac{2}{3}) + 2\omega/9, & \omega + \frac{2}{3} \le x < \omega + 1 \end{cases} .$$



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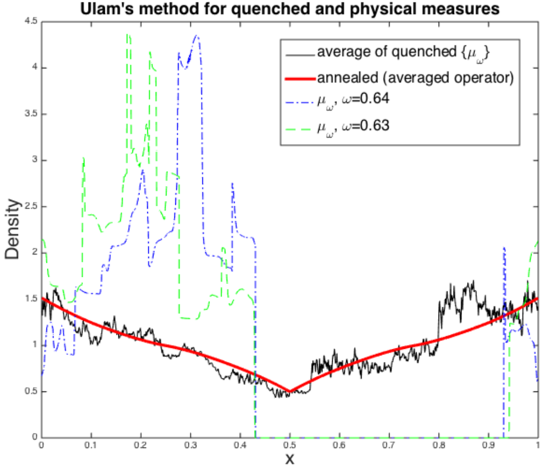

$I := [0,1]$, $\omega \in \Omega := S^1$, $\mathbb{P} :=$Leb, $\sigma(\omega) := \omega + \rho \pmod 1, \rho \notin \mathbb{Q}$.

$$f_\omega(x) := \begin{cases} 2.1\,(x - 2\omega) \pmod 1 & \text{if } \omega \in [0, 1/2), \\ 0.5\,(x - 2(\omega - 0.5)) \pmod 1 & \text{if } \omega \in [1/2, 1). \end{cases}$$

(II)
Stability of Oseledets splittings in an infinite dimensional (Hilbert space) setting

Stochastic Stability of Oseledets Splittings: Setting

- H separable Hilbert space, with basis $e_1, e_2, \dots$.
- Hilbert–Schmidt and strong Hilbert–Schmidt norms, for $A \in H$:

$$\|A\|_{\mathsf{HS}}^2 := \sum_{i,j} \langle Ae_i, e_j \rangle^2, \quad \|A\|_{\mathsf{SHS}}^2 := \sum_{i,j} 2^{2(i+j)} \langle Ae_i, e_j \rangle^2.$$

$$\mathsf{SHS} := \{A \in H : \|A\|_{\mathsf{SHS}} < \infty\} \subset \mathsf{HS} \subset K(H).$$

- Hilbert space cocycle: $(\Omega, \mathbb{P}, \sigma, \mathsf{SHS}, A)$, with
  σ ergodic, $\mathbb{P}$-preserving and invertible;
  $A\colon \Omega \to \mathsf{SHS}$, with log-integrable norm;

$$A_\omega^{(n)} := A(\sigma^{n-1}\omega) A(\sigma^{n-2}\omega) \cdots A(\omega).$$

Stochastic Stability of Oseledets Splittings: Setting

- Lyapunov exponents (with multiplicity):
  $\infty > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq \cdots \geq -\infty$.
- $d_1, d_2, \ldots, d_p, \ldots$ the corresponding multiplicities;
- $D_0 := 0$, $D_i := d_1 + \ldots + d_i$,
  so that $\mu_j = \mu_{j'}$ if $D_{i-1} < j, j' \leq D_i$.
- The notions of singular vectors and singular values apply to compact operators, as in the finite-dimensional case.
  For $A \in K(H)$, let $s_1(A) \geq s_2(A) \geq \ldots$ be the singular values (with multiplicity).
  The maximal logarithmic rate of k-dimensional volume growth is given by

$$\Xi_k(A) := \log(s_1(A) \cdots s_k(A)).$$

Perturbations

- $\bar{\Omega} := \Omega \times \mathsf{SHS}^{\mathbb{Z}}$,
- $\bar{\sigma} := \sigma \times s$, where s is the shift on $\mathsf{SHS}^{\mathbb{Z}}$.
- $\bar{\mathbb{P}} := \mathbb{P} \times \gamma^{\mathbb{Z}}$ where γ is the multi-variate normal distribution on SHS with centred, normal (i, j)th entry with standard deviation $3^{-(i+j)}$, and independent entries.
- For $\epsilon > 0$, define the new cocycle $A^{\epsilon} \colon \bar{\Omega} \to \mathsf{SHS}$, with generator

$$A^{\epsilon}(\omega, (\Delta_n)_{n\in\mathbb{Z}}) = A(\omega) + \epsilon\Delta_0, \quad (\Delta_n \sim \gamma).$$

