Tau-functions of Painlevé Equations and Orthogonal Polynomials

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Painlevé Equations

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 6q^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 2q^3 + zq + a \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{aq^2+b}{z} + cq^3 + \frac{d}{q} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-a)q + \frac{b}{q} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{(q-1)^2}{z^2}\left(aq + \frac{b}{q}\right) + \frac{cq}{z} + \frac{dq(q+1)}{q-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left\{a + \frac{bz}{q^2} + \frac{c(z-1)}{(q-1)^2} + \frac{dz(z-1)}{(q-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

with a, b, c and d arbitrary constants.

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Painlevé σ-Equations

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0 \qquad S_I$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \tfrac{1}{4}\beta^2 \qquad S_{II}$$

$$\left(z\frac{d^2\sigma}{dz^2} - \frac{d\sigma}{dz}\right)^2 + \left[4\left(\frac{d\sigma}{dz}\right)^2 - z^2\right]\left(z\frac{d\sigma}{dz} - 2\sigma\right) + 4z\vartheta_0\vartheta_\infty\frac{d\sigma}{dz} = (\vartheta_0^2 + \vartheta_\infty^2)z^2 \qquad S_{III}$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(\frac{d\sigma}{dz} + 2\vartheta_0\right)\left(\frac{d\sigma}{dz} + 2\vartheta_\infty\right) = 0 \qquad S_{IV}$$

$$\left(z\frac{d^2\sigma}{dz^2}\right)^2 - \left[2\left(\frac{d\sigma}{dz}\right)^2 - z\frac{d\sigma}{dz} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \kappa_j\right) = 0 \qquad S_V$$

$$\frac{d\sigma}{dz}\left[z(z-1)\frac{d^2\sigma}{dz^2}\right]^2 + \left[\frac{d\sigma}{dz}\left\{2\sigma - (2z-1)\frac{d\sigma}{dz}\right\} + \kappa_1\kappa_2\kappa_3\kappa_4\right]^2 = \prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \kappa_j^2\right) \qquad S_{VI}$$

where β, $\vartheta_0, \vartheta_\infty$ and $\kappa_1, \ldots, \kappa_4$ are arbitrary constants.

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial
P_{I}	0	—		
P_{II}	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—
P_{III}	2	**Bessel** $J_\nu(z), I_\nu(z), K_\nu(z)$	1	—
P_{IV}	2	**Parabolic cylinder** $D_\nu(z)$	1	**Hermite** $H_n(z)$
P_{V}	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$
P_{VI}	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$

Properties of the Second Painlevé Equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + a \qquad \mathrm{P_{II}}$$

- **Hamiltonian structure**
- **Airy solutions**

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Hamiltonian Representation

$\mathrm{P_{II}}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial p} = p - q^2 - \tfrac{1}{2}z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial q} = 2qp + a + \tfrac{1}{2} \qquad \mathrm{H_{II}}$$

where $\mathcal{H}_{\mathrm{II}}(q,p,z;a)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{II}}(q,p,z;a) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (a + \tfrac{1}{2})q$$

Eliminating p then q satisfies $\mathrm{P_{II}}$ whilst eliminating q yields

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(a + \tfrac{1}{2})^2 \qquad \mathrm{P_{34}}$$

Theorem (Okamoto [1986])

The function $\sigma(z;a) = \mathcal{H}_{\mathrm{II}} \equiv \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (a + \tfrac{1}{2})q$ *satisfies*

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(a + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

and conversely

$$q(z;a) = \frac{2\sigma''(z) + a + \tfrac{1}{2}}{4\sigma'(z)}, \qquad p(z;a) = -2\frac{\mathrm{d}\sigma}{\mathrm{d}z}$$

are solutions of $\mathrm{H_{II}}$.

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Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n - \tfrac{1}{2} \qquad \mathrm{P_{II}}$$

$$p_n \frac{\mathrm{d}^2 p_n}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2 + 2p_n^3 - zp_n^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2 \sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S_{II}}$$

Theorem

Let

$$\varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(\zeta) + \sin(\vartheta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/3}z$$

with ϑ an arbitrary constant, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ Airy functions, and $\tau_n(z)$ be the Wronskian

$$\tau_n(z;\vartheta) = \det\left[\frac{\mathrm{d}^{j+k}\varphi}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1} = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

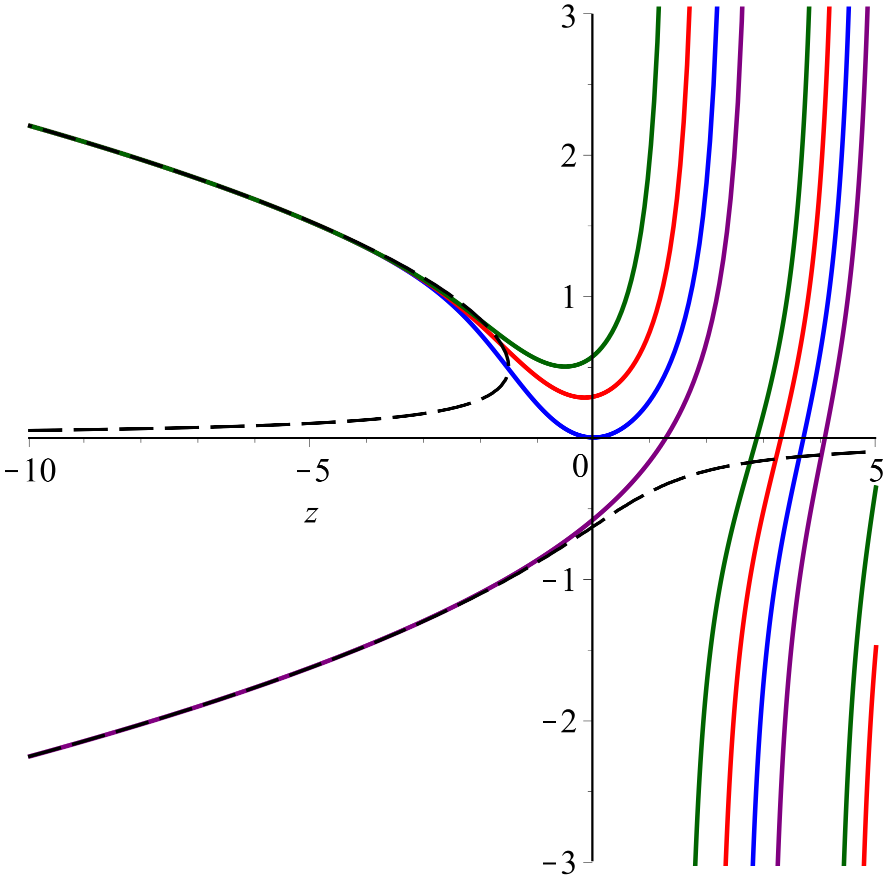
then

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{n-1}(z;\vartheta)}{\tau_n(z;\vartheta)}, \quad p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta), \quad \sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

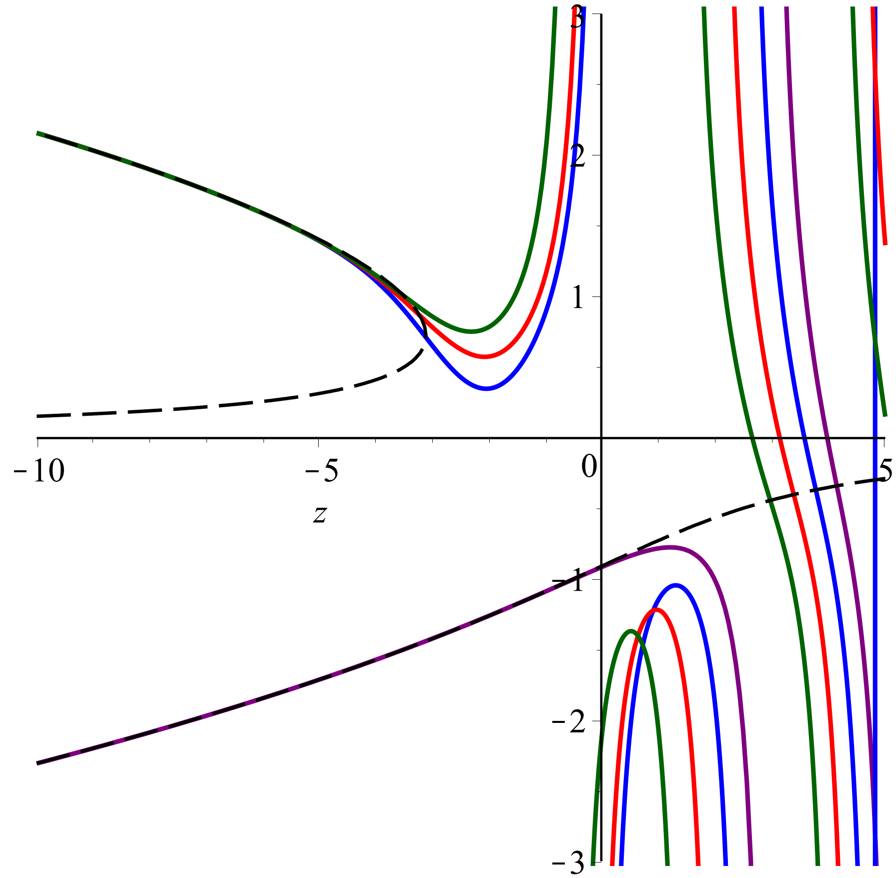
respectively satisfy $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$, with $n \in \mathbb{Z}$.

Airy Solutions of $\mathrm{P_{II}}$

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{n-1}(z;\vartheta)}{\tau_n(z;\vartheta)}$$



$\boldsymbol{n=1}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=2}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

The dashed line is the cubic

$$2q^3 + zq + n - \tfrac{1}{2} = 0$$

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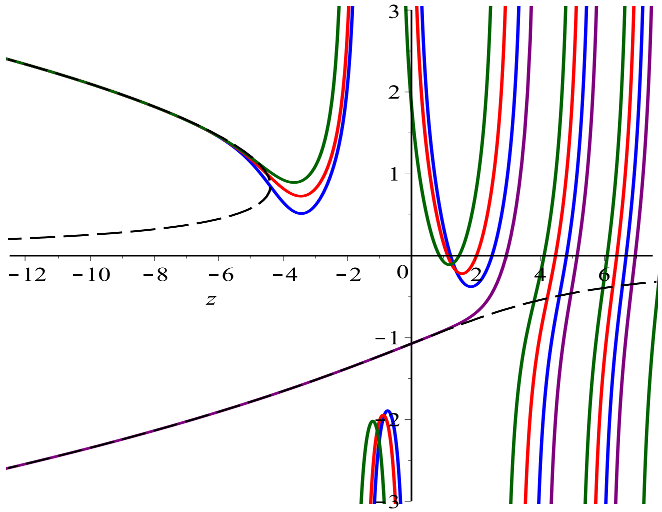
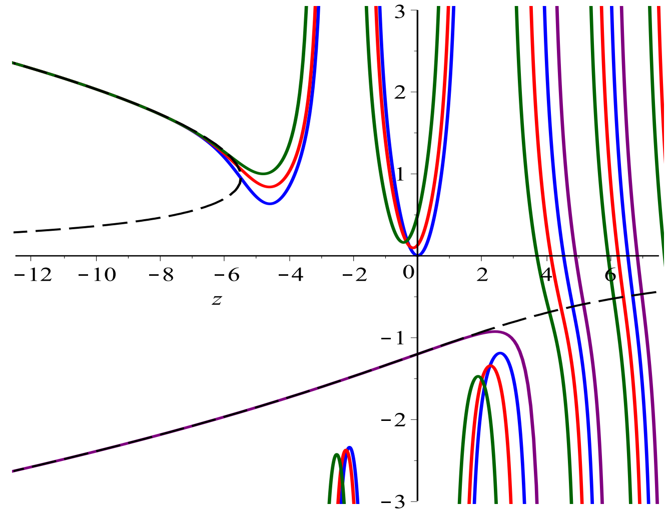
Airy Solutions of $\mathrm{P_{II}}$

