Viscosity solutions for the crystalline mean curvature flow

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Based on joint work with
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Generalized total variation flow

Find $u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that satisfies

$$u_t + F\Big(\nabla u, \operatorname{div}[\nabla_p \sigma(\nabla u)]\Big) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1}$$

- **crystalline anisotropy**

$$\sigma(p) = \max_{\xi_i} p \cdot \xi_i, \qquad \{\sigma \le 1\} \text{ is bounded}$$

- **ellipticity** $F \in C(\mathbb{R}^n \times \mathbb{R})$:

$$F(p,\eta) \ge F(p,\zeta) \quad \text{for all } p \in \mathbb{R}^n \text{ and } \eta \le \zeta$$

Crystalline mean curvature flow

Angenent & Gurtin ’89, Taylor ’91

$\{E_t\}_{\geq 0}$ evolves with normal velocity

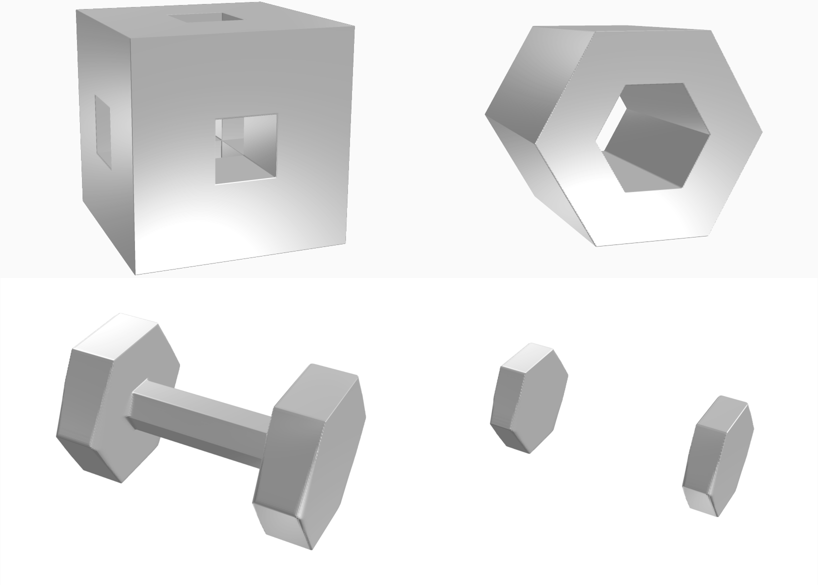
$$V = \beta(\nu)(-\kappa_\sigma + f)$$

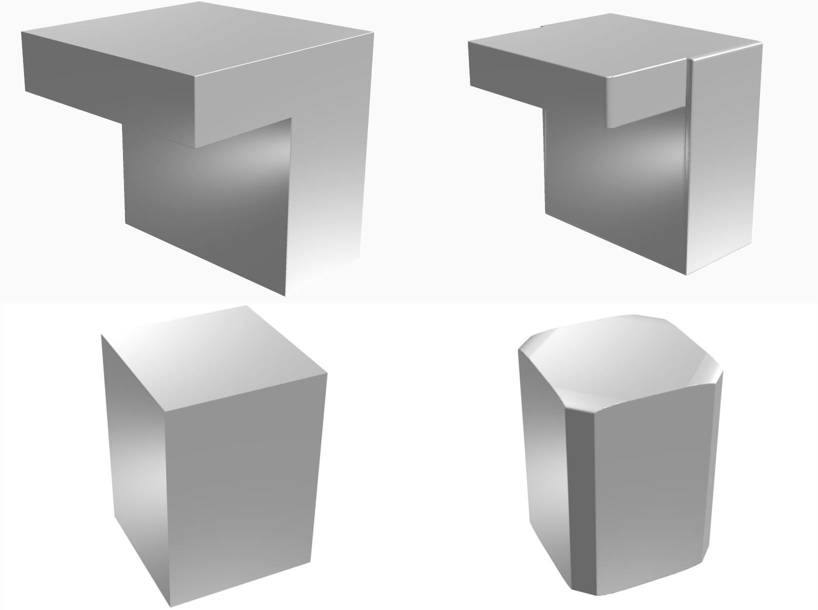
- $\kappa_\sigma := \operatorname{div}_{\partial E_t} \nabla_p \sigma(\nu)$ is the first variation of

$$\int_{\partial E_t} \sigma(\nu)\, dS$$

- level set method

$$E_t = \{u(\cdot, t) < 0\}, \qquad V = -\frac{u_t}{|\nabla u|}, \qquad \nu = \frac{\nabla u}{|\nabla u|}$$





