Numerical approximation of axisymmetric formulations for geometric evolution equations

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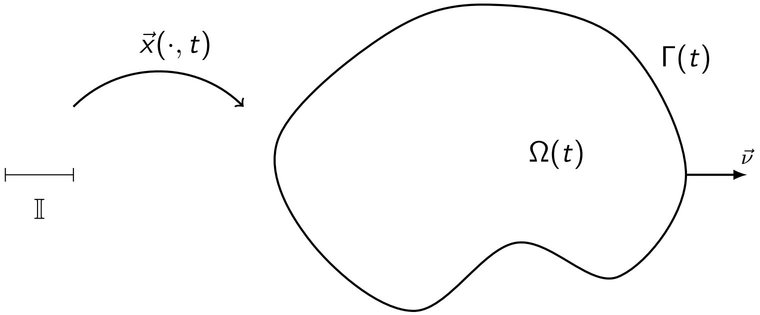
Geometric evolution equations for curves in the plane

Evolving simple (embedded - no intersections) planar closed curve $\Gamma(t)$.

Let $\vec{x}(\rho, t)$, $\rho \in \mathbb{I} := \mathbb{R}/\mathbb{Z}$ (periodic $[0, 1]$), be a parameterization of $\Gamma(t)$.



Let $\Omega(t)$ be the region bounded by $\Gamma(t)$, with outer normal $\vec{\nu}(t)$.

Geometric evolution equations for curves in the plane

On assuming that $|\vec{x}_\rho| > 0$ on $\mathbb{I}$, let s denote arclength, i.e. $\partial_s = \frac{1}{|\vec{x}_\rho|}\,\partial_\rho$.

Then the unit tangent to the curve $\Gamma(t)$ is given by

$$\vec{\tau} = \vec{x}_s = \frac{\vec{x}_\rho}{|\vec{x}_\rho|}\,.$$

As $|\vec{\tau}| = 1$, it holds that

$$0 = \left(|\vec{\tau}|^2\right)_s = (\vec{\tau}\,.\,\vec{\tau})_s = 2\,\vec{\tau}_s\,.\,\vec{\tau}\,,$$

and so $\vec{\tau}_s$ is a multiple of $\vec{\nu}$.

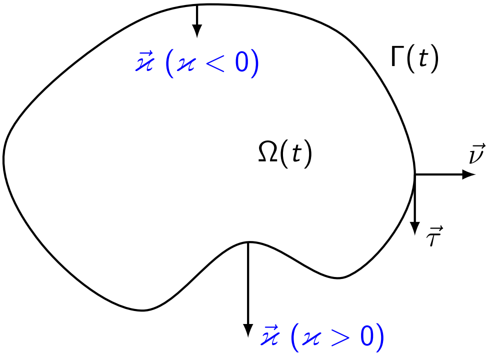
We define the curvature (vector) via

$$\varkappa\,\vec{\nu} = \vec{\varkappa} = \vec{\tau}_s = \vec{x}_{ss} = \frac{1}{|\vec{x}_\rho|}\left(\frac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho\,.$$

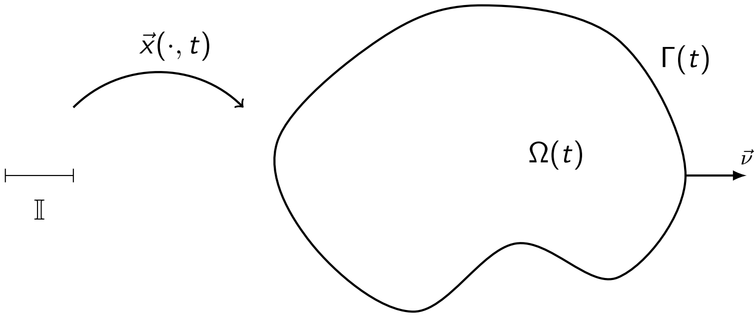
Geometric evolution equations for curves in the plane

$$\varkappa\,\vec{\nu} = \vec{\varkappa} = \vec{\tau}_s = \vec{x}_{ss} = \frac{1}{|\vec{x}_\rho|}\left(\frac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho .$$

As $\vec{\nu}$ is the outward normal, $\varkappa$ is negative if $\Omega(t)$ is locally convex.



Geometric evolution equations for curves in the plane



Clearly, the evolution of $\vec{x}(\cdot, t)$ is described by $\vec{x}_t(\cdot, t)$, which we can decompose into normal and tangential part:

$$\vec{x}_t = (\vec{x}_t \,.\, \vec{\nu})\, \vec{\nu} + (\vec{x}_t \,.\, \vec{\tau})\, \vec{\tau}\,.$$

Of course, the tangential velocity $\vec{x}_t \,.\, \vec{\tau}$ just changes the parameterization $\vec{x}$, but not $\Gamma(t)$. Hence, for the evolution of $\Gamma(t)$, it suffices to prescribe its normal velocity $\mathcal{V} := \vec{x}_t \,.\, \vec{\nu}$.

Geometric evolution equations for curves in the plane

For example:

Mean curvature flow:	$\mathcal{V} = \varkappa$	$(\text{MC})_\Gamma$
Surface diffusion:	$\mathcal{V} = -\varkappa_{ss}$	$(\text{SD})_\Gamma$

These evolution equations have important applications in e.g. Materials Science, and they have the following properties.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Gamma(t)| = -\int_{\Gamma(t)} \mathcal{V}\,\varkappa\,\mathrm{d}s = \begin{cases} -\underbrace{\int_{\Gamma(t)} \varkappa^2\,\mathrm{d}s}_{\|\mathcal{V}\|^2_{L^2(\Gamma(t))}} \leq 0 & (\text{MC})_\Gamma, \\ -\underbrace{\int_{\Gamma(t)} (\varkappa_s)^2\,\mathrm{d}s}_{\|\mathcal{V}\|^2_{H^{-1}(\Gamma(t))}} \leq 0 & (\text{SD})_\Gamma. \end{cases}$$

Here we have introduced $\int_{\Gamma(t)} f\,\mathrm{d}s = \int_{\mathbb{I}} f \circ \vec{x}\,|\vec{x}_\rho|\,\mathrm{d}\rho$, and for simplicity we often do not distinguish between f and $f \circ \vec{x}$.

Geometric evolution equations for curves in the plane

Mean curvature flow:	$\mathcal{V} = \varkappa$	$(\mathrm{MC})_\Gamma$
Surface diffusion:	$\mathcal{V} = -\varkappa_{ss}$	$(\mathrm{SD})_\Gamma$

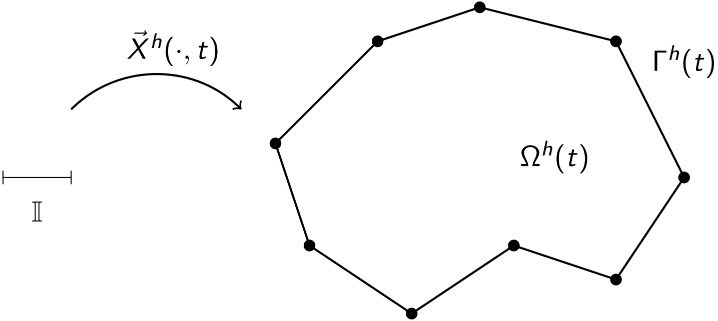
$(\mathrm{MC})_\Gamma$ is the L^2–gradient flow for the energy $|\Gamma(t)|$. (curve shortening flow)

$(\mathrm{SD})_\Gamma$ is the H^{-1}–gradient flow for the energy $|\Gamma(t)|$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Omega(t)| = \int_{\Gamma(t)} \mathcal{V}\,\mathrm{d}s = \begin{cases} \int_{\Gamma(t)} \varkappa\,\mathrm{d}s & = -2\,\pi \quad (\mathrm{MC})_\Gamma\,, \\ -\int_{\Gamma(t)} \varkappa_{ss}\,\mathrm{d}s & = 0 \quad (\mathrm{SD})_\Gamma\,. \end{cases}$$

Numerical approximation

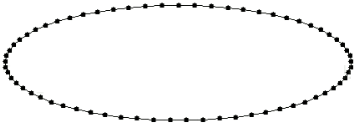
We consider front tracking methods:



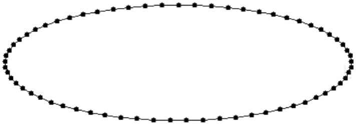
A time stepping scheme approximating $\vec{X}^h_t$ then yields a fully discrete numerical method.