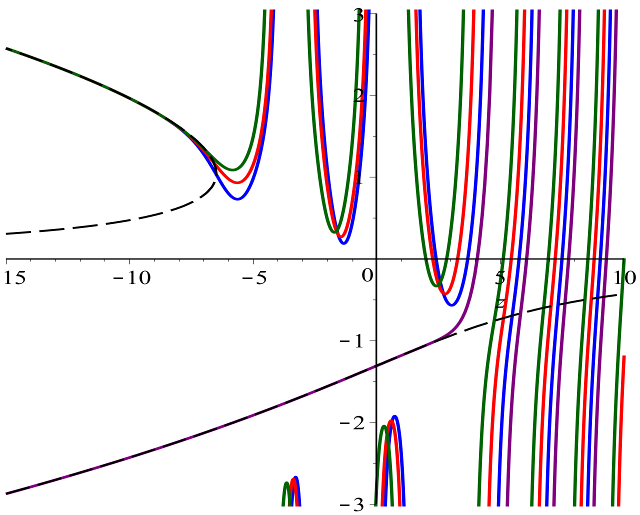
$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}$$



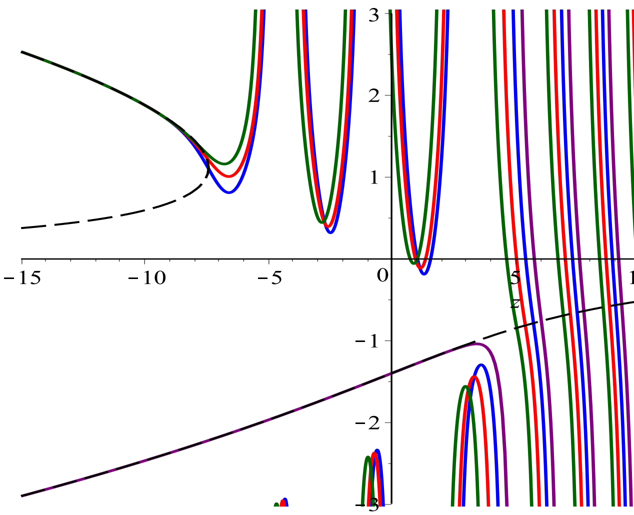
$\boldsymbol{n=3},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=4},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=5},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=6},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

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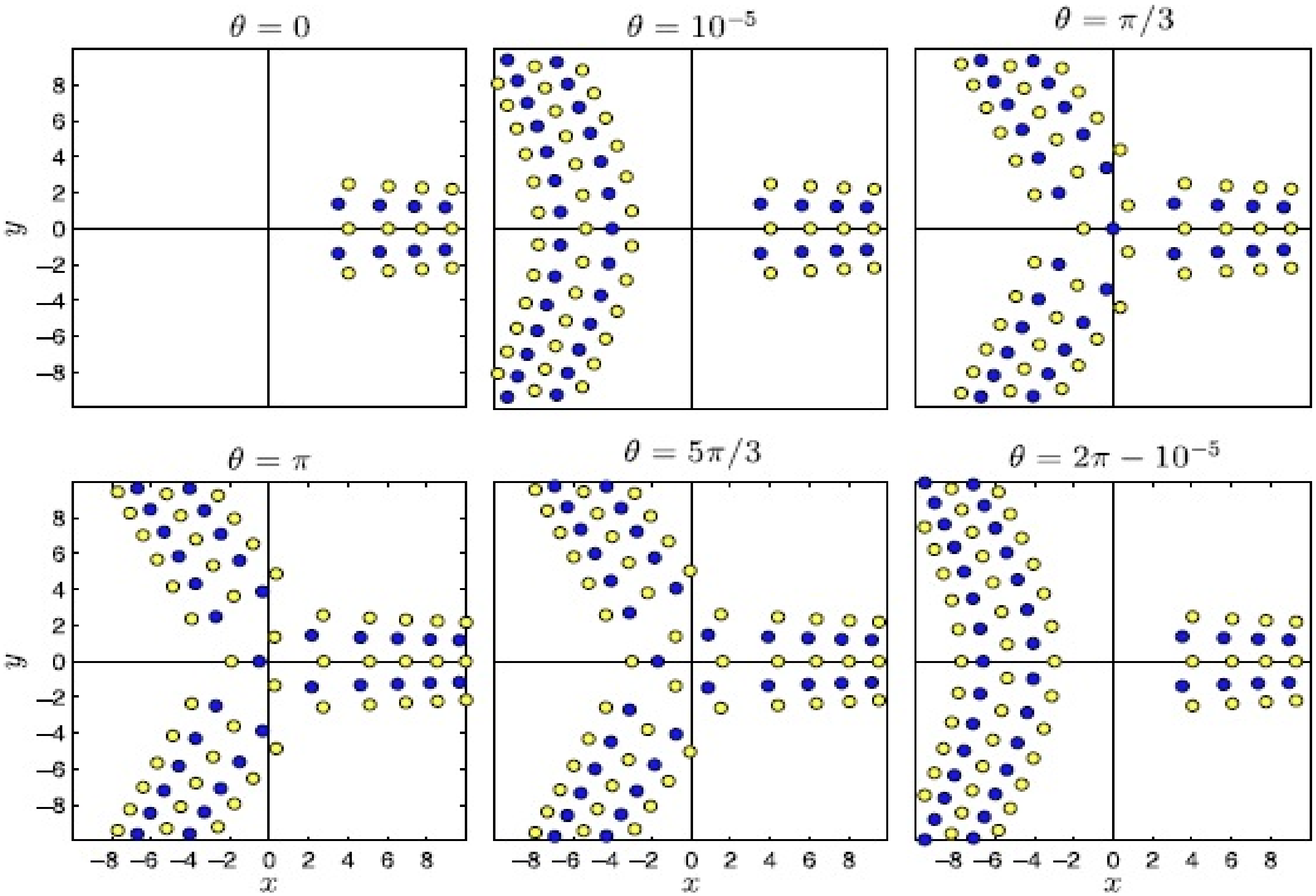
Airy Solutions of $\mathrm{P_{II}}$ with $\alpha = \frac{5}{2}$ $(n = 2)$

(Fornberg & Weideman [2014])

$$q_2(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\mathcal{W}(\varphi,\varphi')}{\mathcal{W}(\varphi,\varphi',\varphi'')}, \qquad \varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}z) + \sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}z)$$

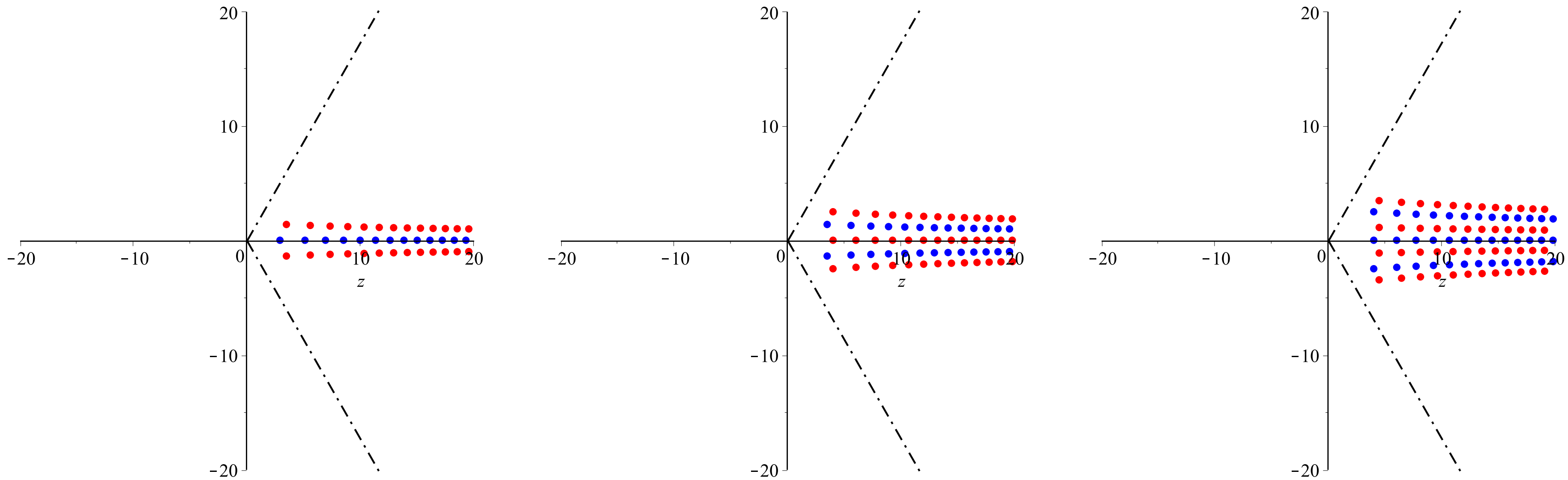


blue/**yellow** denote poles with residue $+1/-1$

Tronquée Solutions of $\mathrm{P_{II}}$ (Airy with $\vartheta = 0$)

$$q_n(z;0) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_{n-1}(z;0)}{\tau_n(z;0)}, \qquad \tau_n(z;0) = \mathcal{W}\left(\varphi_0, \frac{\mathrm{d}\varphi_0}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi_0}{\mathrm{d}z^{n-1}}\right)$$

with $\varphi_0 = \varphi(z;0) = \mathrm{Ai}(-2^{-1/3}z)$

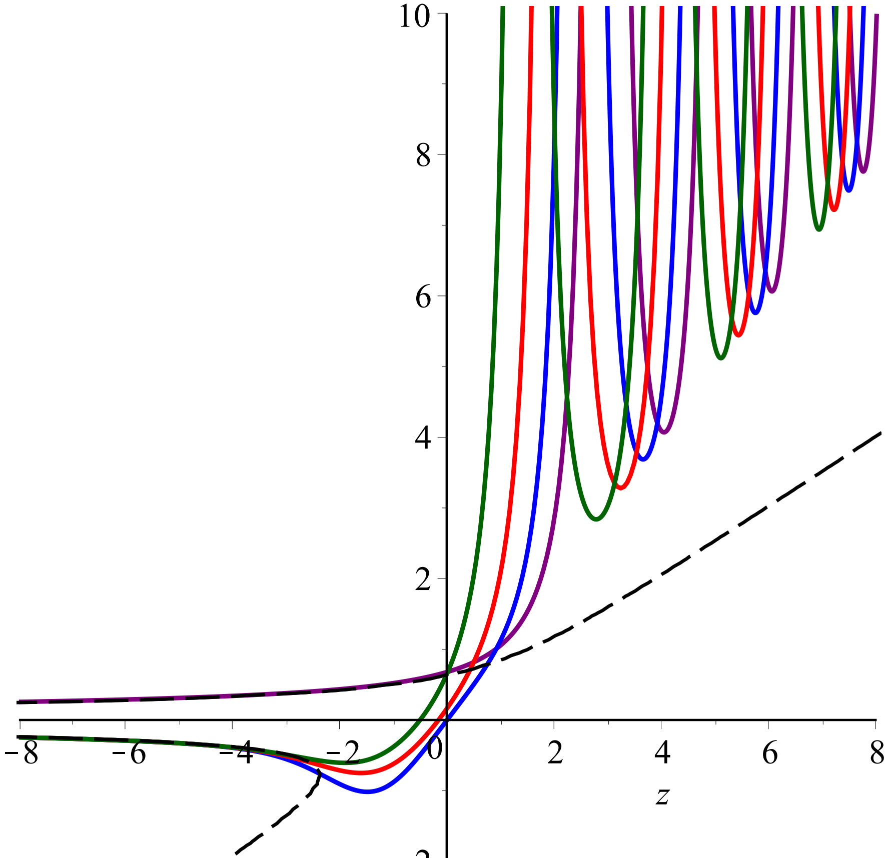


$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$

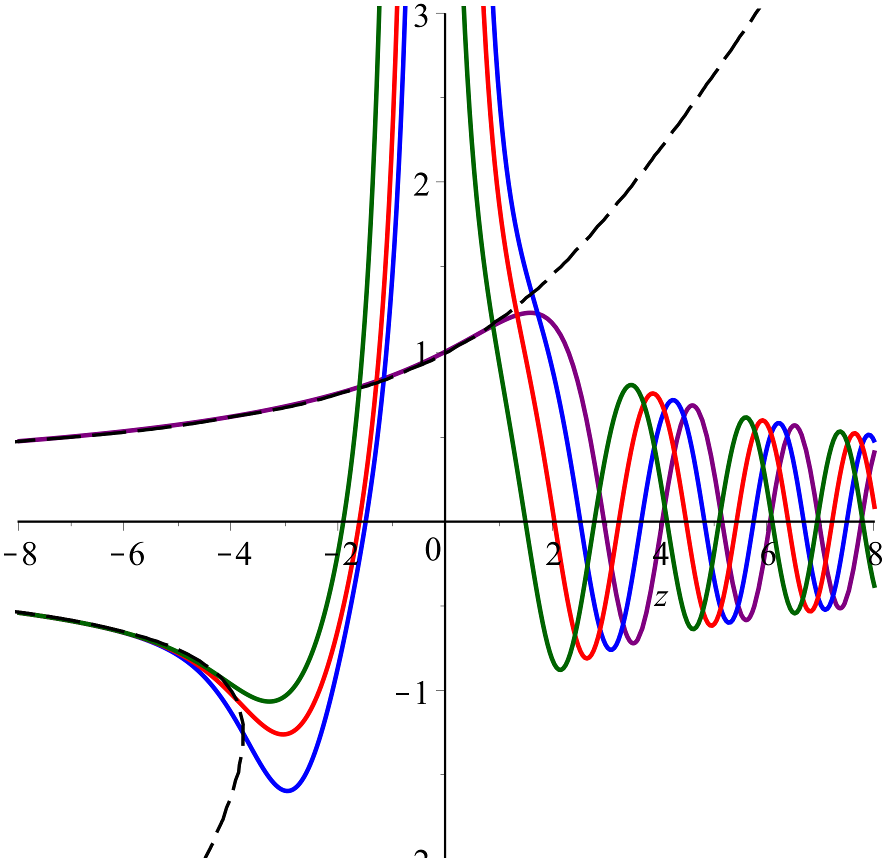
Plots of the poles of $q_n(z;0)$ for $n = 2, 3, 4$; the blue and **red** circles represent poles with residues $+1$ and -1, respectively.

