Free energy landscapes in spherical spin glasses

Eliran Subag

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NYU | COURANT INSTITUTE OF MATHEMATICAL SCIENCES

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Mathematics & Physical Sciences

Spherical spin glasses models

- Given a **"mixture"** $\nu(x) = \sum_{p=1}^{\infty} \gamma_p^2 x^p$, define

$$H_N : \mathbb{S}^N \to \mathbb{R}, \quad \forall N \geq 1$$

$$H_N(\mathbf{x}) = \sum_p \gamma_p \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p}, \quad \mathbf{x} \in \mathbb{S}^N,$$

where $J_{i_1,\ldots,i_p} \sim \text{Normal}(0, N)$ i.i.d.

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  where $J_{i_1,\dots,i_p} \sim \text{Normal}(0, N)$ i.i.d.

- $H_N(\mathbf{x})$ is the Gaussian process that satisfies

$$\mathbb{E} H_N(\mathbf{x}) = 0, \quad \mathbb{E} H_N(\mathbf{x}) H_N(\mathbf{y}) = N\nu(\langle \mathbf{x}, \mathbf{y} \rangle).$$

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- Models with 'Ising spins': replace $\mathbb{S}^N$ with $\Sigma_N = \{\pm 1\}^N$ (and normalize).

The free energy and Parisi's formula

- **Free energy** (at inverse-temperature $\beta > 0$):

$$F_{N,\beta} = \frac{1}{N} \log Z_{N,\beta} = \frac{1}{N} \log \int_{\mathbb{S}^N} e^{\beta H_N(\mathbf{x})} d\mathbf{x}.$$

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Parisi's formula (Parisi '79 [cube], Crisanti-Sommers '92 [sphere])

$$\lim_{N\to\infty} \mathbb{E} F_{N,\beta} = \min_{\mu \in M_1([0,1])} \mathcal{P}_{\nu,\beta}(\mu).$$

- Upper bound proved by **Guerra '03**, lower bound by **Talagrand '06** for even models ($\gamma_p = 0$ for odd p);
  following **Panchenko's '13** proof of ultrametricity, the formula was extended by **Panchenko '14** [cube] and **Chen '13** [sphere] to general mixed models.

The free energy and Parisi's formula

- **Gibbs measure** (at inverse temperature $\beta > 0$):

$$G_N(A) = G_{N,\beta}(A) = \frac{1}{Z_{N,\beta}} \int_A e^{\beta H_N(\mathbf{x})} d\mathbf{x}, \qquad A \subset \mathbb{S}^N.$$

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- For "generic" models,[1] the overlap distribution converges

$$\mu_P(\cdot) = \lim_{N\to\infty} \mathbb{E} G_N^{\otimes 2}\big(\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \in \cdot\,\big),$$

and the limit is the minimizer in Parisi's formula

$$\lim_{N\to\infty} \mathbb{E} F_{N,\beta} = \mathcal{P}_{\nu,\beta}(\mu_P) = \min_{\mu \in M_1([0,1])} \mathcal{P}_{\nu,\beta}(\mu).$$

[1] $\sum_{p \text{ odd}} p^{-1} \mathbf{1}\{\gamma_p \neq 0\} = \sum_{p \text{ even}} p^{-1} \mathbf{1}\{\gamma_p \neq 0\} = \infty$

TAP approach I: the TAP equations

- **Thouless-Anderson-Palmer '77** consider, for the **SK** model ($\nu(x) = x^2$, Ising spins), the local magnetizations $m = (m_i)_{i \le N}$,

$$m := \langle \mathbf{x} \rangle = \int \mathbf{x} dG_{N,\beta},$$

and derived self-consistency equations of the form

$$m_i \approx tanh\Big(\frac{2\beta}{\sqrt{N}} \sum_j J_{ij} m_j + h - \beta^2(1 - q^2) m_i\Big), \qquad i = 1, \ldots, N.$$

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- This was proved rigorously:
  - **Talagrand '03** and **Chatterjee '10** – high-temp. SK,
  - **Auffinger-Jagannath '16** – generic Ising models, at any temp. restricted to pure-states.
  - **Bolthausen '14** proved that in the high-temp. SK model, the unique solution can be obtained as the limit of certain iterative equations.