In practice a crucial role is played by the discrete tangential motion (or lack thereof).

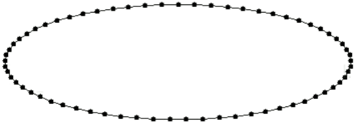
The discrete tangential motion induced by the numerical scheme can lead to coalescence in practice.



DKS



BMN



BGN

BGN formulation

Dziuk, Kuwert, Schätzle (2002) is based on the formulation

$$(\mathrm{SD})_\Gamma \quad \vec{x}_t = -\varkappa_{ss}\,\vec{\nu} \equiv -\vec{\varkappa}_{ss} - \tfrac{3}{2}\,(|\vec{\varkappa}|^2\,\vec{x}_s)_s + \tfrac{1}{2}\,|\vec{\varkappa}|^2\,\vec{\varkappa}, \quad \vec{\varkappa} = \vec{x}_{ss}\,.$$

Bänsch, Morin, Nochetto (2005) is based on the formulation

$$(\mathrm{SD})_\Gamma \quad \vec{x}_t = \mathcal{V}\,\vec{\nu}, \quad \mathcal{V} = -\varkappa_{ss}, \quad \varkappa = \vec{\varkappa}\,.\,\vec{\nu}, \quad \vec{\varkappa} = \vec{x}_{ss}\,.$$

Both approaches have in common that they evolve the parameterization $\vec{x}$ only in the *normal* direction.

We use the following formulation:

$$\vec{x}_t\,.\,\vec{\nu} = \begin{cases} \varkappa & (\mathrm{MC})_\Gamma\,, \\ -\varkappa_{ss} & (\mathrm{SD})_\Gamma\,, \end{cases} \qquad \varkappa\,\vec{\nu} = \vec{x}_{ss}\,.$$

Note that because the tangential component of the velocity $\vec{x}_t$ is not prescribed, there exists a whole *family of solutions* $\vec{x}$, even though the evolution of Γ is uniquely determined.

BGN formulation

Weak formulation:

For smooth test functions $\varphi \in V := H^1(\mathbb{I})$ and $\vec{\varphi} \in \underline{V} := [V]^2$ it holds that

$$\int_\Gamma \vec{x}_t \,.\, \vec{\nu}\, \varphi \,\mathrm{d}s = \begin{cases} \int_\Gamma \varkappa\, \varphi \,\mathrm{d}s & (\mathrm{MC})_\Gamma\,, \\ \int_\Gamma \varkappa_s\, \varphi_s \,\mathrm{d}s & (\mathrm{SD})_\Gamma\,, \end{cases} \qquad \int_\Gamma \varkappa\, \vec{\nu} \,.\, \vec{\varphi} \,\mathrm{d}s + \int_\Gamma \vec{x}_s \,.\, \vec{\varphi}_s \,\mathrm{d}s = 0\,.$$

For the discretization, we approximate $\Gamma(t_m)$ by a polygonal curve Γ^m.

- $V^h \subset V$ and $\underline{V}^h \subset \underline{V}$ are piecewise linear finite element spaces, based on the partitioning $0 = q_0 < q_1 \cdots < q_J = 1$ of $\mathbb{I}$.
- $\Gamma^m = \vec{X}^m(\mathbb{I})$ for $\vec{X}^m \in \underline{V}^h$.
- $(\cdot,\cdot)$ is the L^2–inner product on $\mathbb{I}$.
- $(\cdot,\cdot)^h$ is the mass-lumped L^2–inner product on $\mathbb{I}$, based on $\{q_j\}_{j=0}^J$.

Finite element approximation

$(\mathcal{P}_m)^h_\Gamma$: Find $(\vec{X}^{m+1}, \kappa^{m+1}) \in \underline{V}^h \times V^h$ such that

$$\left(\frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t}, \chi\, \vec{\nu}^m\, |\vec{X}^m_\rho|\right)^h - \begin{cases} \left(\kappa^{m+1}, \chi\, |\vec{X}^m_\rho|\right)^h \\ \left(\kappa^{m+1}_\rho, \chi_\rho\, |\vec{X}^m_\rho|^{-1}\right) \end{cases} = 0 \qquad \forall\, \chi \in V^h\,,$$

$$\left(\kappa^{m+1}\, \vec{\nu}^m, \vec{\eta}\, |\vec{X}^m_\rho|\right)^h + \left(\vec{X}^{m+1}_\rho, \vec{\eta}_\rho\, |\vec{X}^m_\rho|^{-1}\right) = 0 \qquad \forall\, \vec{\eta} \in \underline{V}^h\,.$$

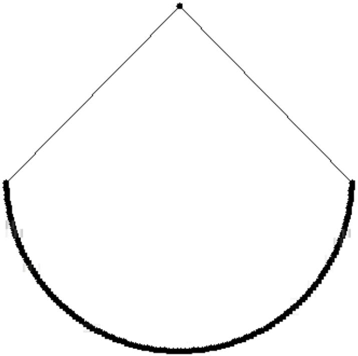
- Existence, Uniqueness
  Under mild assumptions on $\vec{X}^m$, $\exists!\ (\vec{X}^{m+1}, \kappa^{m+1}) \in \underline{V}^h \times V^h$.
- Stability For all $k = 1 \to M$ it holds that

$$|\Gamma^k| + \sum_{m=0}^{k-1} \Delta t \begin{cases} \left(|\kappa^{m+1}|^2, |\vec{X}^m_\rho|\right)^h \\ \left(|\kappa^{m+1}_\rho|^2, |\vec{X}^m_\rho|^{-1}\right) \end{cases} \leq |\Gamma^0|\,.$$

- Area conservation for $(\mathrm{SD})_\Gamma$ for a cont. in time semidiscrete scheme.
- Equidistribution of mesh points for $\vec{X}^h(t)$, where $\vec{X}^h(t)$ is not locally parallel, for any $t > 0$, for a continuous-in-time semidiscrete scheme.

Equidistribution of mesh points

Although equidistribution cannot be shown for the fully discrete scheme, (eventual) equidistribution is observed in practice.



$(J = 128,\ \Delta t = 10^{-7},\ T = 2 \times 10^{-5})$

Geometric evolution equations for surfaces in $\mathbb{R}^3$

Family of evolving hypersurfaces $(\mathcal{S}(t))_{t\in[0,T]}$, without boundary.

Let $\Omega(t)$ be the region bounded by $\mathcal{S}(t)$, with outer normal $\vec{\nu}_{\mathcal{S}}(t)$.

Let $\mathcal{V}_{\mathcal{S}}(t)$ be the normal velocity of $\mathcal{S}(t)$ in the direction $\vec{\nu}_{\mathcal{S}}(t)$, and let $k_{mean} = k_1 + k_2$ denote the mean curvature of $\mathcal{S}(t)$ (sum of principal curvatures k_1 and k_2), so that

$$k_{mean}\,\vec{\nu}_{\mathcal{S}} = \Delta_{\mathcal{S}}\,\vec{\mathrm{id}} \qquad \text{on } \mathcal{S}(t)\,,$$

where $\Delta_{\mathcal{S}} = \nabla_{\mathcal{S}}\,.\,\nabla_{\mathcal{S}}$ is the Laplace–Beltrami operator on $\mathcal{S}(t)$, with $\nabla_{\mathcal{S}}\,.$ and $\nabla_{\mathcal{S}}$ denoting the surface divergence and the surface gradient operators.

As before, for the evolution of $\mathcal{S}(t)$ it suffices to prescribe its normal velocity, e.g.