Airy Solutions of P_{34}

$$p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta)$$



$\boldsymbol{n=1},\ \ \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=2},\ \ \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

The dashed line is the cubic

$$2p^3 - zp^2 - \tfrac{1}{2}n^2 = 0$$

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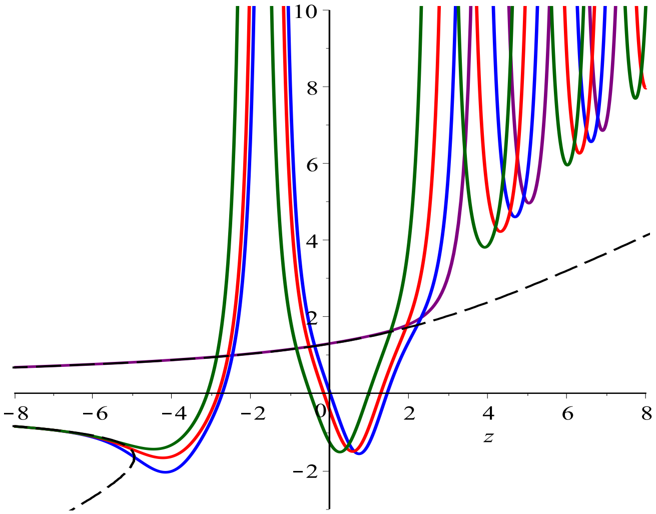
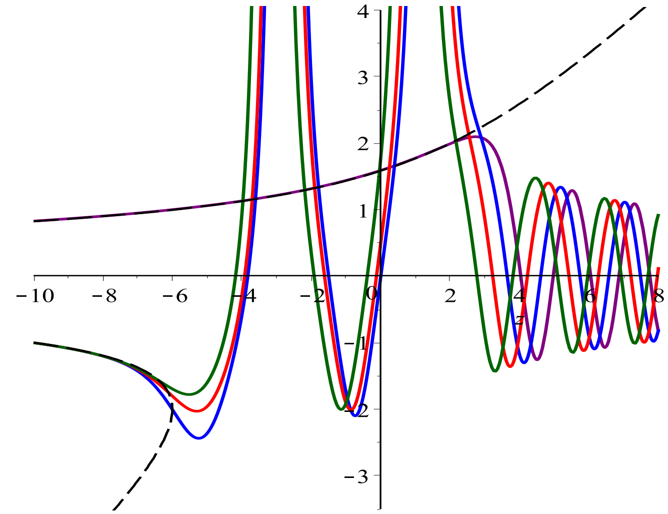
Airy Solutions of P_{34}

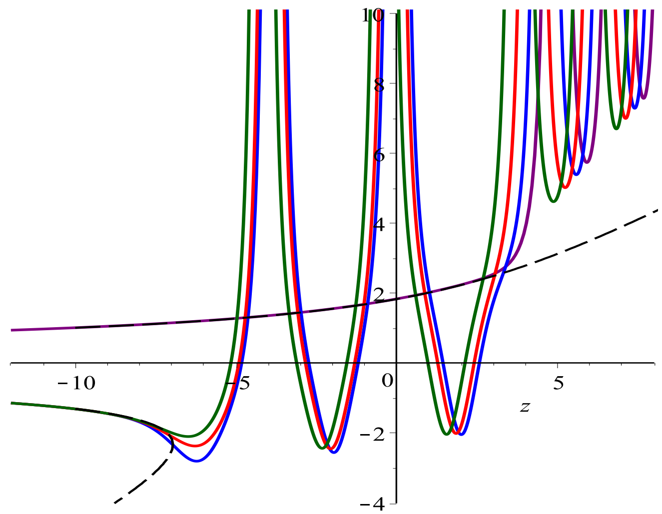
$$p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta)$$



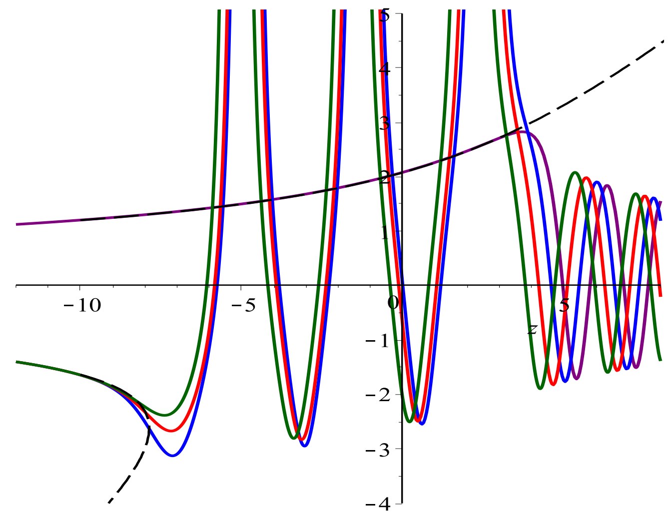
$\boldsymbol{n=3}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=4}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=5}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=6}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

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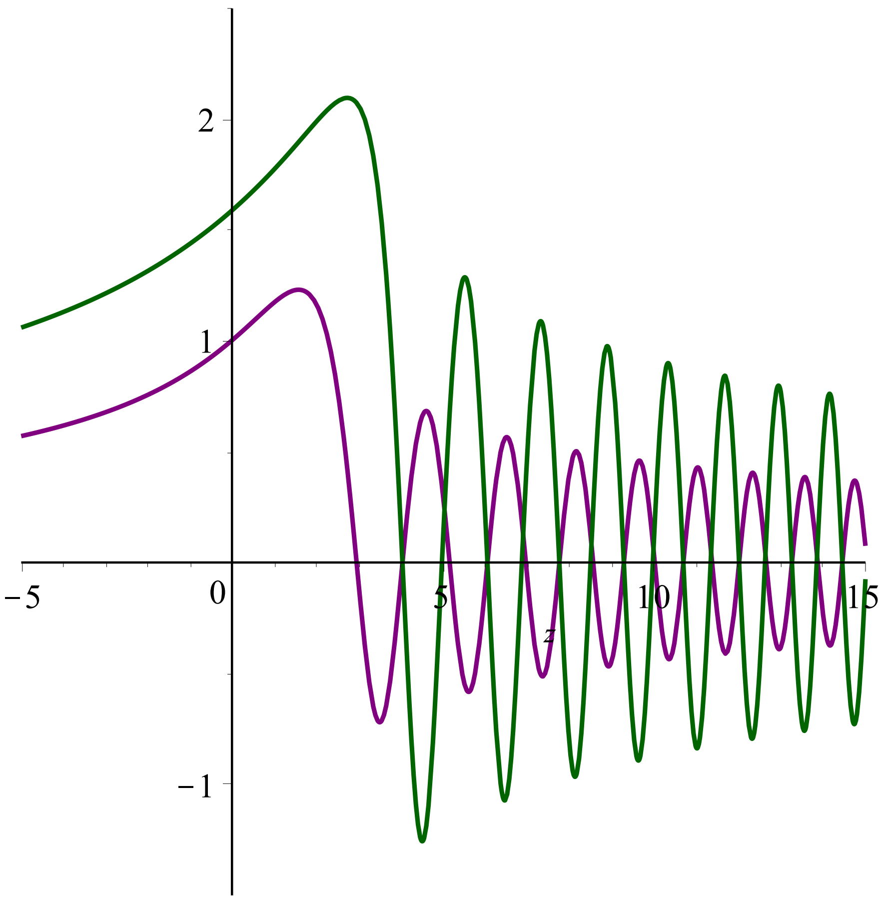
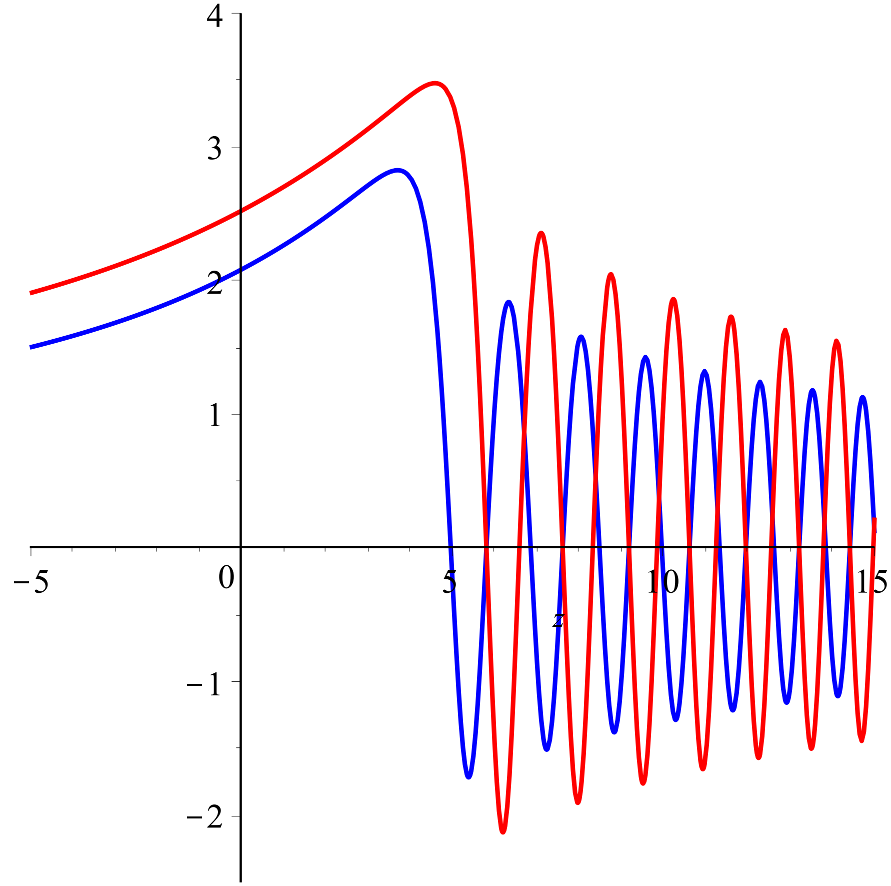
Airy Solutions of P_{34}