TAP approach II: the TAP free energy

- **TAP '77** also associate to the magnetization a free energy of the from

$$F_{\mathrm{TAP}}(m) = \frac{\beta}{N} H_N(m) + f(m),$$

which under a certain convergence condition on $\|m\|$ should give

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- At low temp., there are exp. many solutions ('complexity'> 0).
  For spherical pure p-spin ($\nu(x) = x^p$) this is rigorous:
  Auffinger-Ben Arous-Cerny '12, **Auffinger-Ben Arous '13** – annealed, 1st moment; **S. '17, Ben Arous-S-Zeitouni '18** – quenched, 2nd mom.

TAP approach II: the TAP free energy

The general idea in physics about the low temp phase:

think of $F_{\mathrm{TAP}}(m)$ as a TAP-free-energy on the space of magnetizations;

$$\text{crt. pts. of } F_{\mathrm{TAP}}(m) \leftrightsquigarrow \text{TAP solutions} \leftrightsquigarrow \text{`TAP states'}$$

$$\text{weight of state } m^{\alpha} \leftrightsquigarrow e^{NF_{\mathrm{TAP}}(m^{\alpha})}$$

$$Z_{N,\beta} = e^{NF_{N,\beta}} \approx \sum_{\alpha \leq e^{cN}} e^{NF_{\mathrm{TAP}}(m^{\alpha})}$$

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Focus of the talk: introduce and analyze a free energy landscape

$$\{\mathbf{x} : \|\mathbf{x}\| < 1\} \to \mathbb{R},$$

$$\mathbf{x} \mapsto \mathrm{Band}(\mathbf{x}) \subset \mathbb{S}^N \mapsto F(\mathbf{x}) = \frac{1}{N} \log \int_{\mathrm{Band}(\mathbf{x})} e^{\beta H_N(\mathbf{y})} d\mathbf{y}.$$

(In fact, we'll need to define another free energy
$\mathrm{Band}(\mathbf{x}) \mapsto \tilde{F}(\mathbf{x})$ to get the full picture...)

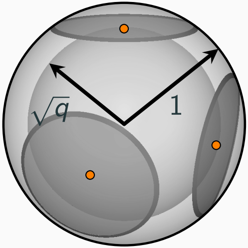
TAP formula for the free energy

TAP formula

Take $q \in (0,1)$ and some $\|\mathbf{x}\| = \sqrt{q}$.

Define

$$\mathrm{Band}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{S}^N : |\langle \mathbf{y} - \mathbf{x}, \mathbf{x}\rangle| < \delta_N\}.$$



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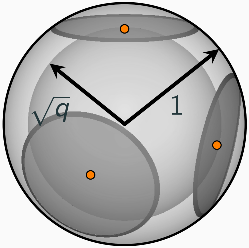

The restriction of $H_N(\cdot)$ to $\mathrm{Band}(\mathbf{x})$ is roughly a spherical model of dimension $N-1$, let ν_q be the corresponding mixture, after we remove the 1-spin component.

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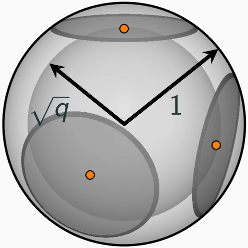

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Notation: $E_\star(q) := \lim_{N\to\infty} \frac{1}{N} \max_{\mathbf{x}\in\sqrt{q}\cdot\mathbb{S}^N} H_N(\mathbf{x})$.

TAP formula

Theorem (S. '18)

For any spherical model and $\beta > 0$:

1. $\lim_{N\to\infty} \mathbb{E} F_{N,\beta} = \max_{q\in[0,1]} \left(\beta E_\star(q) + \frac{1}{2}\log(1-q) + \lim_{N\to\infty} \mathbb{E} F_{N,\beta}(\nu_q) \right).$

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   $$q \in Q \iff \lim_{N\to\infty} \mathbb{E}F_{N,\beta}(\nu_q) = \frac{1}{2}\beta^2\big(\nu(1) - \nu(q) - (1-q)\nu'(q)\big).$$

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   $\implies$ can maximize over Q in (1) and substitute (3).