Mean curvature flow:	$\mathcal{V}_{\mathcal{S}} = k_{mean}$	on $\mathcal{S}(t)$	$(\mathrm{MC})_{\mathcal{S}}$,
Surface diffusion:	$\mathcal{V}_{\mathcal{S}} = -\Delta_{\mathcal{S}}\,k_{mean}$	on $\mathcal{S}(t)$	$(\mathrm{SD})_{\mathcal{S}}$.

Geometric evolution equations for surfaces in $\mathbb{R}^3$

Once again, $(\mathrm{MC})_{\mathcal{S}}$ and $(\mathrm{SD})_{\mathcal{S}}$ are, respectively, the L^2– and H^{-1}–gradient flows of the surface area $|\mathcal{S}(t)|$. In particular, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\mathcal{S}(t)| = -\int_{\mathcal{S}(t)} \mathcal{V}_{\mathcal{S}}\,k_{mean}\,\mathrm{d}\mathcal{H}^2 = \begin{cases} -\underbrace{\int_{\mathcal{S}(t)} k_{mean}^2\,\mathrm{d}\mathcal{H}^2}_{\|\mathcal{V}_{\mathcal{S}}\|^2_{L^2(\mathcal{S}(t))}} \leq 0\,, \\ -\underbrace{\int_{\mathcal{S}(t)} |\nabla_{\mathcal{S}}\,k_{mean}|^2\,\mathrm{d}\mathcal{H}^2}_{\|\mathcal{V}_{\mathcal{S}}\|^2_{H^{-1}(\mathcal{S}(t))}} \leq 0\,, \end{cases}$$

and, for $(\mathrm{SD})_{\mathcal{S}}$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Omega(t)| = \int_{\mathcal{S}(t)} \mathcal{V}_{\mathcal{S}}\,\mathrm{d}\mathcal{H}^2 = -\int_{\mathcal{S}(t)} \Delta_{\mathcal{S}}\,k_{mean}\,\mathrm{d}\mathcal{H}^2 = 0\,.$$

Geometric evolution equations for surfaces in $\mathbb{R}^3$

Based on the weak formulations

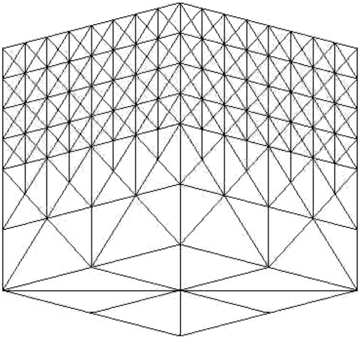
$$\int_{\mathcal{S}(t)} \mathcal{V}_{\mathcal{S}}\,\chi\,\mathrm{d}\mathcal{H}^2 = \begin{cases} \displaystyle\int_{\mathcal{S}(t)} k_{mean}\,\chi\,\mathrm{d}\mathcal{H}^2 & (\mathrm{MC})_{\mathcal{S}} \\ \displaystyle\int_{\mathcal{S}(t)} \nabla_{\mathcal{S}}\,k_{mean}\,.\,\nabla_{\mathcal{S}}\,\chi\,\mathrm{d}\mathcal{H}^2 & (\mathrm{SD})_{\mathcal{S}} \end{cases} \quad \forall\ \chi \in H^1(\mathcal{S}(t))\,,$$

$$\int_{\mathcal{S}(t)} k_{mean}\,\vec{\nu}_{\mathcal{S}}\,.\,\vec{\eta}\,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{S}(t)} \nabla_{\mathcal{S}}\,\vec{\mathrm{id}}\ :\nabla_{\mathcal{S}}\,\vec{\eta}\,\mathrm{d}\mathcal{H}^2 = 0 \qquad \forall\ \vec{\chi} \in [H^1(\mathcal{S}(t))]^3\,,$$

and similarly to $(\mathcal{P}_m)^h_\Gamma$, it is possible to introduce linear, fully discrete surface finite element approximations for $(\mathrm{MC})_{\mathcal{S}}$ and $(\mathrm{SD})_{\mathcal{S}}$ with good mesh properties, and which are unconditionally stable, see BGN (2008).

Tangential distribution of mesh points

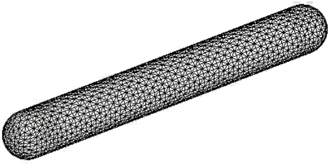
$(\mathrm{SD})_{\mathcal{S}}$

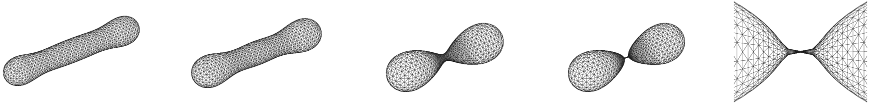


Numerical results

$(\mathrm{SD})_{\mathcal{S}}$ leading to pinch-off.

Rounded cylinder $8 \times 1 \times 1$.





Axisymmetric formulation

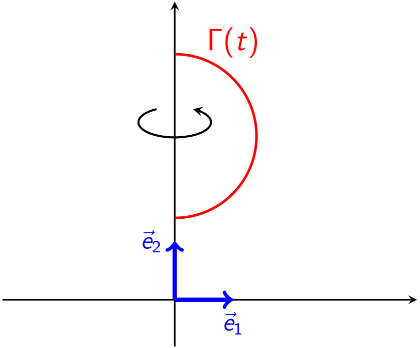
Many evolutions of interest are for surfaces that are axisymmetric, or rotationally symmetric.

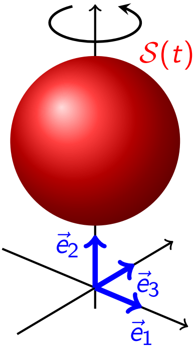
Idea: Exploit axisymmetry in these situations. Based on the BGN formulations for geometric evolution equations for curves, introduce axisymmetric finite element approximations with good distributions of mesh points.

Advantages:

- The PDEs to solve are one-dimensional, not two-dimensional.
- No surface finite elements needed.
- No restrictions due to mesh topology or mesh deformations.

Axisymmetric formulation



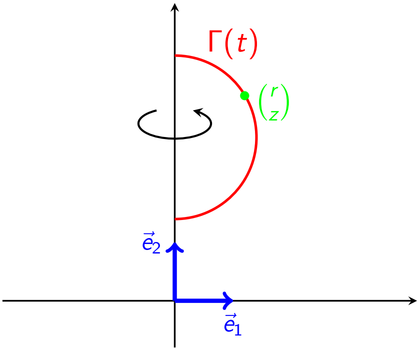


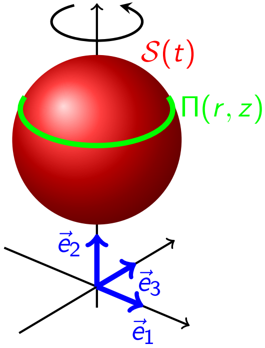
Let $\vec{x}(\cdot, t) : \overline{I} \to \Gamma(t) \subset \mathbb{R}^2$ be a parameterization of $\Gamma(t)$, where either

$$I = \mathbb{I}\,,\ \text{with } \partial I = \emptyset\,, \quad \text{or} \quad I = (0,1)\,,\ \text{with } \partial I = \{0,1\}\,.$$

In the first case, $\mathcal{S}(t)$ is a genus-1 surface, while in the latter case it is a genus-0 surface. Throughout we assume that $\vec{x}(\cdot, t) \,.\, \vec{e}_1 = 0$ on ∂I.

Axisymmetric formulation





On letting $\Pi(r,z) = \{(r\cos\theta, z, r\sin\theta)^T : \theta \in [0, 2\pi)\}$, we have that

$$\mathcal{S}(t) = \bigcup_{(r,z)^T \in \Gamma(t)} \Pi(r,z) = \bigcup_{\rho \in \overline{I}} \Pi(\vec{x}(\rho,t)) .$$

It holds that $\mathcal{V}_{\mathcal{S}} = \vec{x}_t(\rho,t) \,.\, \vec{\nu}(\rho,t)$ on $\Pi(\vec{x}(\rho,t)) \subset \mathcal{S}(t)$.