$$p_n(z;0) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;0)$$



$n = 2$, $n = 4$



$n = 6$, $n = 8$

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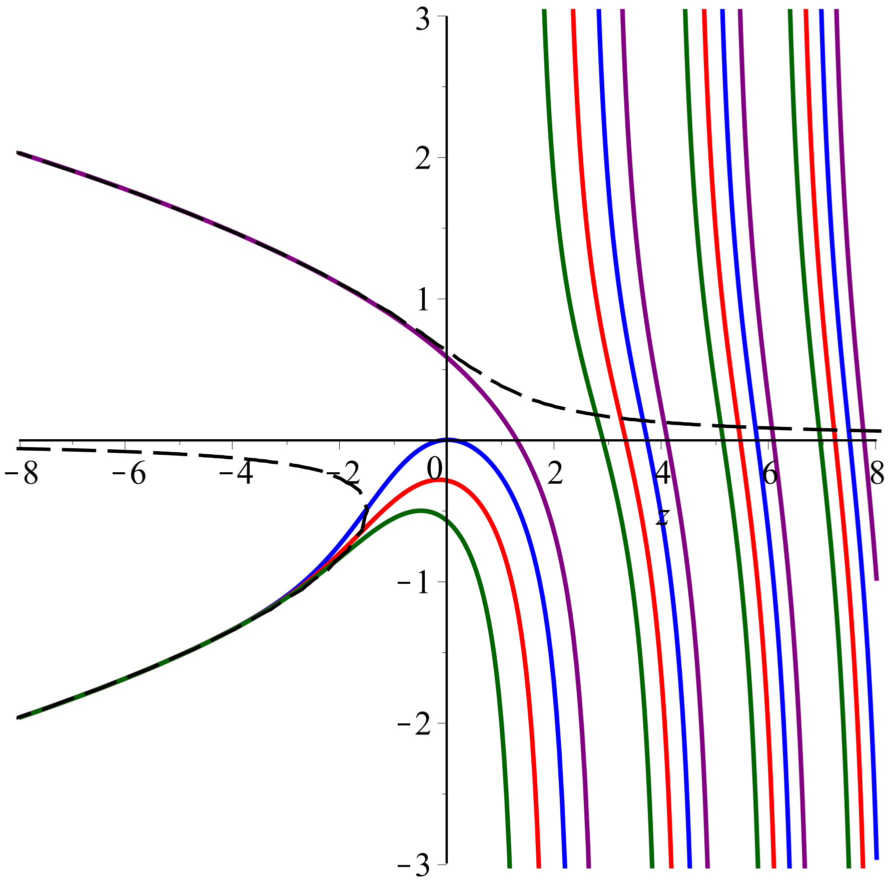
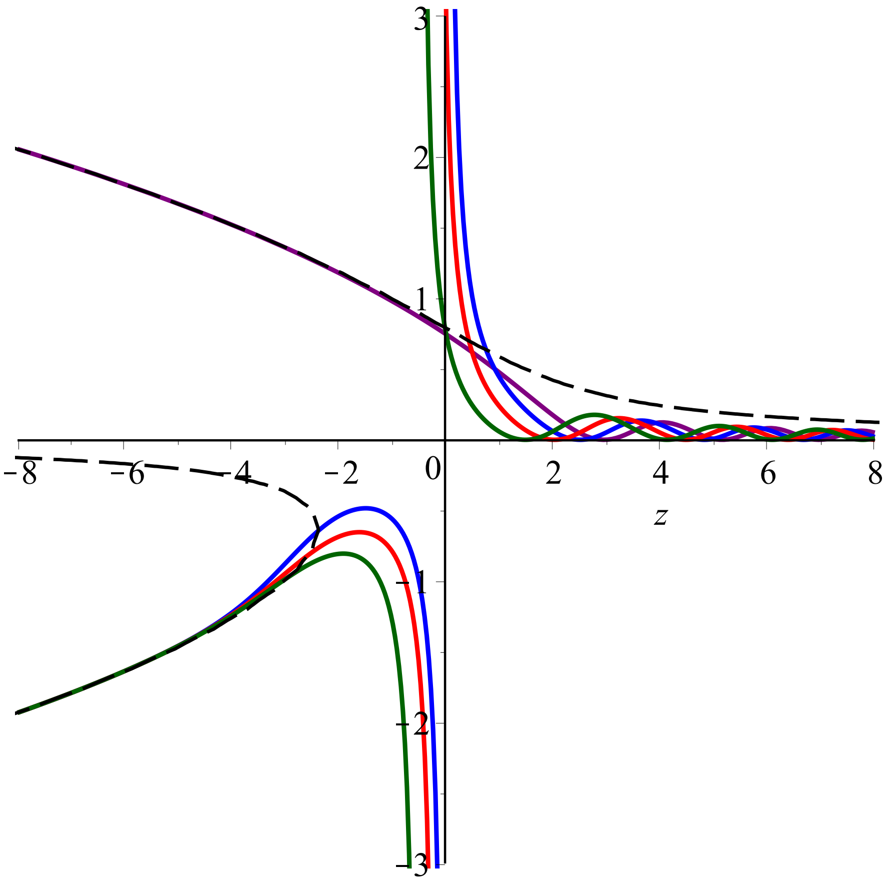
Airy Solutions of $\mathrm{S_{II}}$

$$\sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \tau_n(z;\vartheta)$$



$\boldsymbol{n = 1}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



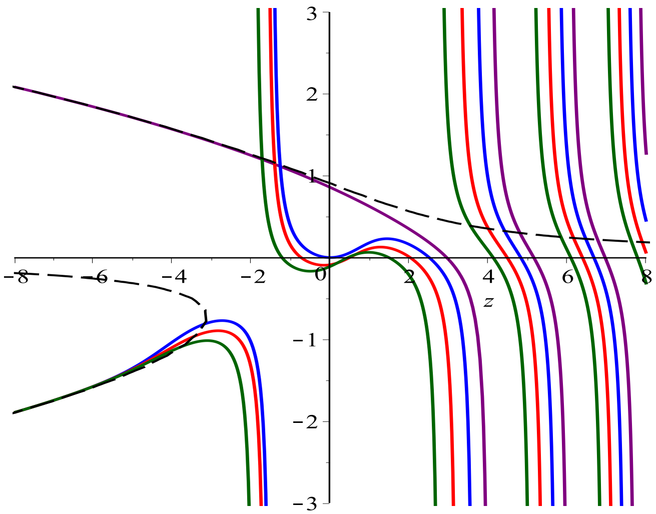
$\boldsymbol{n = 2}, \quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

The dashed line is the cubic

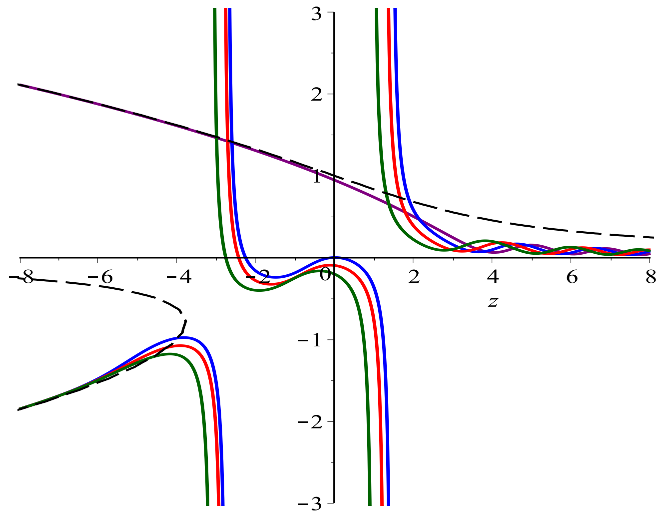
$$2\sigma^3 + z\sigma - \tfrac{1}{2}n = 0$$

Airy Solutions of $\mathrm{S_{II}}$

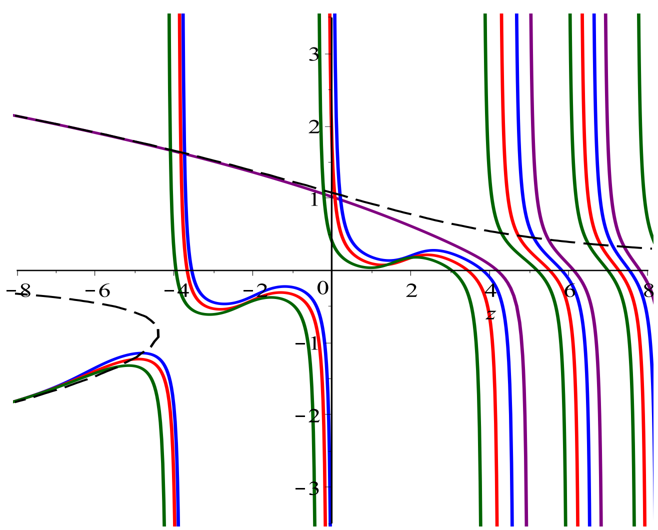
$$\sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \tau_n(z;\vartheta)$$



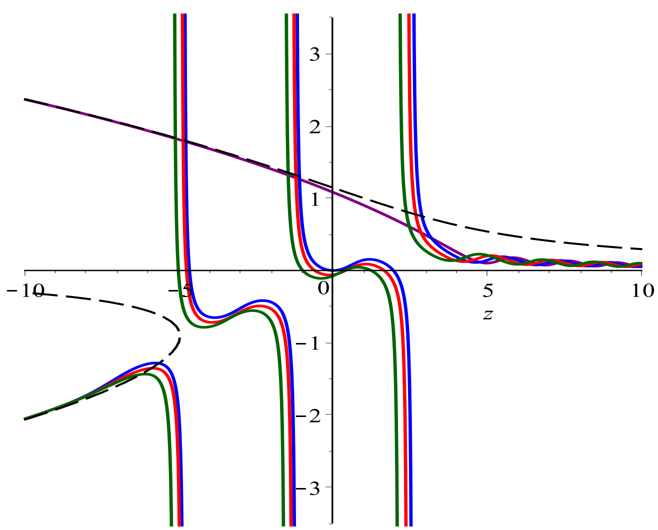
$\boldsymbol{n=3}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=4}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



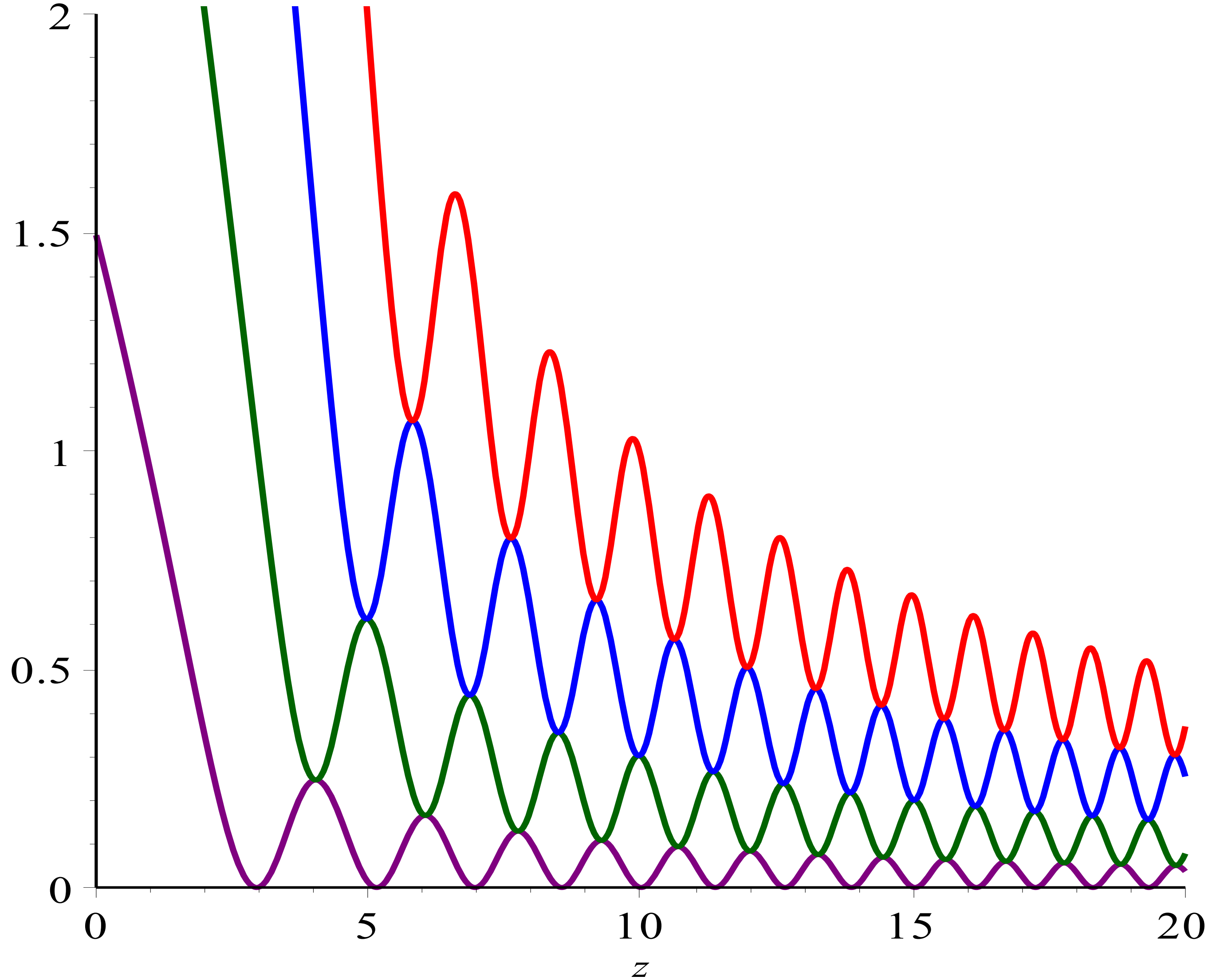
$\boldsymbol{n=5}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=6}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

