Volume preserving Mean-curvature flow with star-shaped sets

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Problem

We are interested in the volume-preserving Mean Curvature Flow, which addresses the evolution of sets Ω_t according to the motion law

$$V = -\kappa + \bar{\lambda}_t \text{ on } \Gamma_t = \partial\Omega_t,$$

where λ_t is chosen such that $\int_{\Gamma_t} V = 0$, i.e. $|\Omega_t|$ does not change over time.

When Ω_t has smooth boundaries, $\bar{\lambda}_t = \frac{1}{Per(\Omega_t)} \int_{\partial\Omega_t} \kappa d\sigma$.

(Partial) Literature

- Huisken (1984) Andrews (2001): smooth, convex flow. Exponential convergence to sphere.
- Escher-Simonett (1998) Short time existence, uniqueness of smooth solutions. Stability near the sphere.
- Smoczyk (1998), Asymptotic behavior under the assumption of star-shaped flow.
- Belletini, Caselles, Chambolle and Novaga (2009): Crystaline flow for convex sets, based on gradient flow structure.
- Mugnai, Seis, and Spadaro (2016) Global weak solutions to the volume-preserving mean-curvature flow via approximate gradient flow. Uniqueness is open.

In general the flow goes through topological changes in finite time, even in two dimensions.

Question: Are there other geometric properties besides convexity that is preserved with the flow?

Star-shapedness is expected to be preserved (see e.g. Saez-Valdinoci 2016), however this is an open question.

A geometric property

For a unit vector π, let $\Pi_\nu(s) := \{x : (x - s) \cdot \nu = 0\}$, and let $\Psi_{s,\nu}$ denote the reflection with respect to $\Pi(s)$. Let us also denote Π^+ and Π^- by the two regions divided by the hyperplane Π, with Π^- the one containing $B_\rho(0)$.

We introduce a stronger version of star-shapedness, called "ρ-reflection":

Definition

A bounded, open set Ω satisfies ρ-reflection if

(i) Ω contains $\overline{B_\rho}(0)$ and

(ii) Ω satisfies that, for $\nu \in S^{n-1}$ and for $s > \rho$,

$$\Psi_{s,\nu}(\Omega \cap \Pi_\nu^+(s)) \subset \Omega \cap \Pi_\nu^-(s).$$

Main questions

When the initial set Ω_0 satisfies ρ-reflection, we ask

(i) Is there a unique evolution of the flow;

(ii) does the evolving set Ω_t satisfy ρ-reflection for later times;

(iii) Does Ω_t converge to a unit ball as time grows?

We need a notion of weak solutions to study the flow, since a priori it is not clear whether the flow does not go through singularities. We will employ two such notions and their coincidence, for the following reason:

- To study the geometric properties of the flow we introduce notions of "viscosity solutions" to enable barrier arguments.
- To obtain energy estimates and discuss long time behavior it is natural to introduce variational structure to the problem.

Energy Structure

The flow can be understood as a gradient flow for the energy $Per(F)$ over sets of fixed volume. In particular it corresponds to the continuum limit of the following discrete-time scheme, where we find a minimizer of

$$Per(F) + \frac{1}{h}\tilde{d}^2(F,E),$$

in each time step, where E is the set from the previous step, over the sets of unit volume. Here the pseudo-distance between sets

$$\tilde{d}^2(F,E) := \int_{E\Delta F} d(x,\partial E)dx$$

is first introduced by Almgren-Taylor-Wang (1993) and Luckhaus-Sturzenhecker (1995).

Even after ruling out singularities, it is difficult to use the scheme (minimizing movements) to obtain the continuum flow in the non-convex setting, since we do not have sufficient information on the total curvature $\int_{\partial F} \kappa dx$.

Approximate solutions

Motivated by [MMS16], we consider the approximate minimizing movement:

$$Per(E) + \frac{1}{\delta}(|F| - 1)^2 + \frac{1}{h}\tilde{d}^2(F, E),$$

where the second term relaxes the requirement that $|F| = 1$. For this energy, formally the discrete solutions converges to the continuous evolution of sets Ω_t whose boundary moves with the velocity law.

$$V = -\kappa + \lambda_\delta(|F|), \quad \lambda_\delta(s) := \frac{2}{\delta}(s - 1).$$

For the mean curvature flow of sets $\{\Omega_t^\delta\}_{t>0}$ with forcing of the type

$$V = -\kappa + \lambda(|\Omega_t|),$$

it is possible to modify viscosity solutions type argument to show that the strong star-shapedness property, ρ-reflection, is preserved over time ([K-Kwon 16]).