TAP formula

Corollary (S. '18)

For any spherical model and $\beta > 0$:

$$\lim_{N\to\infty} \mathbb{E}F_{N,\beta} = \max_{q\in Q}\Big[\beta E_\star(q) + \frac{1}{2}\log(1-q) + \frac{1}{2}\beta^2\big(\nu(1) - \nu(q) - (1-q)\nu'(q)\big)\Big].$$

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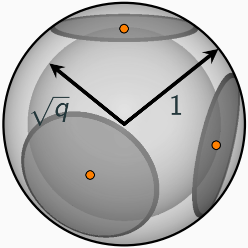
- **Belius-Kistler '18:** spherical pure 2-spin ($\nu(x) = x^2$), prove the same result as the corollary.

Free energy landscapes

Fix $q \in (0,1)$ and some $\delta_N = o(1)$.

For $\mathbf{x} \in \sqrt{q} \cdot \mathbb{S}^N = \{\mathbf{x} : \|\mathbf{x}\| = \sqrt{q}\}$,

$\mathrm{Band}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{S}^N : |\langle \mathbf{y} - \mathbf{x}, \mathbf{x} \rangle| < \delta_N\}$.



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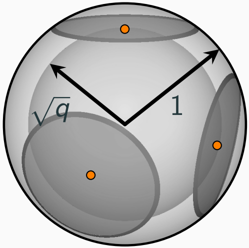

We introduce a free energy landscape on $\sqrt{q} \cdot \mathbb{S}^N$:

$$F_{N,\beta}(\mathbf{x}) = \frac{1}{N} \log \int_{\mathrm{Band}(\mathbf{x})} e^{\beta H_N(\mathbf{y})} d\mathbf{y}.$$

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Lemma: $q \in \mathrm{Supp}(\mu_P) \implies$ w.h.p. there exists a heavy band with

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Proof. [kindly communicated to me by **D. Panchenko**]
If generic, by ultrametricity, can sample many points with $\mathbf{x}_i \cdot \mathbf{x}_j \approx q$; their average is the center $\mathbf{x}$ of a good band.

Otherwise, approximate the model by a sequence of generic models and notice that this property survives the limit, due to continuity properties of μ_P in ν. □

Free energy landscapes II

$$
\begin{aligned}
F_{N,\beta}(\mathbf{x}) &= \frac{1}{N} \log \int_{\mathrm{Band}(\mathbf{x})} e^{\beta H_N(\mathbf{y})} d\mathbf{y} \\
&= \frac{1}{kN} \log \int_{(\mathrm{Band}(\mathbf{x}))^k} e^{\beta \sum_{i \le k} H_N(\mathbf{y}_i)} d\mathbf{y}_1 \cdots d\mathbf{y}_k
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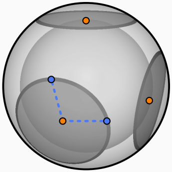
$$\mathrm{Band}(\mathbf{x},k,\rho) := \{(\mathbf{y}_1,\dots,\mathbf{y}_k)\in \mathrm{Band}(\mathbf{x})^k : \forall i\ne j,\ |(\mathbf{y}_i-\mathbf{x})\cdot(\mathbf{y}_j-\mathbf{x})|<\rho\}$$



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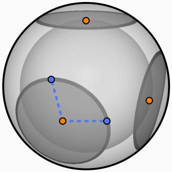

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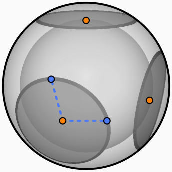

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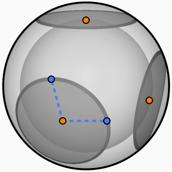

Fix some sequences $k_N\to\infty$, $\rho_N\to 0$ slowly.

$F_{N,\beta}(\mathbf{x},k_N,\rho_N)$ is the second free energy landscape we consider.

Free energy landscapes

Define the centered versions by replacing $H_N(\mathbf{y})$ by $H_N(\mathbf{y}) - H_N(\mathbf{x})$:

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and similarly define $F^{\mathrm{c}}_{N,\beta}(\mathbf{x}, k_N, \rho_N)$, so that

$$F_{N,\beta}(\mathbf{x}) = \frac{\beta}{N} H_N(\mathbf{x}) + F^{\mathrm{c}}_{N,\beta}(\mathbf{x}),$$

$$F_{N,\beta}(\mathbf{x}, k_N, \rho_N) = \frac{\beta}{N} H_N(\mathbf{x}) + F^{\mathrm{c}}_{N,\beta}(\mathbf{x}, k_N, \rho_N).$$