Airy Solutions of S_{II}

$$\sigma_n(z;0) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \det \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} \mathrm{Ai}(-2^{-1/3}z) \right]_{j,k=0}^{n-1}$$



$n = 2$, $n = 4$, $n = 6$, $n = 8$

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Properties of the Fourth Painlevé Equation and the Fourth Painlevé σ-Equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-a)q + \frac{b}{q} \qquad \mathrm{P_{IV}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

- **Hamiltonian Representation**
- **Parabolic Cylinder Function Solutions**

Hamiltonian Representation of $\mathrm{P_{IV}}$

$\mathrm{P_{IV}}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0$$
$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty$$

where $\mathcal{H}_{\mathrm{IV}}(q,p,z;\theta_0,\vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{IV}}(q,p,z;\theta_0,\vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then q satisfies

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}$$

which is $\mathrm{P_{IV}}$ with $a = 1 - \vartheta_0 + 2\vartheta_\infty$ and $b = -2\vartheta_0^2$, whilst eliminating q then p satisfies

$$\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2p}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}$$

and letting $p = -\frac{1}{2}q$ gives $\mathrm{P_{IV}}$ with $a = 2\vartheta_0 - \vartheta_\infty - 1$ and $b = -2\vartheta_\infty^2$.

Theorem (**Okamoto [1986]**)

The function

$$\sigma(z;\theta_0,\vartheta_\infty) = \mathcal{H}_{\rm IV} \equiv 2qp^2 - (q^2+2zq+2\vartheta_0)p + \vartheta_\infty q$$

where q and p satisfy the Hamiltonian system

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}z} &= 4qp - q^2 - 2zq - 2\vartheta_0\\ \frac{\mathrm{d}p}{\mathrm{d}z} &= -2p^2 + 2pq + 2zp - \vartheta_\infty\end{aligned} \qquad \mathrm{H}_{\rm IV}$$

satisfies the second-order, second-degree equation

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S}_{\rm IV}$$

Conversely, if $\sigma(z;\theta_0,\vartheta_\infty)$ is a solution of $\mathrm{S}_{\rm IV}$, then

$$q(z;\theta_0,\vartheta_\infty) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \qquad p(z;\theta_0,\vartheta_\infty) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of the Hamiltonian system $\mathrm{H}_{\rm IV}$.

Parabolic Cylinder Function Solutions of P_{IV}

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right) = \det\left[\frac{\mathrm{d}^{j+k}\varphi_\nu}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1}$$

for $n \geq 1$, *where* $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of P_{IV}

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - a)q + \frac{b}{q}$$

are given by

$$q_{\nu,n}^{[1]}(z;a_1,b_1) = -2z + \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \quad (a_1,b_1) = \left(\varepsilon(2n-\nu), -2(\nu+1)^2\right)$$

$$q_{\nu,n}^{[2]}(z;a_2,b_2) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}, \qquad (a_2,b_2) = \left(-\varepsilon(n+\nu), -2(\nu-n+1)^2\right)$$

$$q_{\nu,n}^{[3]}(z;a_3,b_3) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (a_3,b_3) = \left(\varepsilon(2\nu-n+1), -2n^2\right)$$

Parabolic Cylinder Function Solutions of S_{IV}

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right) = \det\left[\frac{\mathrm{d}^{j+k}\varphi_\nu}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1}$$

for $n \geq 1$*, where* $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of S_{IV}

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0$$

are given by

$$\sigma_{\nu,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_{\nu,n}(z;\varepsilon), \qquad (\vartheta_0, \vartheta_\infty) = \big(\varepsilon(\nu - n + 1), -\varepsilon n\big)$$

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$$\frac{d^2\varphi_\nu}{dz^2} - 2\varepsilon z\frac{d\varphi_\nu}{dz} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- **If $\boldsymbol{\nu \notin \mathbb{Z}}$**

$$\varphi_\nu(z;\varepsilon) = \begin{cases} \left\{\cos(\theta)D_\nu(\sqrt{2}\,z) + \sin(\theta)D_\nu(-\sqrt{2}\,z)\right\}\exp\left(\tfrac{1}{2}z^2\right), & \varepsilon = 1 \\ \left\{\cos(\theta)D_{-\nu-1}(\sqrt{2}\,z) + \sin(\theta)D_{-\nu-1}(-\sqrt{2}\,z)\right\}\exp\left(-\tfrac{1}{2}z^2\right), & \varepsilon = -1 \end{cases}$$

- **If $\boldsymbol{\nu = n \in \mathbb{Z}}$, with** $n \geq 0$

$$\varphi_n(z;\varepsilon) = \begin{cases} \cos(\theta)H_n(z) + \sin(\theta)\exp(z^2)\dfrac{d^n}{dz^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \varepsilon = 1 \\ \cos(\theta)(-i)^n H_n(iz) + \sin(\theta)\exp(-z^2)\dfrac{d^n}{dz^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \varepsilon = -1 \end{cases}$$

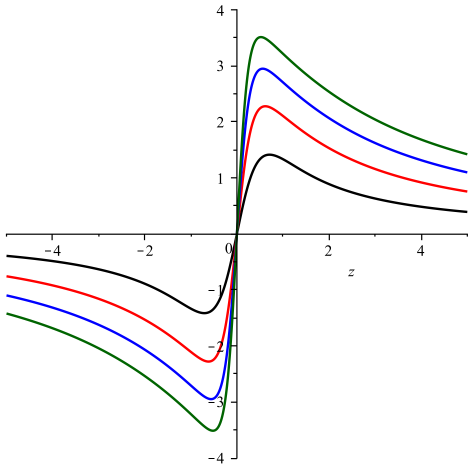
- **If $\boldsymbol{\nu = -n-1 \in \mathbb{Z}}$, with** $n \geq 0$

$$\varphi_{-n-1}(z;\varepsilon) = \begin{cases} \cos(\theta)(-i)^n H_n(iz)\exp(z^2) + \sin(\theta)\dfrac{d^n}{dz^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \varepsilon = 1 \\ \cos(\theta)H_n(z)\exp(-z^2) + \sin(\theta)\dfrac{d^n}{dz^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \varepsilon = -1 \end{cases}$$

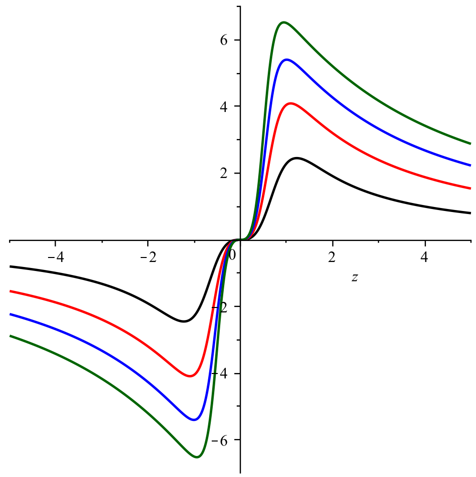
with θ an arbitrary constant, $D_\nu(\zeta)$ the **parabolic cylinder function**, $H_n(z)$ the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

Plots of Bounded Rational Solutions of $\mathrm{S_{IV}}$

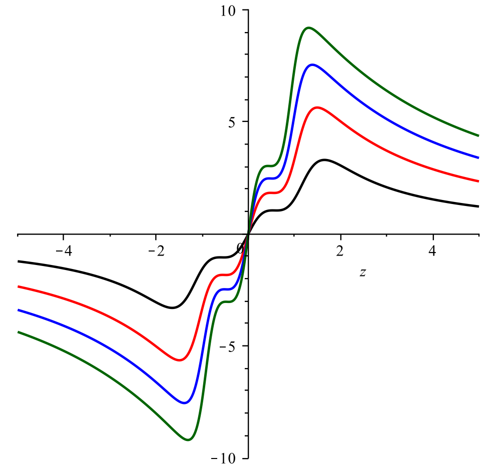
$$\sigma_{m,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\big(H_m(z), H_{m+1}(z), \ldots, H_{m+n-1}(z)\big)$$



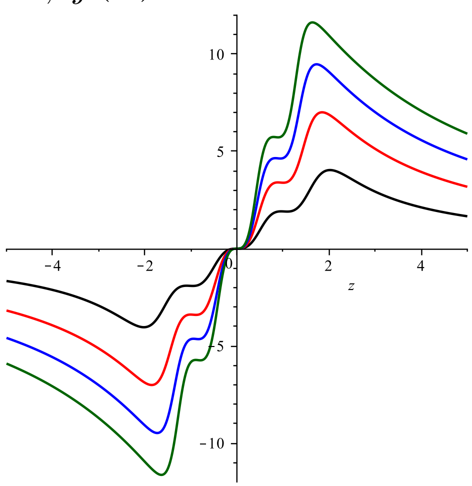
$\sigma_{1,2j}(z), \quad j = 1, 2, 3, 4$



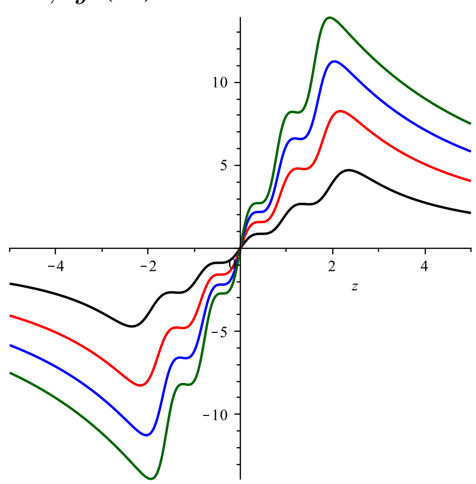
$\sigma_{2,2j}(z), \quad j = 1, 2, 3, 4$



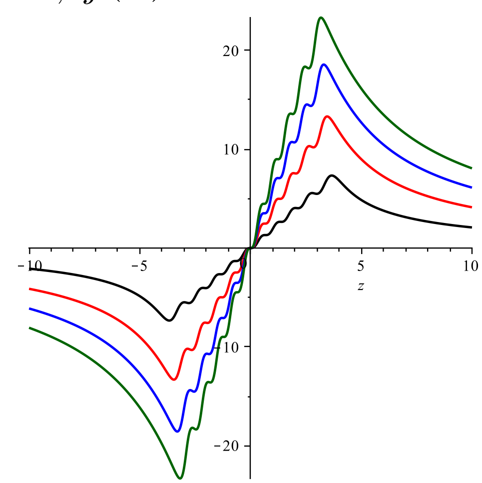
$\sigma_{3,2j}(z), \quad j = 1, 2, 3, 4$



