Regularity and Blow-up of Navier-Stokes Type PDEs using Harmonic and Stochastic Analysis

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1 Overview of the Field

Mathematical analysis on the equations of fluid motion have evolved remarkably ever since the pioneering work of Leray [4] in 1930s. While the fundamental question of the existence and uniqueness of a global solution with finite kinetic energy in the three-dimensional case for the Navier-Stokes equations remains unsolved, over these decades mathematicians have contributed to extraordinary advancement in the study of fluid dynamics.

2 Recent Developments and Open Problems

Very recently, two particular analysis tools have stood out prominently, namely harmonic analysis tools (e.g. [1]) to handle the non-linear terms that involve derivatives, and stochastic analysis tools (e.g. [2, 5]) to study the same equations but forced with noise. Using these tools, we have seen various different results, e.g. global existence of unique solution with dissipative term replaced by a fractional Laplacian or only in selected directions, blow-up of a solution with the structure of the equation slightly modified, and finally the stochastic effects such as whether the noise helps regularize the solution or not.

The objective of this workshop "Regularity and blow-up of Navier-Stokes type PDEs using harmonic and stochastic analysis" at Banff International Research Station (BIRS) was to invite various researchers in fluid dynamics working through different approaches to share their work and provide favorable environment for discussion and collaboration among those with complementary background and training who otherwise would not have had this opportunity to interact. The workshop brought together the stochastic and the deterministic communities to share their finding and have a very open discussion on the future research direction of fluid dynamics and in particular Navier-Stokes equations with the goal of helping each other shed some light on how to be more successful in solving the open questions.

3 Presentation Highlights

The 5-days workshop was scheduled to run everyday. We had deterministic oriented talks by the following experts:

1. Prof. Buckmaster Tristan from Princeton University - Department of Mathematics,

2. Prof. Chae Dongho from Chung-Ang University - Department of Mathematics,

3. Prof. Cheskidov Alexey, University of Illinois at Chicago - Department of Mathematics,

4. Prof. Constantin Peter, Princeton University - Department of Mathematics,

5. Prof. Córdoba Diego, Instituto de Ciencias Matemáticas - Consejo Superior de Investigaciones Cientficas,

6. Prof. Doering Charles, Department of Mathematics, University of Michigan,

7. Prof. Dong Hongjie, Brown University - Department of Mathematics,

8. Prof. Hmidi Taoufik, Université de Rennes1 - Equipe EDP, IRMAR,

9. Prof. Ignatova Mihaela, Princeton University - Department of Mathematics,

10. Prof. Ohkitani Koji, University of Sheffield - Department of Applied Mathematics,

11. Prof. Shvydkoy Roman, University of Illinois at Chicago, Department of Mathematics,

12. Prof. Tarfulea Andrei, University of Chicago - Department of Mathematics,

13. Prof. Titi Edriss, Texas A & M University, Department of Mathematics,

14. Prof. Vasseur Alexis F, University of Texas, Austin - Department of Mathematics,

and the stochastic oriented talks by the following experts:

1. Prof. Ferrario Benedetta, University of Pavia, Department of Mathematics,

2. Prof. Friedlander Susan, University of Southern California - Department of Mathematics,

3. Prof. Glatt-Holtz Nathan, Tulane University - Department of Mathematics,

4. Prof. Iyer Gautam, Carnegie Mellon University - Department of Mathematics,

5. Prof. Maurelli Mario, Technische Universitat Berlin, Institut fur Mathematik,

6. Prof. Romito Marco, Universitá di Pisa, Departimento di Matematica,

7. Prof. Schmalfuss Björn, University of Jena - Institute for stochastic.

It is very rare, if not the first time, to have such an opportunity to intentionally bring experts on the fluid partial differential equations (PDEs) from complementary perspectives and somehow try to bridge our knowledge together. We deliberately tried to order these talks mixed. The talks ranged over not only the Navier-Stokes equations but surface quasi-geostrophic equation from geophysics, flocking models, Kortewegde Vries equation, Boussinesq equation from oceanography, Muskat equation, Lorenz equation, Kuramoto-Sivashinsky equation, Hele-shaw model, equation of mixing, point vortex model, as well as Burgers equation. There were very good dialogues of what stochastic perspectives may be able to offer when speakers from deterministic perspective gave talks, and similarly how some techniques from deterministic case is difficult but may be extended to the stochastic case. In the direction of research of Navier-Stokes type fluid PDEs there are many results which exist in the stochastic case but not the deterministic case and vice versa. It was very refreshing for many participants how the theory of fluid PDEs, considered from a complementary point of view, lead to so many new techniques and remarkable results.