Theory of solutions

- viscosity solutions in $n = 2$: M.-H. Giga, Y. Giga '98–, Giga, Giga, Nakayasu '13, Giga, Giga, Rybka '14
- Bellettini, Caselles, Chambolle, Novaga '05: convex initial data
- Chambolle, Morini, Ponsiglione '17, + Novaga '17 (preprint): well-posedness of the crystalline flow in arbitrary dimension $V = \beta(\nu)(-\kappa_\sigma + f(x,t))$: minimizing movements

Main result: well-posedness

$$\begin{cases} u_t + F\Big(\nabla u, \operatorname{div}[\nabla_p \sigma(\nabla u)]\Big) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = u_0. \end{cases} \tag{1}$$

$u_0 \in C(\mathbb{R}^n)$, constant outside a bounded set, $n \geq 2$

Theorem (Giga, P. '16,'18)

The problem (1) has a unique global viscosity solution $u \in C(\mathbb{R}^n \times [0,\infty))$.

Viscosity solutions

- *satisfy a comparison principle, and*
- *are stable with respect to approximation of σ by smooth anisotropies. (M.-H. Giga, Y. Giga, P. '13,'14)*

Interpretation of $\operatorname{div}[\nabla_p \sigma(\nabla u)]$

The L^2-gradient flow

$$u_t \in -\partial \mathcal{E}(u)$$

of the total variation energy

$$\mathcal{E}(\psi) = \begin{cases} \int_{\mathbb{T}^n} \sigma(D\psi) & \psi \in BV(\mathbb{T}^n) \cap L^2(\mathbb{T}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

For $\psi \in Lip(\mathbb{T}^n)$ (Moll '05)

$$-\partial \mathcal{E}(\psi) = \left\{ \operatorname{div} z \in L^2(\mathbb{T}^n) : z \in L^\infty(\mathbb{T}^n; \mathbb{R}^n),\ z \in \partial \sigma(\nabla \psi) \text{ a.e.} \right\}$$

$$\Lambda[\psi] := \operatorname{div}[\nabla_p \sigma(\nabla \psi)] := \operatorname{div} z_{\min},$$

Viscosity subsolution

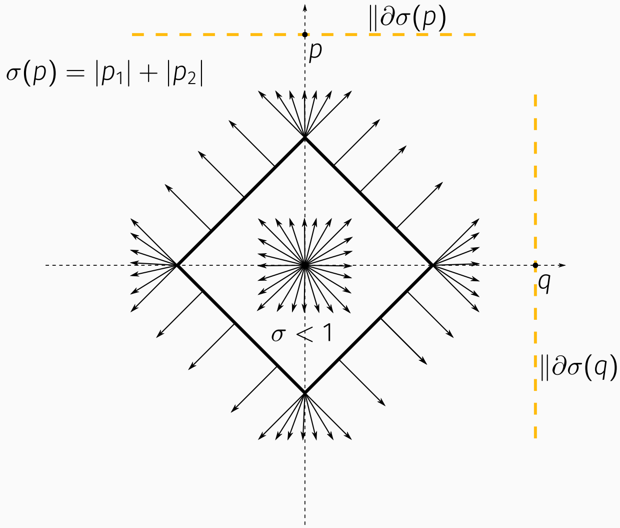
Upper semi-continuous u is a viscosity subsolution of (1) in $Q := \mathbb{R}^n \times (0, T)$ if:

- If $\varphi(x, t) = \hat{\varphi}(x) + g(t)$ with $g \in C^1((0, T))$, admissible stratified faceted function $\hat{\varphi}$, and $u - \varphi(\cdot - h, \cdot)$ has a global maximum at $(\hat{x}, \hat{t})$ for $|h|$ small then

$$\varphi_t(\hat{t}) + F(\nabla\varphi, \operatorname*{ess\,inf}_{x \in B_\delta(\hat{x})} \Lambda(\hat{\varphi})(\hat{x})) \leq 0$$

  for some $\delta > 0$.