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For $q \in \mathrm{Supp}(\mu_P)$, w.p. going to 1: for all $\mathbf{x} \in \sqrt{q} \cdot \mathbb{S}^N$,

$$F_{N,\beta}(\mathbf{x}, k_N, \rho_N) \approx F_{N,\beta}(\mathbf{x}) \approx F_{N,\beta} \iff \frac{1}{N} H_N(\mathbf{x}) \approx E_\star(q).$$

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Free energy landscapes

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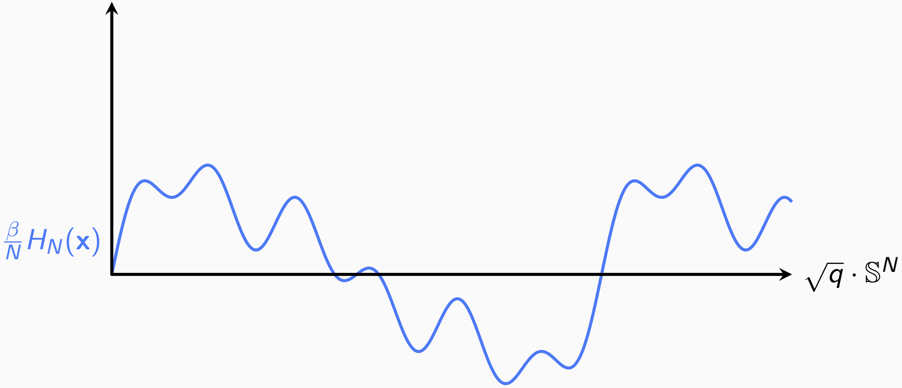

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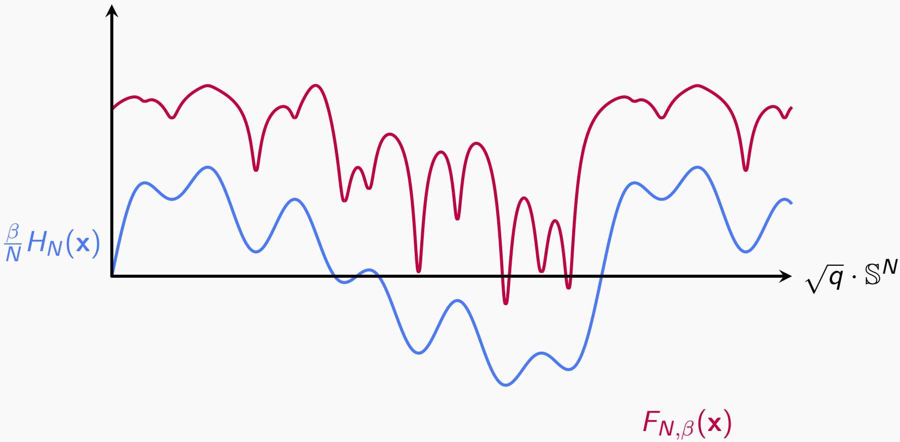