In particular we have that Ω_t^δ is star-shaped with respect to $B_\rho(0)$ and supported in $B_R(0)$, where R, ρ is independent of δ.

On the other hand using the star-shapedness and the velocity law, it is possible to show that Ω_t^δ is unifomly Hölder in time.

Convergence of δ-flow

The equi-continuity of δ-flows with respect to space and time yields the following:

Theorem (K-Kwon)

(a) *The δ-minimizing movement solutions, as the time step size goes to zero, uniformly converge to the continuum flow $(\Omega_t^\delta)_\delta$, which is a unique smooth solution for the velocity law $V = -\kappa + \lambda_\delta$.*

(b) *(Ω_t^δ) converges to a ball exponentially fast.*

(c) *As $\delta \to 0$ the δ-flow $(\Omega_t^\delta)_{t>0}$ locally uniformly converges to a star-shaped flow $(\Omega_t)_{t>0}$. Moreover $|\Omega_t| \equiv 1$ for all times.*

Technical comment on minimizing movement

Technically speaking, we apply the viscosity solutions type argument to the continuum limit of the minimizing movement solutions.

To ensure convergence of discrete movements as time step size goes to zero, we consider the minimizing movements within the class of star-shaped sets,

$$S_{r,M} = \{E : E \text{ is star-shaped with respect to } B_r(0), E \subset B_M(0)\}$$

with $r << 1 << M$, and later show that this constraint can be removed in the continuum limit.

It still remains to characterize Ω_t and its long-time behavior. This has been done in [MMS16] in the setting of distributional solutions, but we aim for a stronger statement given the geometric property.

For this purpose we consider uniform regularity of $\partial\Omega_t^\delta$ with respect to $\delta > 0$.

Regularity of δ-flow

Since $\Omega_t^\delta \in S_{r,R}$ for uniform r and R, its boundary can be locally represented as a Lipschitz graph $x_n = u(x', t)$. Then the graph solves

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\,\mathrm{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) + \lambda(t)\sqrt{1+|Du|^2}.$$

From here one can obtain uniform $C^{1,1}$ estimates for u that only depends on the Lipschitz constant for u and the L^2 norm of λ. From $C^{1,1}$ to further regularity as well as uniqueness Escher and Simonnet [ES98].

On the other hand we can show that λ_δ is uniformly bounded in $L^1([0, T])$, adopting the perturbation argument in [MMS16].

Heuristic derivation of the L^2 bound, [MMS16]

We have, for $E_\delta(F) := Per(F) + \frac{1}{\delta}(1-|F|)^2$,

$$E_\delta(\Omega_0) \geq \int_0^T -\frac{d}{dt}E_\delta(\Omega_t^\delta) = \int_0^t \int_\Gamma V^2 dt = \int_0^T \int_\Gamma (-\kappa + \lambda_\delta(t))^2 dt.$$

Thus

$$\|\lambda_\delta(t))\|_{L^2([0,T]} \leq E(0) + (\int_0^t \int_\Gamma \kappa^2 dt)^{1/2},$$

and the latter term can be bounded using the minimizing property of discrete scheme.

Properties of the limiting flow

Theorem

Given that the initial set Ω_0 is smooth, we have

(a) Ω_t *is star-shaped, smooth and preserves its volume over time.*

(b) Ω_t *is the unique solution of*

$$(V) \qquad V = -\kappa + \bar{\lambda}_t.$$

If Ω_0 is not smooth then we still can show that Ω_t satisfies the velocity law (V) in the viscosity sense, however in this case we do not know whether $\bar{\lambda}_t$ is unique.

Long time behavior

From the minimizing scheme and the uniform convergence of the anti-derivative of λ_δ, we have

$$(t-s)d_H^2(\Omega_t,\Omega_s) \leq E(\Omega_t) - E(\Omega_s) \text{ for any } 0 < s < t,$$

where $E(F) := Per(F)$. From this and the fact that $\Omega_t \in S_{r,R}$ we have that

(a) The sequence of space-time family of sets $(\Omega_{t+n})_{t>0}$ converges to $(\Omega_\infty)_{t>0}$;
(b) Ω_∞ is stationary;
(c) $\partial\Omega_\infty$ satisfies a prescribed mean curvature equation.

From (c) it follows that Ω_∞ is a ball.

Lastly, Escher and Simonnet [ES98] states that once the set is sufficiently close to a ball in $C^{1,1}$ norm, it exponentially fast converges to it. Thus we obtain:

Theorem

Ω_t converges exponentially fast to a unit ball.

Thank you for your attention!