Energy slicing



Admissible stratified faceted function at $\hat{p}$

Facet dimension $k := \dim \partial\sigma(\hat{p})$

- Stratified: Decomposition of x

$x' \in \mathbb{R}^k$	$x'' \in \mathbb{R}^{n-k}$
$\parallel \partial\sigma(\hat{p})$	$\perp \partial\sigma(\hat{p})$

$$\hat{\varphi}(x) = \psi(x') + f(x'') + \hat{p} \cdot x$$

faceted	smooth
$\psi \in Lip(\mathbb{R}^k)$	$f \in C^1(\mathbb{R}^{n-k})$
$\nabla\psi(0) = 0$	$\nabla f(0) = 0$

- Admissible: $\Lambda_{\hat{p}}[\psi]$ is defined

Comparison principle

Theorem

Suppose u and v are a subsolution and supersolution in $Q = \mathbb{R}^n \times (0, T)$, respectively.

$$\text{If} \qquad u \leq v \text{ at } t = 0 \qquad \text{then} \qquad u \leq v \qquad \text{in } Q.$$

Comparison principle: proof

Doubling of variables argument with an extra shift parameter which "flattens" the solutions at the contact point:

$$u(x,t) - v(y,s) - \frac{|x - y - \zeta|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon}, \qquad |\zeta| \le \kappa(\varepsilon).$$

Gradient at the contact point

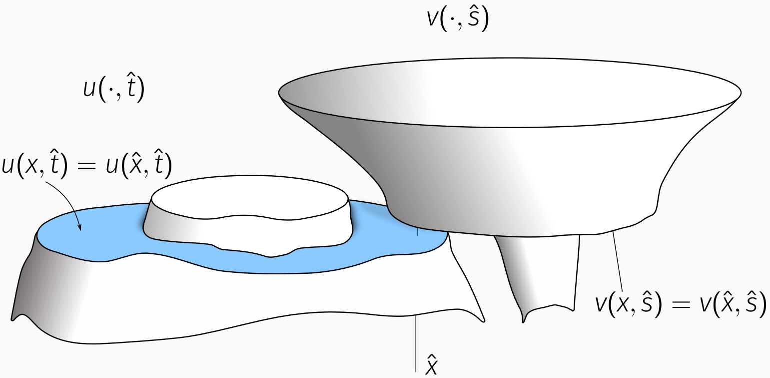
$$\frac{\hat{x} - \hat{y} - \zeta}{\varepsilon}.$$

There exists an open ball of ζ such that

$$\partial\sigma\left(\frac{\hat{x} - \hat{y} - \zeta}{\varepsilon}\right) \text{ is constant.}$$

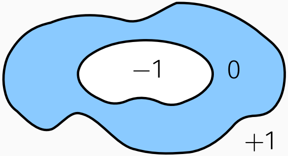
u and v are **constant** in the direction parallel to $\partial\sigma$ at the contact point.

Comparison principle: proof



Abstract facet

$$\chi : \mathbb{R}^k \to \{1, 0, -1\}$$



is called a facet

Facet χ is admissible if there is

$$\psi \in Lip(\mathbb{R}^k) \text{ such that } \qquad \operatorname{sign} \psi = \chi \qquad \text{and } \Lambda[\psi] \text{ is defined.}$$

Theorem (Comparison principle)

$$\chi_1 \le \chi_2 \qquad \Rightarrow \qquad \Lambda[\psi_1] \le \Lambda[\psi_2] \quad \textit{a.e. on } \{\chi_1 = \chi_2 = 0\}.$$

Density result

Theorem (Giga, P. '18)

For any $r > 0$, any facet χ, $\{\chi \leq 0\}$ bounded, there exists an admissible facet $\tilde{\chi}$ such that

$$\chi(x) \leq \tilde{\chi}(x) \leq \sup_{|y-x|\leq r} \chi(y)$$

Proof.

1. Assume $\chi \geq 0$.
2. Solve the resolvent problem

$$\psi + a\partial\mathcal{E}(\psi) \ni d \qquad \text{in } L^2(\mathbb{T}^k)$$

with $d(x) = \operatorname{dist}\left(x, \left\{\sup_{\frac{r}{4}} \chi = 0\right\}\right) - \operatorname{dist}\left(x, \left\{\sup_{\frac{r}{4}} \chi > 0\right\}\right)$

3. Define

$$\tilde{\chi} := \mathbf{1}_{\{\psi > 0\}}.$$

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Thank you for your attention!

1. Y. Giga, N. Požár, *Approximation of General Facets by Regular Facets with Respect to Anisotropic Total Variation Energies and Its Application to Crystalline Mean Curvature Flow*, Comm. Pure Appl. Math. **71** (2018), no. 7, 1461–1491

2. Y. Giga, N. Požár, *A level set crystalline mean curvature flow of surfaces*, Adv. Differential Equations **21** (2016), no. 7–8, 631–698