$\sigma_{4,2j}(z), \quad j = 1, 2, 3, 4$



$\sigma_{5,2j}(z), \quad j = 1, 2, 3, 4$

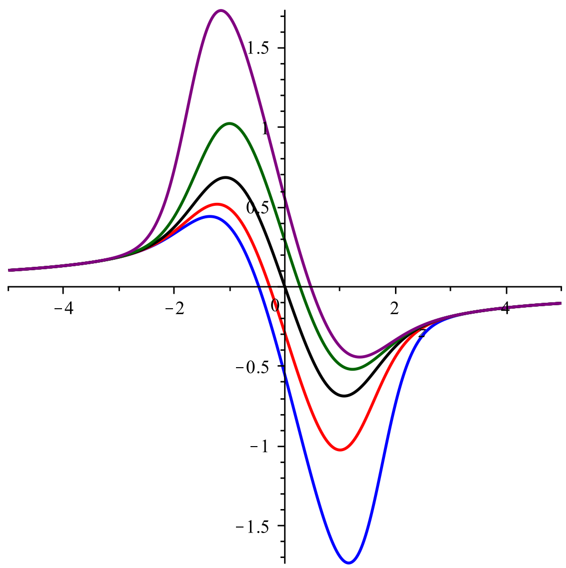


$\sigma_{10,2j}(z), \quad j = 1, 2, 3, 4$

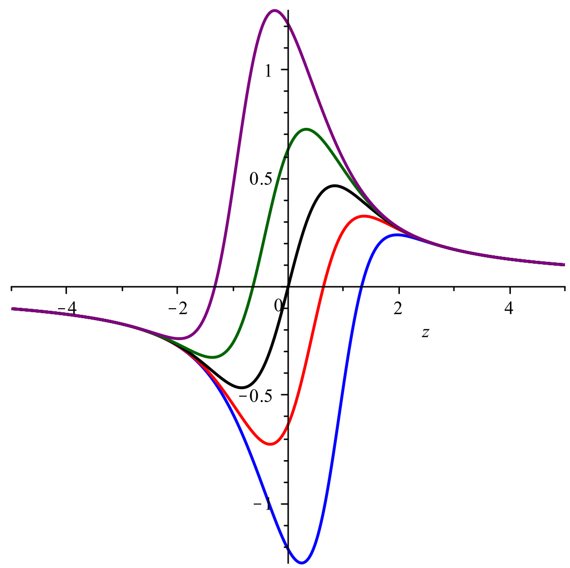
Plots of Bounded Special Function Solutions of $\mathrm{S_{IV}}$

$$\sigma_{\nu,n}(z) = -2nz + \frac{\mathrm{d}}{\mathrm{d}z}\ln \mathcal{W}\big(\varphi_\nu, \varphi_\nu', \ldots, \varphi_\nu^{(n-1)}\big)$$

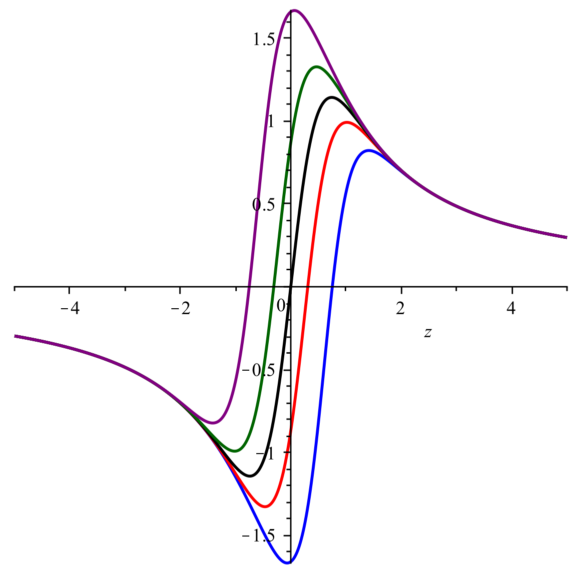
$$\varphi_\nu(z) = \Big\{\cos(\theta) D_{-\nu}\big(\sqrt{2}\,z\big) + \sin(\theta) D_{-\nu}\big(-\sqrt{2}\,z\big)\Big\}\exp\left(\tfrac{1}{2}z^2\right), \quad 0 < \theta < \tfrac{1}{2}\pi, \;\; \nu > 0$$



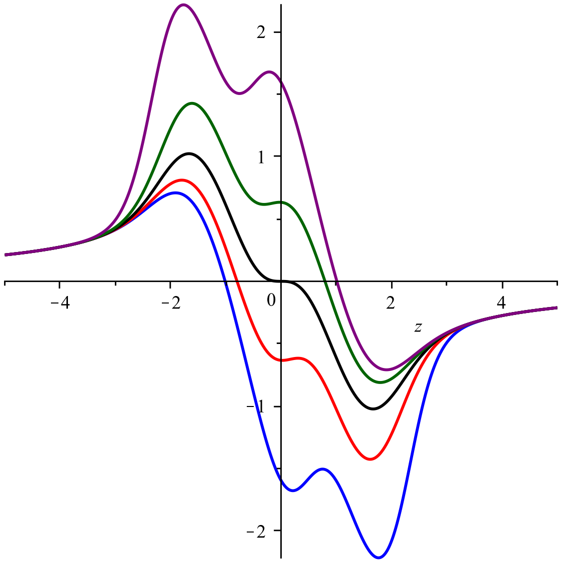
$\sigma_{1/2,1}(z)$



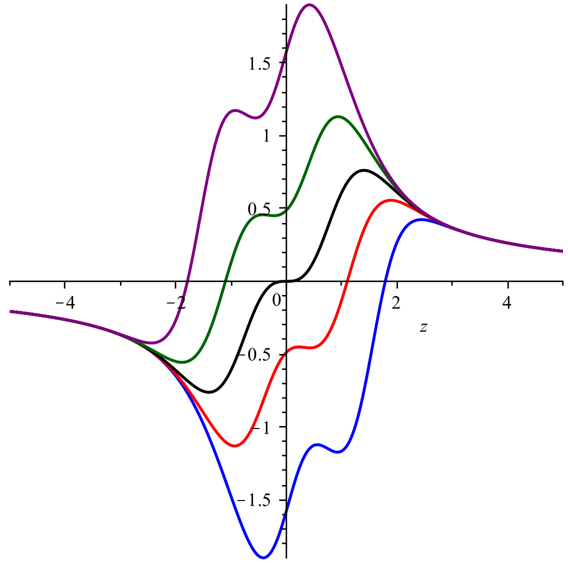
$\sigma_{3/2,1}(z)$



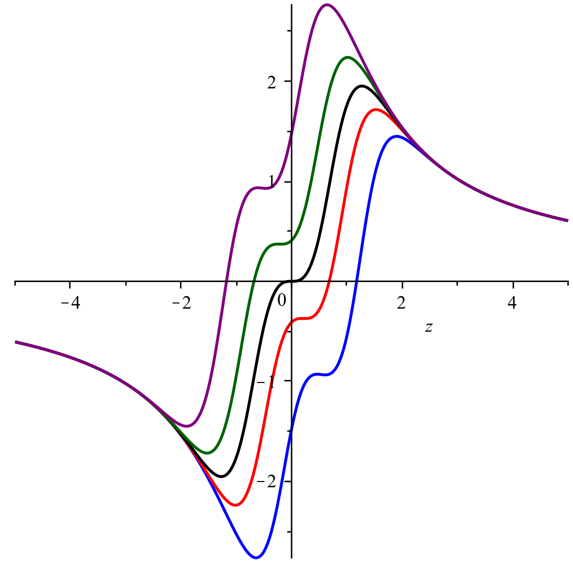
$\sigma_{5/2,1}(z)$



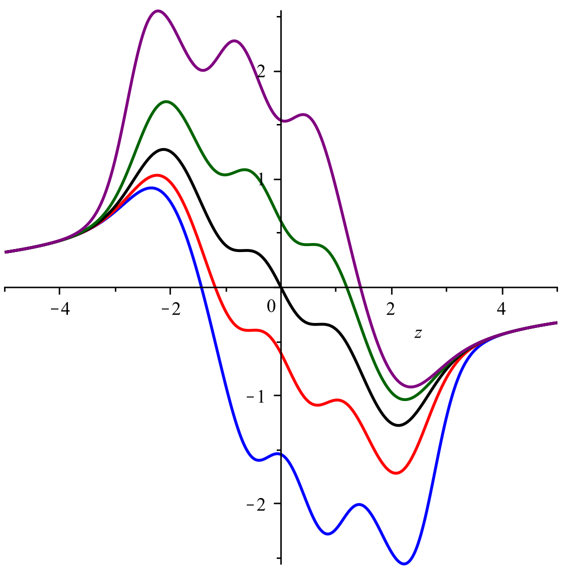
$\sigma_{1/2,2}(z)$



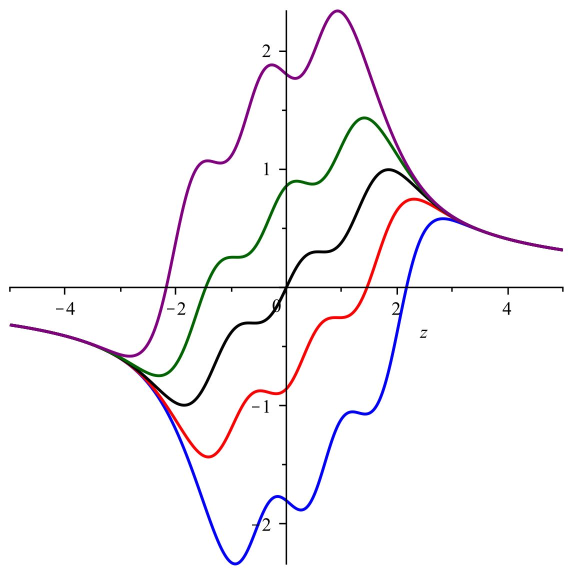
$\sigma_{3/2,2}(z)$



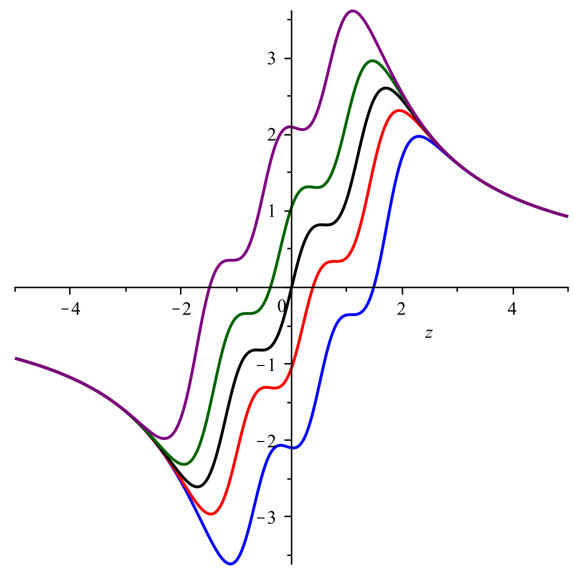
$\sigma_{5/2,2}(z)$



$\sigma_{1/2,3}(z)$



$\sigma_{3/2,3}(z)$



$\sigma_{5/2,3}(z)$

Orthogonal Polynomials

- Some History
- Monic orthogonal polynomials
- Semi-classical orthogonal polynomials

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to **Shohat [1939]** and **Freud [1976]**.
- **Fokas, Its & Kitaev [1991, 1992]** identified these integrable equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$w(x;t) = \exp\left(-x^4 + tx^2\right), \qquad x, t \in \mathbb{R},$$

and showed that the coefficients in the **three-term recurrence relation** can be expressed in terms of solutions of

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n$$

which is **discrete** $\mathrm{P_I}$ ($\mathrm{dP_I}$), as shown by **Bonan & Nevai [1984]**, and

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}t}\right)^2 + \frac{3}{2}q_n^3 + 4tq_n^2 + 2(t^2 + \tfrac{1}{2}n)q_n - \frac{n^2}{2q_n}$$

which is $\mathrm{P_{IV}}$ with $a = -\frac{1}{2}n$ and $b = -\frac{1}{2}n^2$.

Freud [1976] considered orthogonal polynomials $P_n(x)$ with respect to the weight

$$w(x) = |x|^\rho \exp(-|x|^m), \qquad \rho > -1, \quad m > 0$$

which satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x)$$

and gave recurrence relations for β_n in the cases $m = 2, 4, 6$

- $m = 2$ (Hermite polynomials when $\rho = 0$)

$$2\beta_n = n + \tfrac{1}{2}\rho\big[1 - (-1)^n\big]$$

- $m = 4$

$$4\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1}) = n + \tfrac{1}{2}\rho\big[1 - (-1)^n\big]$$

  which an autonomous dP$_{\rm I}$

- $m = 6$

$$\begin{aligned}6\beta_n\big(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} \\ + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}\big) = n + \tfrac{1}{2}\rho\big[1 - (-1)^n\big]\end{aligned}$$

Later studies include **Lew & Quarles [1983]**, **Bonan & Nevai [1984]**, **Fokas, Its & Kitaev [1991, 1992]**, **Clarke & Shizgal [1993]**, **Magnus [1995]**, ...