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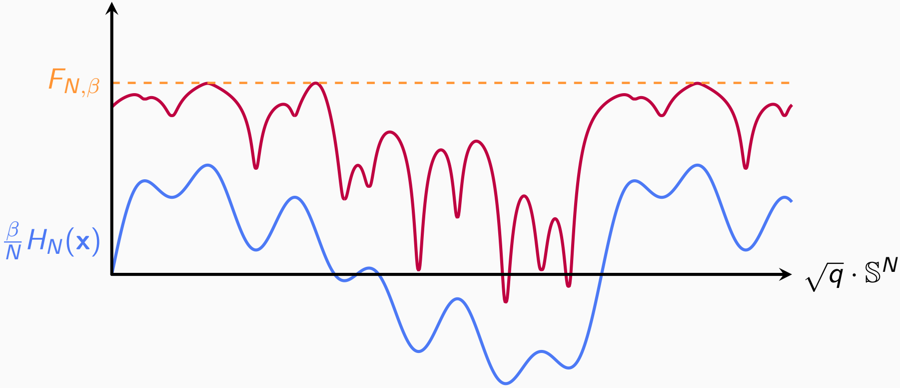
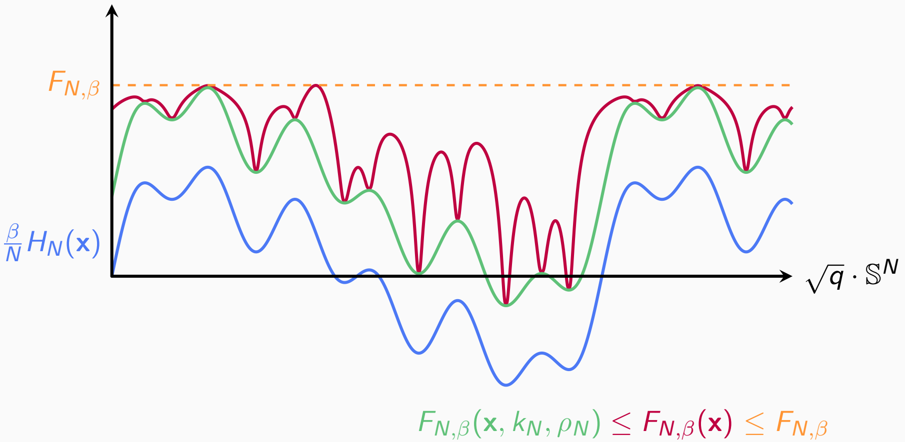


$F_{N,\beta}(\mathbf{x}) \leq F_{N,\beta}$

Free energy landscapes



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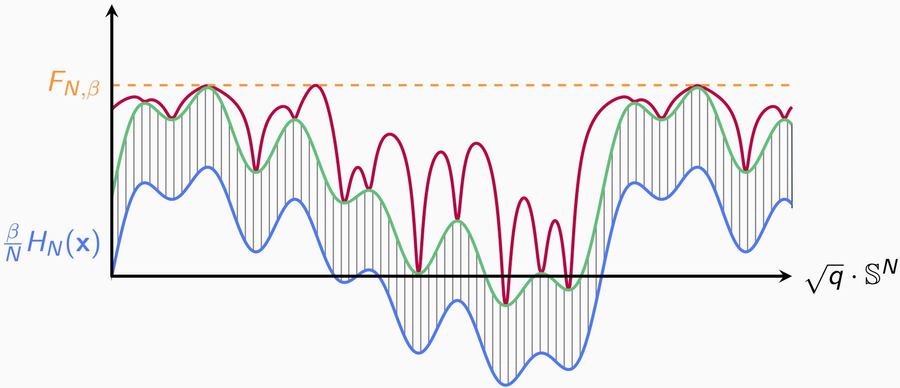

$$\frac{\beta}{N} H_N(\mathbf{x}) + \mathbb{E} F^{\mathrm{c}}_{N,\beta}(\mathbf{x}, k_N, \rho_N) \approx F_{N,\beta}(\mathbf{x}, k_N, \rho_N) \leq F_{N,\beta}(\mathbf{x}) \leq F_{N,\beta}$$

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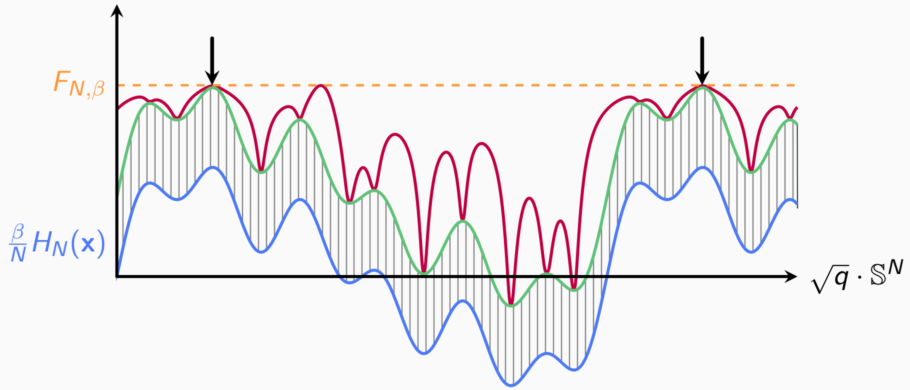