- Goal: compare splittings of $\mathcal{R} = (\Omega, \mathbb{P}, \sigma, A)$ and $\mathcal{R}_{\epsilon} = (\bar{\Omega}, \bar{\mathbb{P}}, \bar{\sigma}, A^{\epsilon})$, as $\epsilon \to 0$.

Stochastic Stability of Oseledets Splittings

Theorem (Froyland–GT–Quas, to appear)

(i) *Convergence of Lyapunov exponents:*
Let the Lyapunov exponents of the perturbed matrix cocycle $(\bar{\Omega}, \bar{P}, \bar{\sigma}, A^\epsilon)$ *be*

$$\mu_1^\epsilon \geq \mu_2^\epsilon \geq \ldots \geq \mu_d^\epsilon,$$

with multiplicity. Then $\mu_i^\epsilon \to \mu_i$ *for each* i *as* $\epsilon \to 0$.

(ii) *Convergence in probability of Oseledets spaces:*
Let $\mathcal{N} = (\mu_i - \delta, \mu_i + \delta)$, *with* $\mu_i > -\infty$ *and* $\mu_j \notin \mathcal{N}$ *if* $\mu_j \neq \mu_i$.
Let ϵ_0 *be such that for each* $\epsilon \leq \epsilon_0$,
$\mu_j^\epsilon \in \mathcal{N}$ *for each* $D_{i-1} < j \leq D_i$.
For $\epsilon < \epsilon_0$, *let* $Y_i^\epsilon(\bar{\omega})$ *denote the sum of the Oseledets spaces of* A^ϵ *having exponents in* $\mathcal{N}$.
Then $Y_i^\epsilon(\bar{\omega})$ *converges in probability to* $Y_i(\omega)$ *as* $\epsilon \to 0$.
(Convergence in the Grassmannian of H*.)*

Strategy of the Proof: Stability of Lyapunov Exponents

Goal: obtain a lower bound for the sum of the k top perturbed Lyapunov exponents (maximal logarithmic growth rate of k-volumes).

- For $\epsilon > 0$, define a block length, $N \sim |\log \epsilon|$.
- For large n, estimate the top exponents of the product $A^{\epsilon (nN)}_{\bar{\omega}}$, a perturbed block of length nN.
- Replace the (sub-additive) logarithmic k-volume growth, $\Xi_k(\cdot)$ by a related approximately super-additive quantity,

$$\tilde{\Xi}_k(A) = \mathbb{E}\Xi_k(\Pi_k \Delta A \Delta' \Pi_k),$$

  where Π_k is the orthogonal projection onto $\langle e_1, \ldots, e_k \rangle$, and $\Delta, \Delta' \sim \gamma$ are independent.
- Use this super-additivity to split $A^{\epsilon (nN)}_{\bar{\omega}}$ into good *super-blocks* (of length a multiple of N) and bad blocks (of length $N-2$):

$$\Xi_k(A^{\epsilon (nN)}_{\bar{\omega}}) \gtrsim \tilde{\Xi}_k(A^{\epsilon (nN)}_{\bar{\omega}}) \gtrsim \sum \tilde{\Xi}_k(\text{blocks}).$$

Strategy of the Proof: Stability of Lyapunov Exponents

- Show $\Xi_k(G^\epsilon) \gtrsim \Xi_k(G)$, where G represents a good super-block and G^ϵ its perturbed version.
- Show $\mathbb{E}\tilde{\Xi}_k(B^\epsilon) \gtrsim \tilde{\Xi}_k(B)$ where B is a bad block and B^ϵ is its perturbed version.
- Show $\tilde{\Xi}_k(B) \gtrsim \Xi_k(B)$ and $\tilde{\Xi}_k(G^\epsilon) \gtrsim \Xi_k(G^\epsilon)$.
- Re-assemble the pieces using *sub-additivity* of Ξ_k and account for the errors.