For the exponential weight

$$w(x) = \exp(-x^4 + tx^2)$$

the monic orthogonal polynomials satisfy the recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n(t)P_{n-1}(x)$$

Theorem (**Shohat [1939]**, **Freud [1976]**, **Bonan & Nevai [1984]**)
The recurrence coefficients for the weight $w(x) = \exp(-x^4 + tx^2)$ *satisfy*

$$4\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1} - \tfrac{1}{2}t) = n, \qquad n \geq 1$$

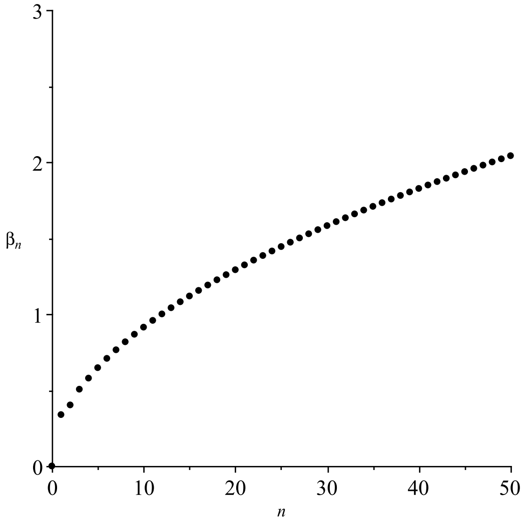
with initial conditions

$$\beta_0 = 0, \qquad \beta_1 = \frac{\int_{-\infty}^{\infty} x^2 \exp(-x^4 + tx^2)\,\mathrm{d}x}{\int_{-\infty}^{\infty} \exp(-x^4 + tx^2)\,\mathrm{d}x} = \Phi(t)$$

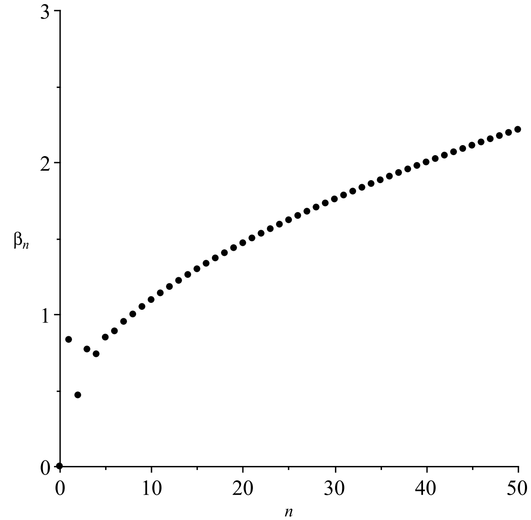
Theorem (**Freud [1976]**)
The recurrence coefficients for the weight $w(x) = \exp(-x^4 + tx^2)$ *satisfy*

$$\lim_{n\to\infty} \frac{\beta_n}{\sqrt{n}} = \frac{1}{\sqrt{12}}$$

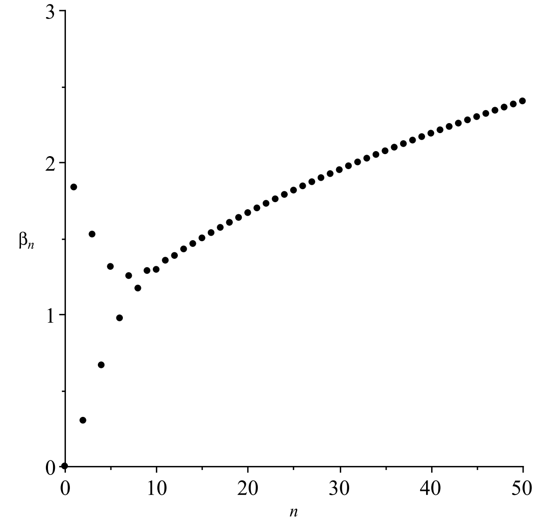
$$4\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1} - \tfrac{1}{2}t) = n, \qquad \beta_0 = 0, \quad \beta_1 = \Phi(t)$$



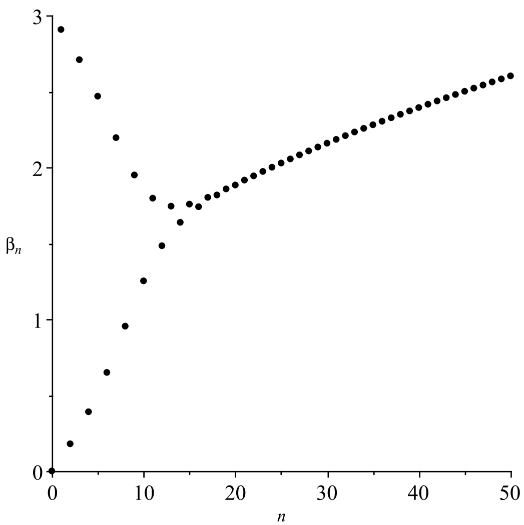
$t = 0$



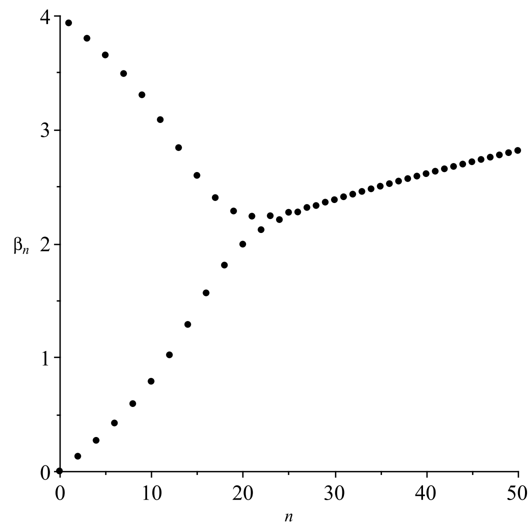
$t = 2$



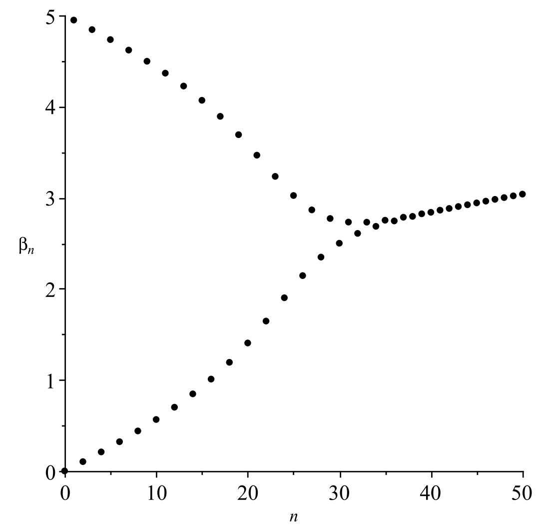
$t = 4$



$t = 6$



$t = 8$



$t = 10$

18.32 OP's with Respect to Freud Weights

A *Freud weight* is a weight function of the form

18.32.1 $$w(x) = \exp(-Q(x)), \quad -\infty < x < \infty,$$

where $Q(x)$ is real, even, nonnegative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. No explicit expressions for the corresponding OP's are available. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky (2001) and Nevai (1986). For a uniform asymptotic expansion in terms of Airy functions (§9.2) for the OP's in the case $Q(x) = x^4$ see Bo and Wong (1999).

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $w(x)$, such that

$$\int_a^b P_m(x)P_n(x)\,w(x)\,\mathrm{d}x = h_n\delta_{m,n}, \qquad h_n > 0, \qquad m, n = 0, 1, 2, \ldots$$

One of the important properties that orthogonal polynomials have is that they satisfy the **three-term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \displaystyle\int_a^b x^k\,w(x)\,\mathrm{d}x$ are the **moments** of the weight $w(x)$.

If the weight has the form

$$w(x;t) = w_0(x)\exp(tx)$$

where the integrals $\displaystyle\int_{-\infty}^{\infty} x^k w_0(x)\exp(tx)\,\mathrm{d}x$ exist for all $k \geq 0$.

- The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_n - \beta_{n+1}, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

- The kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k w_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} w_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$$

- Since $\mu_k(t) = \dfrac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$, then $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) = \det\left[\frac{\mathrm{d}^{j+k}\mu_0}{\mathrm{d}t^{j+k}}\right]_{j,k=0}^{n-1}$$

$$\widetilde{\Delta}_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-2}\mu_0}{\mathrm{d}t^{n-2}}, \frac{\mathrm{d}^{n}\mu_0}{\mathrm{d}t^{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta_n(t)$$

Generalized Freud Weight

$$w(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}, \quad \nu > -1$$

- **PAC, K Jordaan & A Kelil**, "A generalized Freud weight", *Stud. Appl. Math.*, **136** (2016) 288–320
- **PAC & K Jordaan**, "Properties of generalized Freud polynomials", *J. Approx. Theory*, **225** (2018) 148–175

Generalized Freud weight

For the **generalized Freud weight**

$$w(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}$$

the moments are

$$\mu_0(t;\nu) = \int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x = \int_0^{\infty} y^{\nu+1} \exp\left(-y^2 + ty\right) \mathrm{d}y$$
$$= 2^{-(\nu+1)/2}\Gamma(\nu+1)\, \exp(\tfrac{1}{8}t^2) D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)$$

$$\mu_{2n}(t;\nu) = \int_{-\infty}^{\infty} x^{2n}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x$$
$$= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x\right) = \frac{\mathrm{d}^n \mu_0}{\mathrm{d}t^n}$$

$$\mu_{2n+1}(t;\nu) = \int_{-\infty}^{\infty} x^{2n+1}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x = 0$$

for $n = 1, 2, \ldots$, where $D_\nu(\zeta)$ is the **parabolic cylinder function**.

When $\nu = n \in \mathbb{Z}^+$, then

$$D_{-n-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) = \tfrac{1}{2}\sqrt{2\pi}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\left[1 + \operatorname{erf}\left(\tfrac{1}{2}t\right)\right] \exp\left(\tfrac{1}{8}t^2\right)\right\},$$

where $\operatorname{erf}(z)$ is the **error function**.

Since $\mu_{2j+1}(t;\nu)=0$ then using ideas due to **Hubert [2017]**

$$\Delta_{2n}=\begin{vmatrix} \mu_0 & 0 & \mu_2 & \dots & \mu_{2n-2} & 0 \\ 0 & \mu_2 & 0 & \dots & 0 & \mu_{2n} \\ \mu_2 & 0 & \mu_4 & \dots & \mu_{2n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{2n-2} & 0 & \mu_{2n} & \dots & \mu_{4n-4} & 0 \\ 0 & \mu_{2n} & 0 & \dots & 0 & \mu_{4n-2} \end{vmatrix}=\mathcal{A}_n\mathcal{B}_n$$

$$\Delta_{2n+1}=\begin{vmatrix} \mu_0 & 0 & \mu_2 & \dots & 0 & \mu_{2n} \\ 0 & \mu_2 & 0 & \dots & \mu_{2n} & 0 \\ \mu_2 & 0 & \mu_4 & \dots & 0 & \mu_{2n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_{2n} & 0 & \dots & \mu_{4n-2} & 0 \\ \mu_{2n} & 0 & \mu_{2n+2} & \dots & 0 & \mu_{4n} \end{vmatrix}=\mathcal{A}_{n+1}\mathcal{B}_n$$

where

$$\mathcal{A}_n=\begin{vmatrix} \mu_0 & \mu_2 & \dots & \mu_{2n-2} \\ \mu_2 & \mu_4 & \dots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n-2} & \mu_{2n} & \dots & \mu_{4n-4} \end{vmatrix}, \qquad \mathcal{B}_n=\begin{vmatrix} \mu_2 & \mu_4 & \dots & \mu_{2n} \\ \mu_4 & \mu_6 & \dots & \mu_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n} & \mu_{2n+2} & \dots & \mu_{4n-2} \end{vmatrix}$$

which can be written as Wronskians since $\mu_{2j}=\dfrac{\mathrm{d}^j\mu_0}{\mathrm{d}t^j}$.