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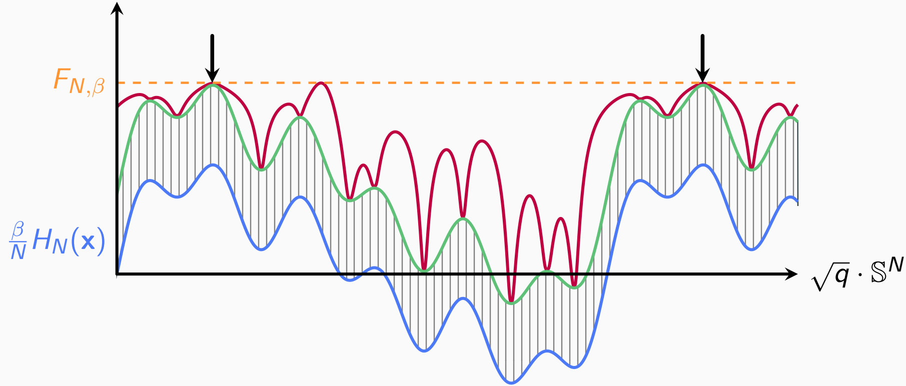

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* Essentially, $N^{-1} \nabla H_N(\mathbf{x}) \approx 0 \iff F_{N,\beta}(\mathbf{x}, k_N, \rho_N) \approx F_{N,\beta}(\mathbf{x})$.
Recall that TAP solutions $\leftrightsquigarrow \frac{\partial}{\partial m} F_{TAP}(m)$...

Proposition (S. '18)

W.p. going to 1, uniform concentration of the centered free energy:

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Discretize by continuity and use union bound to get uniformity. ☐

Proof – equality of free energies $\iff$ maximality

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Denoting equality/inequality up to $o(1)$ w.h.p. by $\approx / \lesssim$,

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For $q \in \mathrm{Supp}(\mu_P)$, $\exists \mathbf{x}_0$ as in **3)**

$$\implies F_{N,\beta} \approx \beta E_\star(q) + \mathbb{E} F^{\mathrm{c}}_{N,\beta}(\mathbf{x}_0, k_N, \rho_N).$$

Thus, $\quad \frac{1}{N} H_N(\mathbf{x}) \approx E_\star(q) \implies F_{N,\beta}(\mathbf{x}, k_N, \rho_N) \approx F_{N,\beta}(\mathbf{x}) \approx F_{N,\beta}.$

Talagrand's pure states decomposition

Pure states decomposition

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- Denote

$$q_\star = \max \mathrm{Supp}(\mu_P) < 1.$$

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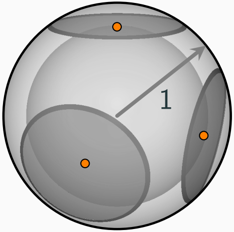
* This decomposition can also be directly derived from the ultrametricity property proved by **Panchenko '13**; and a similar one was derived by **Jagannath '17** even when $\alpha_\star = 0$.

Pure states decomposition

$$m^k = \frac{1}{G_N(A_k)} \int_{A_k} \mathbf{x} dG_N(\mathbf{x}),$$

$$\mathrm{Band}(m) = \left\{ \mathbf{x} \in \mathbb{S}^N : |\langle \mathbf{x} - \mathbf{x}_\star, \mathbf{x}_\star \rangle| \leq \delta_N \right\}.$$



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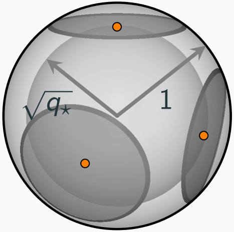

(easy) Lemma

A_k as in Talagrand's decomposition.

$$\lim_{N\to\infty} \mathbb{E}\left| \|m^k\| - \sqrt{q_\star} \right| = 0,$$

$$\lim_{N\to\infty} G_{N,\beta}(A_k \triangle \mathrm{Band}(m^k)) = 0.$$

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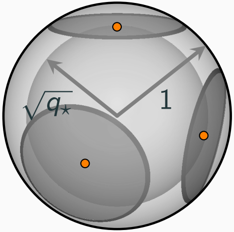

Theorem (S. '18)

A_k as in Talagrand's decomposition.

$$\lim_{N\to\infty} \frac{1}{N} H_N(m^k) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\mathbf{x}\in\sqrt{q_\star}\mathbb{S}^N} H_N(\mathbf{x}).$$

Thank You!