Between each talk, we typically had breaks of 30 minutes during which participants and speakers were able to engage themselves in deeper discussions through the board outside the lecture rooms. For example, after a talk by a certain speaker, a participant was able to share his progress in a very similar direction and they found out that they had very similar results but proven through different perspectives. Due to their discussion, these two participants will be able to acknowledge each others' works in their papers, which is important in the communication through journal publications.

4   Scientific Progress Made

We considered that it will be important to have time during which instead of one speaker gives a talk about his/her results, every participant including speakers and non-speakers feel free to share anything, even non-significant computations within their research which have given them concerns or questions. In the direction research of fluid PDEs, there remain so many open problems and unexplored directions of research, which, only if they were shared, will catch much attention and we may quickly see rapid developments. Therefore we reserved Tuesday 16:30 - 17:30, Thursday 16:30 - 17:30, as well as Friday 9:00 - 10:00 and 10:30 - 11:30 for the Discussion periods. During these four rare opportunities, we all sat together, allowed speakers to give additional comments beyond their presentations, sometimes corrections, and also shared open problems as well as general questions.

In particular, an expert in the deterministic PDEs in fluid raised a question of extending the well known result concerning the ergodicity of the two-dimensional Navier-Stokes equations with a hypoelliptic noise on a two-dimensional torus ([3]) to a more general bounded domain. The group of experts on the fluid PDEs forced by noise were able to respond by elaborating on the results which have been obtained, those which have not obtained but are possibly achievable, and finally those which are very difficult and completely open. Again, this was a very good dialogue which was only possible because the groups of researchers on fluid PDEs from complementary perspectives have come together and felt comfortable enough to share their questions honestly and hear feedback directly with one another.

Moreover, on the Thursday 16:30 - 17:30 session, led particularly by the experts with much experience such as Prof. Peter Constantin, Prof. Edriss Titi and Prof. Charles Doering, participants were encouraged to share their open problems, write them on the board and hear thoughts from other participants. In particular, problems concerning exponential decay of a solution to an equation in mixing by Prof. Gautam Iyer, point vortex dynamics by Prof. Marco Romito and numerical analysis by Prof. Adam Larios were written on the board. There were many more problems proposed as very interesting, difficult but perhaps possible candidates for research in the near future, and the board was filled completely and very quickly. Thereafter, many shared their thoughts about relevant results, possible techniques to tackle such problems and difficulty in a very friendly manner. Such a fruitful discussion of sharing problems rather than new results can give us all a better understanding of the directions of research which will be of interest to a wide community of researchers.

Another noteworthy advantage of our workshop was that due to the prestige of the BIRS, its kind accommodation, and the fact that this was not a conference over a few days but an entire week, we were able to attract many renown experts from overseas outside the United States and Canada such as Asia and Europe. Even outside lectures times, during the tours by BIRS, on our way to daily meals at the Vistas Dining Room, walks to the Town of Banff in the evening as well as the trip to Lake Louise on the free Wednesday afternoon, the participants were able to interact with researchers in same field of study with whom they are typically able to communicate only through E-mails or reading papers. In a very relaxed, friendly fashion, every participant was able to share their research results, problems they are facing, and hear feedback from many others, all of which have been extremely helpful and productive for not only research but also motivation and gaining wider perspectives. It is actually very important to spend time together in such non-research activities and have conversations even outside research. It may lead to long term collaboration as well as friendship, helping in particular the young participants build their careers as mathematicians in this international community of fluid dynamics PDEs.

5   Outcome of the Meeting

This 5-day workshop has opened a new bridge between the deterministic and stochastic community of fluid PDEs. Every participant has enjoyed their time at BIRS fully, as confirmed verbally directly from participants. Due to excellent presentations and deep discussions which took place, many long-term collaborations among the participants who met each others for the first time in this workshop are strongly expected.

References

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