Axisymmetric formulation

For the principal curvatures of $\mathcal{S}(t)$, also called in-plane and azimuthal curvatures, it holds that

$$k_1 = \varkappa(\rho, t) \quad \text{and} \quad k_2 = -\frac{\vec{\nu}(\rho, t) \,.\, \vec{e}_1}{\vec{x}(\rho, t) \,.\, \vec{e}_1} \qquad \text{on } \Pi(\vec{x}(\rho, t)) \subset \mathcal{S}(t)\,,$$

where we recall that $\varkappa$ denotes the curvature of $\Gamma(t)$.

Clearly, for a smooth surface with bounded principal curvatures it follows that

$$\vec{\nu}(\cdot, t) \,.\, \vec{e}_1 = 0 \text{ on } \partial I \quad \iff \quad \vec{x}_\rho(\cdot, t) \,.\, \vec{e}_2 = 0 \text{ on } \partial I\,.$$

Hence, for $\rho_0 \in \partial I$, it holds that

$$\lim_{\rho \to \rho_0} \frac{\vec{\nu}(\rho, t) \,.\, \vec{e}_1}{\vec{x}(\rho, t) \,.\, \vec{e}_1} = \lim_{\rho \to \rho_0} \frac{\vec{\nu}_\rho(\rho, t) \,.\, \vec{e}_1}{\vec{x}_\rho(\rho, t) \,.\, \vec{e}_1} = \vec{\nu}_s(\rho_0, t) \,.\, \vec{\tau}(\rho_0, t) = -\varkappa(\rho_0, t)\,.$$

$$(\mathrm{MC})_{\mathcal{S}} \qquad \vec{x}_t \,.\, \vec\nu = \varkappa - \frac{\vec\nu \,.\, \vec{e}_1}{\vec{x} \,.\, \vec{e}_1}\,, \qquad \varkappa\,\vec\nu = \vec{x}_{ss} \quad \text{on } I\,,$$

with $\vec{x}_t \,.\, \vec{e}_1 = 0$ and $\vec{x}_s \,.\, \vec{e}_2 = 0$ on ∂I.

Let

$$\underline{V}_\partial = \{\vec\eta \in [H^1(I)]^2 : \vec\eta \,.\, \vec{e}_1 = 0 \quad \text{on } \partial I\}\,.$$

Weak formulation:

$(\mathcal{A})$: Let $\vec{x}(0) \in \underline{V}_\partial$. For $t \in (0, T]$ find $\vec{x}(t) \in [H^1(I)]^2$, with $\vec{x}_t(t) \in \underline{V}_\partial$, and $\varkappa(t) \in L^2(I)$ such that

$$\int_I \vec{x}_t \,.\, \vec\nu\,\chi\,|\vec{x}_\rho| \,\mathrm{d}\rho = \int_I \left(\varkappa - \frac{\vec\nu \,.\, \vec{e}_1}{\vec{x} \,.\, \vec{e}_1}\right)\chi\,|\vec{x}_\rho| \,\mathrm{d}\rho \qquad \forall\ \chi \in L^2(I)\,,$$

$$\int_I \varkappa\,\vec\nu \,.\, \vec\eta\,|\vec{x}_\rho| \,\mathrm{d}\rho + \int_I (\vec{x}_\rho \,.\, \vec\eta_\rho)\,|\vec{x}_\rho|^{-1} \,\mathrm{d}\rho = 0 \qquad \forall\ \vec\eta \in \underline{V}_\partial\,.$$

Mean curvature flow

Clearly, it holds that

$$|\mathcal{S}(t)| = E(\vec{x}(t)) := 2\,\pi \int_I \vec{x}(\rho,t)\,.\,\vec{e}_1\,|\vec{x}_\rho(\rho,t)|\,\mathrm{d}\rho\,.$$

Choosing $\vec{\eta} = (\vec{x}\,.\,\vec{e}_1)\,\vec{x}_t \in \underline{V}_\partial$ and $\chi = (\vec{x}\,.\,\vec{e}_1)\,(\vec{x}_t\,.\,\vec{\nu})$ we obtain that

$$\begin{aligned}
\frac{1}{2\,\pi}\,\frac{\mathrm{d}}{\mathrm{d}t}\,E(\vec{x}(t)) &= \int_I \vec{x}_t\,.\,\vec{e}_1\,|\vec{x}_\rho| + \vec{x}\,.\,\vec{e}_1\,\frac{(\vec{x}_t)_\rho\,.\,\vec{x}_\rho}{|\vec{x}_\rho|}\,\mathrm{d}\rho \\
&= \int_I \vec{x}_t\,.\,[\vec{e}_1 - (\vec{e}_1\,.\,\vec{\tau})\,\vec{\tau}]\,|\vec{x}_\rho|\,\mathrm{d}\rho - \int_I (\vec{x}\,.\,\vec{e}_1)\,\varkappa\,\vec{\nu}\,.\,\vec{x}_t\,|\vec{x}_\rho|\,\mathrm{d}\rho \\
&= \int_I (\vec{x}_t\,.\,\vec{\nu})\,(\vec{e}_1\,.\,\vec{\nu})\,|\vec{x}_\rho|\,\mathrm{d}\rho - \int_I (\vec{x}\,.\,\vec{e}_1)\,\varkappa\,\vec{x}_t\,.\,\vec{\nu}\,|\vec{x}_\rho|\,\mathrm{d}\rho \\
&= -\int_I \vec{x}\,.\,\vec{e}_1\left[\varkappa - \frac{\vec{\nu}\,.\,\vec{e}_1}{\vec{x}\,.\,\vec{e}_1}\right]\vec{x}_t\,.\,\vec{\nu}\,|\vec{x}_\rho|\,\mathrm{d}\rho \\
&= -\int_I \vec{x}\,.\,\vec{e}_1\,(\vec{x}_t\,.\,\vec{\nu})^2\,|\vec{x}_\rho|\,\mathrm{d}\rho \leq 0\,.
\end{aligned}$$

Unforuntately, this cannot be mimicked at the discrete level.

Given a $\kappa^{m+1} \in V^h$, we define $\mathfrak{K}^m(\kappa^{m+1}) \in V^h$ such that

$$[\mathfrak{K}^m(\kappa^{m+1})](q_j) = \begin{cases} \dfrac{\vec{\omega}^m(q_j) \,.\, \vec{e}_1}{\vec{X}^m(q_j) \,.\, \vec{e}_1} & q_j \in \bar{I} \setminus \partial I\,, \\ -\kappa^{m+1}(q_j) & q_j \in \partial I\,, \end{cases}$$

where the vertex normal $\vec{\omega}^m \in \underline{V}^h$ is the mass-lumped L^2–projection of the normal $\vec{\nu}^m$ of Γ^m onto $\underline{V}^h$.

$(\mathcal{A}_m)^h$: Find $\vec{X}^{m+1} \in \underline{V}^h_\partial = \underline{V}^h \cap \underline{V}_\partial$ and $\kappa^{m+1} \in V^h$ such that

$$\left(\frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t}, \chi\, \vec{\nu}^m\, |\vec{X}^m_\rho|\right)^h = \left(\kappa^{m+1} - \mathfrak{K}^m(\kappa^{m+1}), \chi\, |\vec{X}^m_\rho|\right)^h \quad \forall\, \chi \in V^h\,,$$

$$\left(\kappa^{m+1}\, \vec{\nu}^m, \vec{\eta}\, |\vec{X}^m_\rho|\right)^h + \left(\vec{X}^{m+1}_\rho, \vec{\eta}_\rho\, |\vec{X}^m_\rho|^{-1}\right) = 0 \quad \forall\, \vec{\eta} \in \underline{V}^h_\partial\,.$$

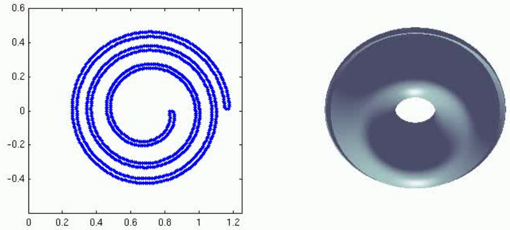
Properties of the scheme $(\mathcal{A}_m)^h$:

- Existence, Uniqueness
  Under mild assumptions on $\vec{X}^m$, $\exists!\ (\vec{X}^{m+1}, \kappa^{m+1}) \in \underline{V}^h \times V^h$.
- No Stability proof
  Even for $\partial I = \emptyset$, it does not seem possible to prove stability. However, in practice the discrete energy is always monotonically deceasing.
- Equidistribution of mesh points for $\vec{X}^h(t)$, where $\vec{X}^h(t)$ is not locally parallel, for any $t > 0$, for a continuous-in-time semidiscrete scheme.