Strategy of the Proof: Stability of Oseledets Spaces

- Assume $\mu_k > 0 > \mu_{k+1}$. Let $\delta_0 < 1$, $E_k^\epsilon(\bar{\omega}) = \oplus_{j=1}^k Y_j^\epsilon(\bar{\omega})$ and

$$U_\epsilon = \left\{\bar{\omega} \colon \angle\big(E_k^\epsilon(\bar{\omega}), E_k(\omega)\big) > 2\delta_0\right\}, \quad W_\epsilon = \bar{\sigma}^{-N} U_\epsilon \cap \bar{G}.$$

  To show: $\forall 0 < \eta < 1$ and small $\epsilon > 0$, $\bar{\mathbb{P}}(W_\epsilon) < \eta$.
- (Convergence of $Y_k^\epsilon(\bar{\omega})$ to $Y_k^0(\omega)$ then follows from the identity $Y_k^\epsilon(\bar{\omega}) = E_k^\epsilon(\bar{\omega}) \cap F_{k-1}^\epsilon(\bar{\omega})$ and duality.)
- If $\bar{\omega} \in \bar{G}$, and $\angle(E_k^\epsilon(\bar{\sigma}^N \bar{\omega}), E_k(\sigma^N \omega)) > 2\delta$, then $\perp(E_k^\epsilon(\bar{\omega}), F_k(A_\omega^{(N)})) < 4\delta^{-1} e^{-(\mu_k - \tau)N}$.
- If ϵ is sufficiently small so that $4\delta^{-1} + 2 < e^{k\tau N}$, $\bar{\omega} \in \bar{G}$ and $\perp(E_k^\epsilon(\bar{\omega}), F_k(A_\omega^{(N)})) < 4\delta^{-1} e^{-(\mu_k - \tau)N}$, we have

$$\Xi_k(A^{\epsilon}{}_{\bar{\omega}}^{(N)}|_{E_k^\epsilon(\bar{\omega})}) \leq (\mu_1 + \ldots + \mu_{k-1} + 2k\tau)N.$$

Strategy of the Proof: Stability of Oseledets Spaces

- $$\begin{aligned}\mu_1^\epsilon + \ldots + \mu_k^\epsilon &= \lim_{n\to\infty} \frac{1}{n} \int \Xi_k(A^{\epsilon(n)}_{\bar\omega}|_{E_k^\epsilon(\bar\omega)})\, d\bar{\mathbb{P}}(\bar\omega) \\ &\le \frac{1}{N}\int_{W_\epsilon} \Xi_k(A^{\epsilon(N)}_{\bar\omega}|_{E_k^\epsilon(\bar\omega)})\, d\bar{\mathbb{P}}(\bar\omega) + \frac{1}{N}\int_{W_\epsilon^c} \Xi_k(A^{\epsilon(N)}_{\bar\omega})\, d\bar{\mathbb{P}}(\bar\omega) \\ &\le (\mu_1 + \ldots + \mu_{k-1} + 2k\tau)\bar{\mathbb{P}}(W_\epsilon) + (\mu_1 + \cdots + \mu_k)\bar{\mathbb{P}}(W_\epsilon^c) + 2\tau.\end{aligned}$$

  Hence,

  $$\mu_k \bar{\mathbb{P}}(W_\epsilon) \le (\mu_1 + \ldots + \mu_k) - (\mu_1^\epsilon + \ldots + \mu_k^\epsilon) + 4k\tau.$$

  In particular, using convergence of the Lyapunov exponents, for sufficiently small ϵ, we have $\bar{\mathbb{P}}(W_\epsilon) \le 5k\tau/\mu_k < \eta$.