Mean curvature flow

Numerical result for $(\mathcal{A}_m)^h$

Unwinding spiral torus.



$(J = 1024,\ \Delta t = 10^{-7},\ T = 0.0267)$

Mean curvature flow

Idea for stable scheme: Use the mean curvature of $\mathcal{S}(t)$,

$$\varkappa_{\mathcal{S}} = \varkappa - \frac{\vec{\nu}\,.\,\vec{e}_1}{\vec{x}\,.\,\vec{e}_1} \quad \text{on } I\,,$$

as a variable in the weak formulation, where we note that

$$\begin{aligned}
(\vec{x}\,.\,\vec{e}_1)\,\varkappa_{\mathcal{S}}\,\vec{\nu} &= (\vec{x}\,.\,\vec{e}_1)\,\varkappa\,\vec{\nu} - (\vec{e}_1\,.\,\vec{\nu})\,\vec{\nu} = (\vec{x}\,.\,\vec{e}_1)\,\vec{\varkappa} + (\vec{e}_1\,.\,\vec{\tau})\,\vec{\tau} - \vec{e}_1 \\
&= (\vec{x}\,.\,\vec{e}_1)\,\vec{\tau}_s + (\vec{x}_s\,.\,\vec{e}_1)\,\vec{\tau} - \vec{e}_1 = [(\vec{x}\,.\,\vec{e}_1)\,\vec{\tau}]_s - \vec{e}_1 \\
&= [(\vec{x}\,.\,\vec{e}_1)\,\vec{x}_s]_s - \vec{e}_1\,.
\end{aligned}$$

$(\mathcal{C})$: Let $\vec{x}(0) \in \underline{V}_\partial$. For $t \in (0, T]$ find $\vec{x}(t) \in [H^1(I)]^2$, with $\vec{x}_t(t) \in \underline{V}_\partial$, and $\varkappa_{\mathcal{S}}(t) \in L^2(I)$ such that

$$\int_I (\vec{x}\,.\,\vec{e}_1)\,(\vec{x}_t\,.\,\vec{\nu})\,\chi\,|\vec{x}_\rho|\,\mathrm{d}\rho = \int_I (\vec{x}\,.\,\vec{e}_1)\,\varkappa_{\mathcal{S}}\,\chi\,|\vec{x}_\rho|\,\mathrm{d}\rho \quad \forall\ \chi \in L^2(I)\,,$$

$$\int_I (\vec{x}\,.\,\vec{e}_1)\,\varkappa_{\mathcal{S}}\,\vec{\nu}\,.\,\vec{\eta}\,|\vec{x}_\rho|\,\mathrm{d}\rho + \int_I \left[\vec{\eta}\,.\,\vec{e}_1 + \vec{x}\,.\,\vec{e}_1\,\frac{\vec{x}_\rho\,.\,\vec{\eta}_\rho}{|\vec{x}_\rho|^2}\right]|\vec{x}_\rho|\,\mathrm{d}\rho = 0 \quad \forall\ \vec{\eta} \in \underline{V}_\partial\,.$$

Mean curvature flow

Choosing $\vec{\eta} = \vec{x}_t$ and $\chi = \varkappa_{\mathcal{S}}$ yields that

$$\frac{1}{2\,\pi}\,\frac{\mathrm{d}}{\mathrm{d}t}\,E(\vec{x}(t)) = \int_I \left[\vec{x}_t\,.\,\vec{e}_1 + \vec{x}\,.\,\vec{e}_1\,\frac{(\vec{x}_t)_\rho\,.\,\vec{x}_\rho}{|\vec{x}_\rho|^2}\right]|\vec{x}_\rho|\;\mathrm{d}\rho$$

$$= -\int_I (\vec{x}\,.\,\vec{e}_1)\,(\vec{x}_t\,.\,\vec{\nu})\,\varkappa_{\mathcal{S}}\,|\vec{x}_\rho|\;\mathrm{d}\rho$$

$$= -\int_I \vec{x}\,.\,\vec{e}_1\,|\varkappa_{\mathcal{S}}|^2\,|\vec{x}_\rho|\;\mathrm{d}\rho\,.$$

This stability proof goes directly across to the natural semidiscrete scheme $(\mathcal{C}_h)$, i.e.

$$\frac{1}{2\,\pi}\,\frac{\mathrm{d}}{\mathrm{d}t}\,E(\vec{X}^h(t)) = -\left(\vec{X}^h\,.\,\vec{e}_1\,|\kappa^h_{\mathcal{S}}|^2, |\vec{X}^h_\rho|\right) \leq 0\,.$$

$(\mathcal{C}_{m,\star})$: Let $\vec{X}^0 \in \underline{V}^h_\partial$. For $m = 0, \ldots, M-1$, find $\vec{X}^{m+1} \in \underline{V}^h_\partial$ and $\kappa_\mathcal{S}^{m+1} \in V^h$ such that

$$\left(\vec{X}^m \,.\, \vec{e}_1 \, \frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t}, \chi \, \vec{\nu}^m \, |\vec{X}^m_\rho|\right) = \left((\vec{X}^m \,.\, \vec{e}_1) \, \kappa_\mathcal{S}^{m+1}, \chi \, |\vec{X}^m_\rho|\right) \qquad \forall \, \chi \in V^h \,,$$

$$\left((\vec{X}^m \,.\, \vec{e}_1) \, \kappa_\mathcal{S}^{m+1} \, \vec{\nu}^m, \vec{\eta} \, |\vec{X}^m_\rho|\right) + \left(\vec{\eta} \,.\, \vec{e}_1, |\vec{X}^{m+1}_\rho|\right) + \left((\vec{X}^m \,.\, \vec{e}_1) \, \vec{X}^{m+1}_\rho, \vec{\eta}_\rho \, |\vec{X}^m_\rho|^{-1}\right) = 0 \quad \forall \, \vec{\eta} \in \underline{V}^h_\partial \,.$$

$(\mathcal{C}_{m,\star})$ is a (mildly) nonlinear scheme. The nonlinearity is necessary in order to be able to prove stability for the fully discrete scheme, via choosing $\chi = \Delta t \, \kappa_\mathcal{S}^{m+1}$ and $\vec{\eta} = \vec{X}^{m+1} - \vec{X}^m \in \underline{V}^h_\partial$.

Properties of the scheme $(\mathcal{C}_{m,\star})$:

- No Existence, Uniqueness proof
  Nonlinear scheme. In practice, a Newton method always converges within three iterations.
- Stability

$$E(\vec{X}^{m+1}) + 2\,\pi\,\Delta t \left(\vec{X}^m \,.\, \vec{e}_1 \,|\kappa_{\mathcal{S}}^{m+1}|^2, |\vec{X}^m_\rho|\right) \leq E(\vec{X}^m)\,.$$

- Nontrivial tangential motion
  The ratio

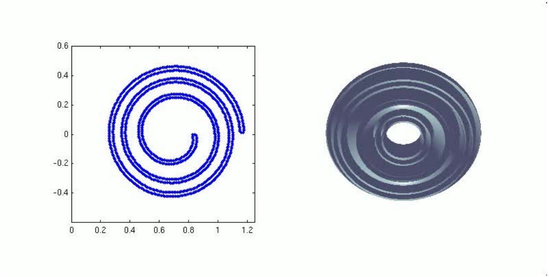
$$\mathfrak{r}^m = \frac{\max_{j=1\to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|}{\min_{j=1\to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|}$$

  of largest element/smallest element of Γ^m is bounded in practice. The ratio becomes smaller for smaller time steps, but is always significantly larger than 1.

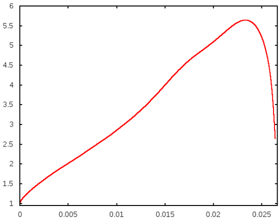
Mean curvature flow

Numerical result for $(\mathcal{C}_{m,\star})$

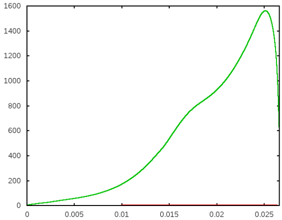
Unwinding spiral torus. $(J = 1024,\ \Delta t = 10^{-7},\ T = 0.0267)$



$\mathfrak{r}^m$ for $(\mathcal{A}_m)^h$:



$\mathfrak{r}^m$ for $(\mathcal{C}_{m,\star})$:



Surface diffusion

$$\mathcal{V}_\mathcal{S} = -\Delta_\mathcal{S}\, k_{mean} \quad \text{on} \quad \mathcal{S}(t)$$

On recalling the weak formulation

$$\int_{\mathcal{S}(t)} \mathcal{V}_\mathcal{S}\, \chi \,\mathrm{d}\mathcal{H}^2 = \int_{\mathcal{S}(t)} \nabla_\mathcal{S}\, k_{mean} \,.\, \nabla_\mathcal{S}\, \chi \,\mathrm{d}\mathcal{H}^2 \quad \forall\ \chi \in H^1(\mathcal{S}(t))\,,$$

and on noting that

$$\nabla_\mathcal{S}\, k_{mean} = [\varkappa_S(\rho, t)]_s\, \vec\tau \qquad \text{on } \Pi(\vec x(\rho, t)) \subset \mathcal{S}(t)\,,$$

we obtain the following weak formulation in the axisymmetric setting:
$(\mathcal{F})$: Let $\vec x(0) \in \underline{V}_\partial$. For $t \in (0, T]$ find $\vec x(t) \in [H^1(I)]^2$, with $\vec x_t(t) \in \underline{V}_\partial$, and $\varkappa_\mathcal{S}(t) \in H^1(I)$ such that

$$\int_I (\vec x \,.\, \vec e_1)\, (\vec x_t \,.\, \vec\nu)\, \chi\, |\vec x_\rho| \,\mathrm{d}\rho = \int_I (\vec x \,.\, \vec e_1)\, [\varkappa_\mathcal{S}]_\rho\, \chi_\rho\, |\vec x_\rho|^{-1} \,\mathrm{d}\rho \quad \forall\ \chi \in H^1(I)\,,$$

$$\int_I (\vec x \,.\, \vec e_1)\, \varkappa_\mathcal{S}\, \vec\nu \,.\, \vec\eta\, |\vec x_\rho| \,\mathrm{d}\rho + \int_I \left[\vec\eta \,.\, \vec e_1 + \vec x \,.\, \vec e_1\, \frac{\vec x_\rho \,.\, \vec\eta_\rho}{|\vec x_\rho|^2}\right] |\vec x_\rho| \,\mathrm{d}\rho = 0 \quad \forall\ \vec\eta \in \underline{V}_\partial\,.$$

Surface diffusion

Integration by parts yields the following strong formulation:

$$(\mathrm{SD})_{\mathcal{S}} \qquad \vec{x}_t \,.\, \vec{\nu} = -\frac{1}{\vec{x}\,.\,\vec{e}_1}\,[\vec{x}\,.\,\vec{e}_1\,[\varkappa_{\mathcal{S}}]_s]_s = -[\varkappa_{\mathcal{S}}]_{ss} - \frac{\vec{x}_s\,.\,\vec{e}_1}{\vec{x}\,.\,\vec{e}_1}\,[\varkappa_{\mathcal{S}}]_s \quad \text{on } I\,,$$

with $\vec{x}_t \,.\, \vec{e}_1 = 0$ and $\vec{x}_s \,.\, \vec{e}_2 = (\varkappa_{\mathcal{S}})_s = 0$ on ∂I.

Of course, choosing $\chi = 2\,\pi$ in $(\mathcal{F})$ yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Omega(t)| = \int_{\mathcal{S}(t)} \mathcal{V}_{\mathcal{S}}\,\mathrm{d}\mathcal{H}^2 = 2\,\pi \int_I (\vec{x}\,.\,\vec{e}_1)\,\vec{x}_t\,.\,\vec{\nu}\,|\vec{x}_\rho|\,\mathrm{d}\rho = 0\,.$$

Moreover, on choosing $\chi = \varkappa_{\mathcal{S}}$ and $\vec{\eta} = \vec{x}_t$ we obtain that

$$\frac{1}{2\,\pi}\,\frac{\mathrm{d}}{\mathrm{d}t}\,E(\vec{x}(t)) = -\int_I \vec{x}\,.\,\vec{e}_1\,|(\varkappa_{\mathcal{S}})_\rho|^2\,|\vec{x}_\rho|^{-1}\,\mathrm{d}\rho \leq 0\,.$$

It is possible to mimic these two properties on the discrete level.

$(\mathcal{F}_{m,\star})$: Let $\vec{X}^0 \in \underline{V}^h_\partial$. For $m = 0, \ldots, M-1$, find $\vec{X}^{m+1} \in \underline{V}^h_\partial$ and $\kappa_{\mathcal{S}}^{m+1} \in V^h$ such that

$$\left(\vec{X}^m \,.\, \vec{e}_1 \, \frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t}, \chi \, \vec{\nu}^m \, |\vec{X}^m_\rho|\right) = \left((\vec{X}^m \,.\, \vec{e}_1) \, [\kappa_{\mathcal{S}}^{m+1}]_\rho, \chi_\rho \, |\vec{X}^m_\rho|^{-1}\right) \quad \forall \chi \in V^h,$$

$$\left((\vec{X}^m \,.\, \vec{e}_1) \, \kappa_{\mathcal{S}}^{m+1} \, \vec{\nu}^m, \vec{\eta} \, |\vec{X}^m_\rho|\right) + \left(\vec{\eta} \,.\, \vec{e}_1, |\vec{X}^{m+1}_\rho|\right) + \left((\vec{X}^m \,.\, \vec{e}_1) \, \vec{X}^{m+1}_\rho, \vec{\eta}_\rho \, |\vec{X}^m_\rho|^{-1}\right) = 0 \quad \forall \vec{\eta} \in \underline{V}^h_\partial .$$

Stability proof via choosing $\chi = \Delta t \, \kappa_{\mathcal{S}}^{m+1}$ and $\vec{\eta} = \vec{X}^{m+1} - \vec{X}^m \in \underline{V}^h_\partial$ as before.

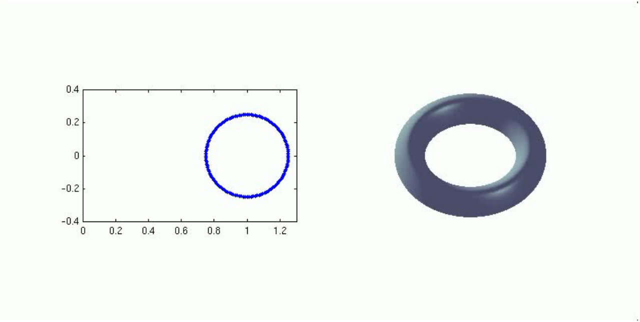
Properties of the scheme $(\mathcal{F}_{m,\star})$:

- No Existence, Uniqueness proof
  Nonlinear scheme. In practice, a Newton method always converges within three iterations.
- Stability

$$E(\vec{X}^{m+1}) + 2\,\pi\,\Delta t\left(\vec{X}^{m}\,.\,\vec{e}_1\,|[\kappa_{\mathcal{S}}^{m+1}]_\rho|^2, |\vec{X}_\rho^{m}|^{-1}\right) \leq E(\vec{X}^{m})\,.$$

- Volume conservation for continuous-in-time semidiscrete scheme $(\mathcal{F}_h)$.
- Nontrivial tangential motion
  The ratio $\mathfrak{r}^m$ of largest element/smallest element of Γ^m is bounded in practice, asymptotically approaching a value significantly larger than 1, but smaller than 10.

A torus evolving towards a sphere.



$(J = 128,\ \Delta t = 10^{-6},\ T = 0.0239)$

Surface diffusion

It holds that

$$(\mathrm{SD})_{\mathcal{S}} \quad (\vec{x}\,.\,\vec{e}_1)\,\vec{x}_t\,.\,\vec{\nu} = -\left[\vec{x}\,.\,\vec{e}_1\,[\varkappa_{\mathcal{S}}]_s\right]_s = -\left[\vec{x}\,.\,\vec{e}_1\left[\varkappa - \frac{\vec{\nu}\,.\,\vec{e}_1}{\vec{x}\,.\,\vec{e}_1}\right]_s\right]_s \quad \text{on } I\,.$$

Hence an alternative weak formulation, that will induce an equidistribution property on the discrete level, is given as follows.

$(\mathcal{E})$: Let $\vec{x}(0) \in \underline{V}_\partial$. For $t \in (0, T]$ find $\vec{x}(t) \in [H^1(I)]^2$, with $\vec{x}_t(t) \in \underline{V}_\partial$, and $\varkappa(t) \in H^1(I)$ such that

$$\int_I (\vec{x}\,.\,\vec{e}_1)\,\vec{x}_t\,.\,\vec{\nu}\,\chi\,|\vec{x}_\rho|\,\mathrm{d}\rho = \int_I \vec{x}\,.\,\vec{e}_1\left[\varkappa - \frac{\vec{\nu}\,.\,\vec{e}_1}{\vec{x}\,.\,\vec{e}_1}\right]_\rho \chi_\rho\,|\vec{x}_\rho|^{-1}\,\mathrm{d}\rho \quad \forall\, \chi \in H^1(I)\,,$$

$$\int_I \varkappa\,\vec{\nu}\,.\,\vec{\eta}\,|\vec{x}_\rho|\,\mathrm{d}\rho + \int_I (\vec{x}_\rho\,.\,\vec{\eta}_\rho)\,|\vec{x}_\rho|^{-1}\,\mathrm{d}\rho = 0 \quad \forall\,\vec{\eta} \in \underline{V}_\partial\,.$$

$(\mathcal{E}_m)^h$: Find $\vec X^{m+1} \in \underline{V}^h_\partial = \underline{V}^h \cap \underline{V}_\partial$ and $\kappa^{m+1} \in V^h$ such that

$$\left(\vec X^m \,.\, \vec e_1 \,\frac{\vec X^{m+1} - \vec X^m}{\Delta t}, \chi\, \vec\nu^m \,|\vec X^m_\rho|\right)^h$$

$$= \left(\vec X^m \,.\, \vec e_1 \left[\kappa^{m+1} - \mathfrak{K}^m(\kappa^{m+1})\right]_\rho, \chi_\rho \,|\vec X^m_\rho|^{-1}\right) \qquad \forall\, \chi \in V^h\,,$$

$$\left(\kappa^{m+1}\,\vec\nu^m, \vec\eta\,|\vec X^m_\rho|\right)^h + \left(\vec X^{m+1}_\rho, \vec\eta_\rho\,|\vec X^m_\rho|^{-1}\right) = 0 \quad \forall\, \vec\eta \in \underline{V}^h_\partial\,,$$

where we have recalled

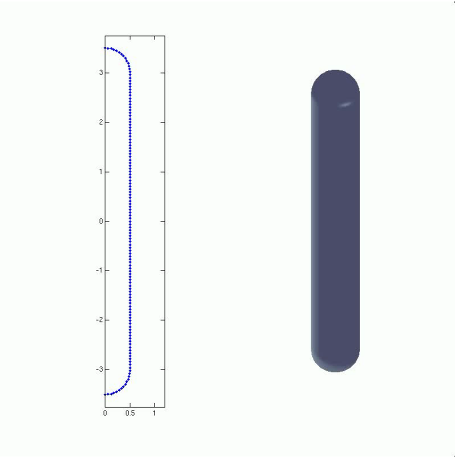
$$[\mathfrak{K}^m(\kappa^{m+1})](q_j) = \begin{cases} \dfrac{\vec\omega^m(q_j)\,.\,\vec e_1}{\vec X^m(q_j)\,.\,\vec e_1} & q_j \in \bar I \setminus \partial I\,, \\ -\kappa^{m+1}(q_j) & q_j \in \partial I\,. \end{cases}$$

Mean curvature flow

Properties of the scheme $(\mathcal{E}_m)^h$:

- Existence, Uniqueness
  Under mild assumptions on $\vec{X}^m$, $\exists!\ (\vec{X}^{m+1}, \kappa^{m+1}) \in \underline{V}^h \times V^h$.
- No Stability proof
  Even for $\partial I = \emptyset$, it does not seem possible to prove stability. However, in practice the discrete energy is always monotonically deceasing.
- Approximate volume conservation for continuous-in-time semidiscrete scheme $(\mathcal{E}_h)^h$.
- Equidistribution of mesh points for $\vec{X}^h(t)$, where $\vec{X}^h(t)$ is not locally parallel, for any $t > 0$, for a continuous-in-time semidiscrete scheme $(\mathcal{E}_h)^h$.

A rounded cylinder of dimension $7 \times 1 \times 1$ evolving to a sphere.

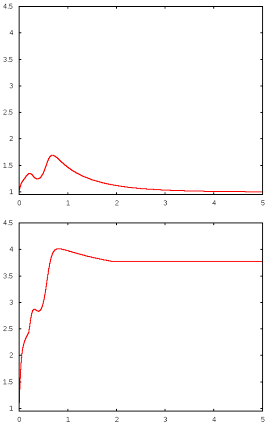


$(J = 128,\ \Delta t = 10^{-4},\ T = 0.8)$

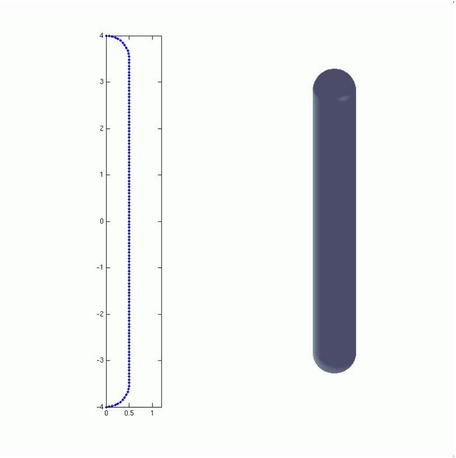
$\mathfrak{r}^m$

$(\mathcal{E}_m)^h$

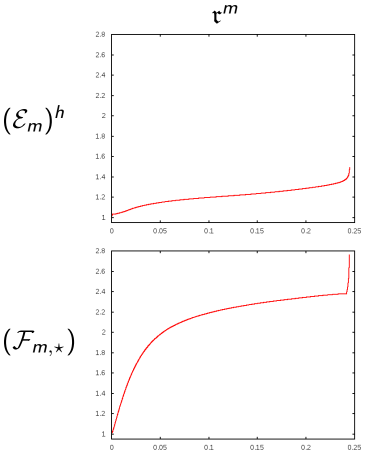
$(\mathcal{F}_{m,\star})$



A rounded cylinder of dimension $8 \times 1 \times 1$ leading to pinch-off.



$(J = 128,\ \Delta t = 10^{-4},\ T = 0.245)$



Outlook

Generalizations and further work:

- $(\text{MC})_{\mathcal{S}}$ and $(\text{SD})_{\mathcal{S}}$ for open surfaces $\mathcal{S}$ with boundary $\partial\mathcal{S}$.
  - Dirichlet boundary conditions.
  - Freeslip boundary conditions on hyperplanes parallel to $\mathbb{R} \times \{0\} \times \mathbb{R}$.
    - Contact angle conditions.
  - Freeslip boundary conditions on boundary of infinite cylinder.
    - Contact angle conditions.
- More general curvatures flows for closed surfaces $\mathcal{S}$.
  - Gauss curvature flow
  - Inverse mean curvature flow
  - Nonlinear mean curvature flows
- Willmore flow/Helfrich flow for closed surfaces $\mathcal{S}$.
- Willmore flow/Helfrich flow for open surfaces $\mathcal{S}$ with boundary $\partial\mathcal{S}$.
  - Clamped boundary conditions.
  - Navier boundary conditions.
  - Semifree boundary conditions.
  - Free boundary conditions.
- . . .

References

1. J. W. Barrett, H. Garcke, and R. Nürnberg, *A parametric finite element method for fourth order geometric evolution equations*, J. Comput. Phys., **222** (2007), 441–467.
2. ——, *On the parametric finite element approximation of evolving hypersurfaces in* $\mathbb{R}^3$, J. Comput. Phys., **227** (2008), 4281–4307.
3. ——, *Variational discretization of axisymmetric curvature flows*, (2018), arXiv 1805.04322.
4. ——, *Finite element methods for fourth order axisymmetric geometric evolution equations*, (2018), in preparation.

Choosing $\chi = \Delta t\, \kappa_{\mathcal{S}}^{m+1}$ and $\vec\eta = \vec X^{m+1} - \vec X^m \in \underline{V}_\partial^h$ yields that

$$
\begin{aligned}
-\Delta t \left(\vec X^m \,.\, \vec e_1\, |\kappa_{\mathcal{S}}^{m+1}|^2, |\vec X^m_\rho|\right) \\
&= \left(\vec X^{m+1} - \vec X^m, \vec e_1\, |\vec X^{m+1}_\rho|\right) \\
&\qquad + \left((\vec X^m \,.\, \vec e_1)\,(\vec X^{m+1}_\rho - \vec X^m_\rho), \vec X^{m+1}_\rho\, |\vec X^m_\rho|^{-1}\right) \\
&\geq \left(\vec X^{m+1} - \vec X^m, \vec e_1\, |\vec X^{m+1}_\rho|\right) + \left(\vec X^m \,.\, \vec e_1, |\vec X^{m+1}_\rho| - |\vec X^m_\rho|\right) \\
&= \left(\vec X^{m+1} \,.\, \vec e_1, |\vec X^{m+1}_\rho|\right) - \left(\vec X^m \,.\, \vec e_1, |\vec X^m_\rho|\right) \\
&= \frac{1}{2\,\pi}\, E(\vec X^{m+1}) - \frac{1}{2\,\pi}\, E(\vec X^m)\,,
\end{aligned}
$$

where we have used the inequality $(\vec a - \vec b)\,.\,\vec a \geq (|\vec a| - |\vec b|)\,|\vec b|$ for $\vec a$, $\vec b \in \mathbb{R}^2$.

Mean curvature flow

An alternative approximation considers the curvature vector of $\mathcal{S}(t)$,

$$\vec{\varkappa}_{\mathcal{S}} = \varkappa_{\mathcal{S}}\,\vec{\nu} \qquad \text{on } I\,,$$

as a variable in the weak formulation. A fully discrete scheme is then:

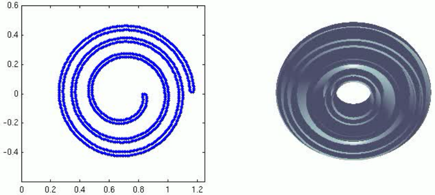
$(\mathcal{D}_{m,\star})$: Let $\vec{X}^0 \in \underline{V}^h_\partial$. For $m = 0, \ldots, M-1$, find $\vec{X}^{m+1} \in \underline{V}^h_\partial$ and $\vec{\kappa}^{m+1}_{\mathcal{S}} \in \underline{V}^h$ such that

$$\left(\vec{X}^m \,.\, \vec{e}_1\,\frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t}, \vec{\chi}\,|\vec{X}^m_\rho|\right) = \left((\vec{X}^m \,.\, \vec{e}_1)\,\vec{\kappa}^{m+1}_{\mathcal{S}}, \vec{\chi}\,|\vec{X}^m_\rho|\right) \quad \forall\ \vec{\chi} \in \underline{V}^h\,,$$

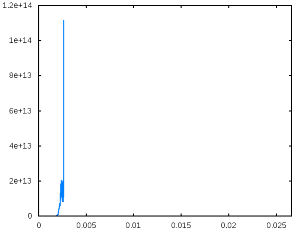
$$\left((\vec{X}^m \,.\, \vec{e}_1)\,\vec{\kappa}^{m+1}_{\mathcal{S}}, \vec{\eta}\,|\vec{X}^m_\rho|\right) + \left(\vec{\eta}\,.\,\vec{e}_1, |\vec{X}^{m+1}_\rho|\right) + \left((\vec{X}^m \,.\, \vec{e}_1)\,\vec{X}^{m+1}_\rho, \vec{\eta}_\rho\,|\vec{X}^m_\rho|^{-1}\right) = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h_\partial\,.$$

$(\mathcal{D}_{m,\star})$ can also be shown to be unconditionally stable.
However, in practice it leads to very nonuniform meshes and coalescence.

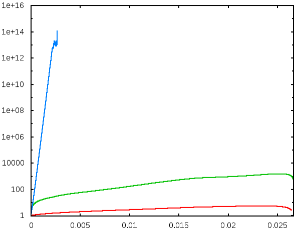
Unwinding spiral torus. $(J = 1024,\ \Delta t = 10^{-7},\ T = 0.0267)$



$\mathfrak{r}^m$ for $(\mathcal{D}_{m,\star})$, $(\mathcal{C}_{m,\star})$, $(\mathcal{A}_m)^h$



$\log_{10} \mathfrak{r}^m$ for $(\mathcal{D}_{m,\star})$, $(\mathcal{C}_{m,\star})$, $(\mathcal{A}_m)^h$



Mean curvature flow

Convergence experiment

A true solution for $(\mathrm{MC})_{\mathcal{S}}$ is given by a sphere of radius $r(t)$, with

$$r(t) = [1 - 4\,t]^{\frac{1}{2}}\,, \qquad t \in [0, \tfrac{1}{4})\,.$$

		$(\mathcal{A}_m)^h$		$(\mathcal{C}_{m,\star})$	
J	h_{Γ^0}	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC
32	1.0792e-01	7.3110e-04	—	3.7596e-03	—
64	5.3988e-02	1.8422e-04	1.990129	1.1565e-03	1.702088
128	2.6997e-02	4.6098e-05	1.998974	3.5226e-04	1.715328
256	1.3499e-02	1.1525e-05	2.000044	1.0672e-04	1.722902
512	6.7495e-03	2.8813e-06	1.999975	3.2277e-05	1.725252

$\|\Gamma - \Gamma^h\|_{L^\infty} = \max_{m=1,\ldots,M} \max_{j=0,\ldots,J} \left| |\vec{X}^m(q_j)| - r(t_m) \right|$ over the time interval $[0, \frac{1}{8}]$.

We set $\Delta t = 0.1\, h_{\Gamma^0}^2$.

A true solution for $(\mathrm{MC})_{\mathcal{S}}$ is given by a sphere of radius $r(t)$, with

$$r(t) = [1 - 4\,t]^{\frac{1}{2}}\,, \qquad t \in [0, \tfrac{1}{4})\,.$$

		$(\mathcal{A}_m)^h$		$(\mathcal{D}_{m,\star})$	
J	h_{Γ^0}	$\\|\Gamma - \Gamma^h\\|_{L^\infty}$	EOC	$\\|\Gamma - \Gamma^h\\|_{L^\infty}$	EOC
32	1.0792e-01	7.3110e-04	—	3.6916e-03	—
64	5.3988e-02	1.8422e-04	1.990129	1.0449e-03	1.822245
128	2.6997e-02	4.6098e-05	1.998974	2.9111e-04	1.844024
256	1.3499e-02	1.1525e-05	2.000044	8.0222e-05	1.859594
512	6.7495e-03	2.8813e-06	1.999975	2.1916e-05	1.872013

$\|\Gamma - \Gamma^h\|_{L^\infty} = \max_{m=1,\ldots,M} \max_{j=0,\ldots,J} \left| |\vec{X}^m(q_j)| - r(t_m) \right|$ over the time interval $[0, \frac{1}{8}]$.

We set $\Delta t = 0.1\,h_{\Gamma^0}^2$.