Theorem (**PAC, Jordaan & Kelil [2016]**)

The recurrence coefficient $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t),$$

is given by

$$\beta_{2n}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\nu+1)}{\tau_n(t;\nu)}, \qquad \beta_{2n+1}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}(t;\nu)}{\tau_n(t;\nu+1)}$$

where $\tau_n(t;\nu)$ is the Wronskian given by

$$\tau_n(t;\nu) = \mathcal{W}\left(\phi_\nu, \frac{\mathrm{d}\phi_\nu}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\phi_\nu}{\mathrm{d}t^{n-1}}\right)$$

with

$$\phi_\nu(t) = \mu_0(t;\nu) = \frac{\Gamma(\nu+1)}{2^{(\nu+1)/2}}\ \exp\left(\tfrac{1}{8}t^2\right) D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)$$

Remark: The function $S_n(t;\nu) = \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\tau_n(t;\nu)$ satisfies

$$4\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 - \left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \nu\right) = 0$$

which is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation, so

$$\beta_{2n}(t;\nu) = S_n(t;\nu+1) - S_n(t;\nu), \qquad \beta_{2n+1}(t;\nu) = S_{n+1}(t;\nu) - S_n(t;\nu+1)$$

Theorem

The recurrence coefficients $\beta_n(t)$ *satisfy the equation*

$$\frac{\mathrm{d}^2\beta_n}{\mathrm{d}t^2} = \frac{1}{2\beta_n}\left(\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}a_n)\beta_n + \frac{b_n}{16\beta_n} \qquad (1)$$

which is equivavlent to $\mathrm{P_{IV}}$, *where the parameters* a_n *and* b_n *are given by*

$$a_{2n} = -2\nu - n - 1, \qquad a_{2n+1} = \nu - n$$
$$b_{2n} = -2n^2, \qquad b_{2n+1} = -2(\nu + n + 1)^2$$

Further $\beta_n(t)$ *satisfies the nonlinear difference equation*

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\nu+1)[1-(-1)^n]}{8\beta_n} \qquad (2)$$

which is the general **discrete** $\mathrm{P_I}$.

Remark: The link between the differential equation (1) and the difference equation (2) is given by the **Bäcklund transformations**

$$\beta_{n+1} = \frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}, \qquad \beta_{n-1} = -\frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}$$

with $c_n = \tfrac{1}{2}n + \tfrac{1}{4}(2\nu+1)[1-(-1)^n]$.

The first few recurrence coefficients are:

$$\beta_1(t) = \Phi_\nu$$

$$\beta_2(t) = -\frac{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1}{2\Phi_\nu}$$

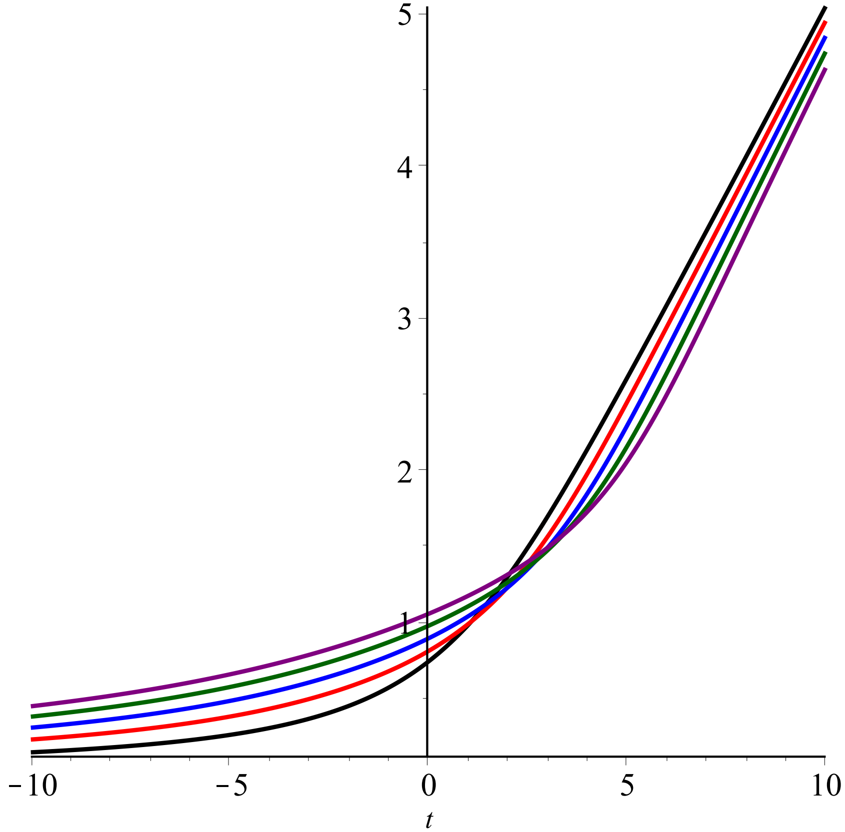
$$\beta_3(t) = -\frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} - \frac{\nu+1}{2\Phi_\nu}$$

$$\beta_4(t) = \frac{t}{2(\nu+2)} + \frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} + \frac{(\nu+1)(t^2+2\nu+4)\Phi_\nu + (\nu+1)^2 t}{2(\nu+2)[2(\nu+2)\Phi_\nu^2 - (\nu+1)t\Phi_\nu - (\nu+1)^2]}$$

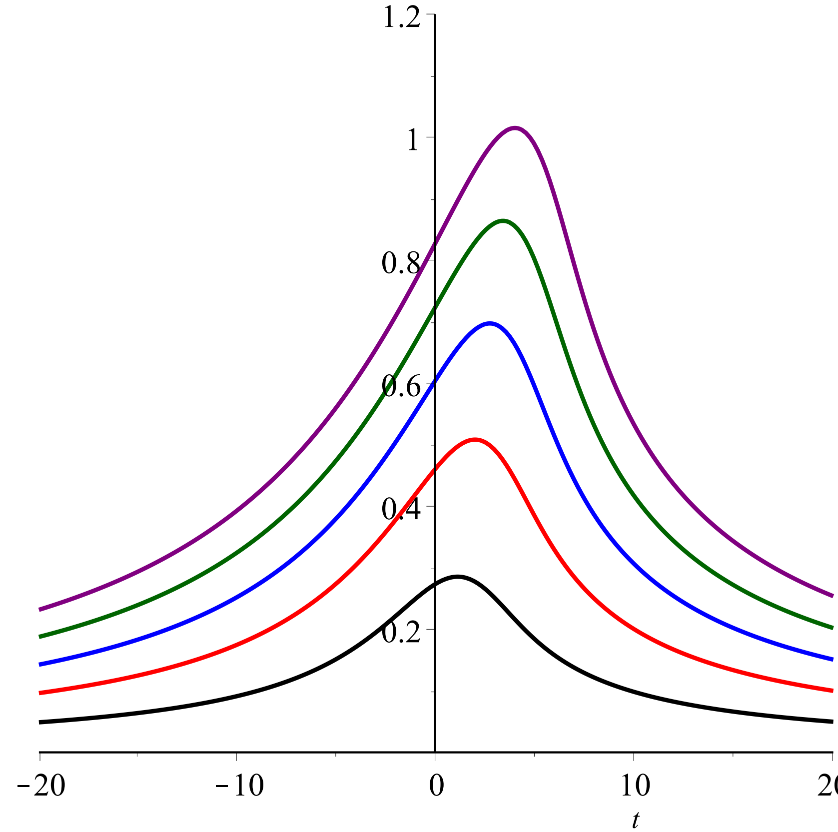
$$\beta_5(t) = -\frac{2\nu t}{\nu+1} - \frac{2(\nu+1)}{t} - \frac{2\nu(2t^2+\nu+1)\Phi_\nu - 4\nu^2 t}{(\nu+1)[(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2]} - \frac{2[\nu t^2 + (\nu+1)(2\nu+1)]\Phi_\nu^2 + 2\nu t(t^2+4\nu+5)\Phi_\nu - 4\nu^2 t^2 - 8\nu^2(\nu+1)}{t[t\Phi_\nu^3(t) + (2t^2 - 2\nu + 1)\Phi_\nu^2 - 6\Phi_\nu \nu t + 4\nu^2]}$$

where

$$\Phi_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln\left\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)\exp\left(\tfrac{1}{8}t^2\right)\right\} = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\nu}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}.$$



$\beta_{2n-1}(t;\tfrac{1}{2}), \quad n = 1, 2, \ldots, 5$



$\beta_{2n}(t;\tfrac{1}{2}), \quad n = 1, 2, \ldots, 5$

Plots of the recurrence coefficients $\beta_{2n-1}(t;\tfrac{1}{2})$ **and** $\beta_{2n}(t;\tfrac{1}{2})$, $n = 1, 2, \ldots, 5$, **for** $n = 1$ **(black),** $n = 2$ **(red),** $n = 3$ **(blue),** $n = 4$ **(green) and** $n = 5$ **(purple).**

Lemma (PAC, Jordaan & Kelil [2016])

As $t \to \infty$, the recurrence coefficient $\beta_n(t;\nu)$ has the asymptotic expansion

$$\beta_{2n}(t;\nu) = \frac{n}{t} - \frac{2n(2\nu - n + 1)}{t^3} + \mathcal{O}\big(t^{-5}\big)$$

$$\beta_{2n+1}(t;\nu) = \frac{t}{2} + \frac{\nu - n}{t} - \frac{2(\nu^2 - 4\nu n + n^2 - \nu - n)}{t^3} + \mathcal{O}\big(t^{-5}\big)$$

for $n \in \mathbb{N}$. Further, as $t \to -\infty$

$$\beta_{2n}(t;\nu) = -\frac{n}{t} + \frac{2n(2\nu + 3n + 1)}{t^3} + \mathcal{O}\big(t^{-5}\big)$$

$$\beta_{2n+1}(t;\nu) = -\frac{\nu + n + 1}{t} + \frac{2(\nu + n + 1)(\nu + 3n + 2)}{t^3} + \mathcal{O}\big(t^{-5}\big)$$

Conjecture

- $\beta_{2n+1}(t;\nu)$ *is a monotonically increasing function of* t.
- $\beta_{2n+2}(t;\nu) > \beta_{2n}(t;\nu)$, *for all* t.
- $\beta_{2n}(t;\nu)$ *has one maximum at* $t = t^*_{2n}$, *with* $t^*_{2n+2} > t^*_{2n}$.

Theorem (PAC & Jordaan [2018])

For $\nu > -1$ and $\beta_0 = 0$, there exists a ***unique*** $\beta_1(t;\nu) > 0$ *such that* $\{\beta_n(t;\nu)\}_{n\in\mathbb{N}}$ *defined by the nonlinear discrete equation*

$$\beta_n\left(\beta_{n+1} + \beta_n + \beta_{n-1} - \tfrac{1}{2}t\right) = \tfrac{1}{4}[n + (2\nu+1)\Delta_n] \tag{1}$$

with $\Delta_n = \frac{1}{2}[1-(-1)^n]$, is a ***positive sequence*** *and the solution arises when*

$$\beta_1(t;\nu) = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\nu}\left(-\frac{1}{2}\sqrt{2}\,t\right)}{D_{-\nu-1}\left(-\frac{1}{2}\sqrt{2}\,t\right)}$$

Consider the discrete equation (1) with initial conditions $\beta_0 = 0$ and

$$\beta_1 = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\left[\frac{\cos(\theta)D_{-\nu}\left(-\frac{1}{2}\sqrt{2}\,t\right) - \sin(\theta)D_{-\nu}\left(\frac{1}{2}\sqrt{2}\,t\right)}{\cos(\theta)D_{-\nu-1}\left(-\frac{1}{2}\sqrt{2}\,t\right) + \sin(\theta)D_{-\nu-1}\left(\frac{1}{2}\sqrt{2}\,t\right)}\right]$$

with $0 \le \theta \le \frac{1}{2}\pi$ a parameter (if $\frac{1}{2}\pi < \theta < \pi$ then β_1 has a pole at a finite value of t), which is the general solution of the Riccati equation

$$\frac{\mathrm{d}\Phi_\nu}{\mathrm{d}t} = -\Phi_\nu^2 + \tfrac{1}{2}t\Phi_\nu + \tfrac{1}{2}(\nu+1)$$

$$\Phi_\nu(t;\theta) = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\left[\frac{\cos(\theta)D_{-\nu}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) - \sin(\theta)D_{-\nu}\big(\tfrac{1}{2}\sqrt{2}\,t\big)}{\cos(\theta)D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + \sin(\theta)D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)}\right]$$

Since the parabolic cylinder function $D_\nu(z)$ has the asymptotics

$$D_\nu(z) = \begin{cases} z^\nu \exp(-\tfrac{1}{4}z^2)\big\{1+\mathcal{O}(z^{-2})\big\}, & \textbf{as} \quad z\to\infty, \\ \dfrac{\sqrt{2\pi}}{\Gamma(-\nu)}(-z)^{-\nu-1}\exp(\tfrac{1}{4}z^2)\big\{1+\mathcal{O}(z^{-2})\big\}, & \textbf{as} \quad z\to-\infty, \end{cases}$$

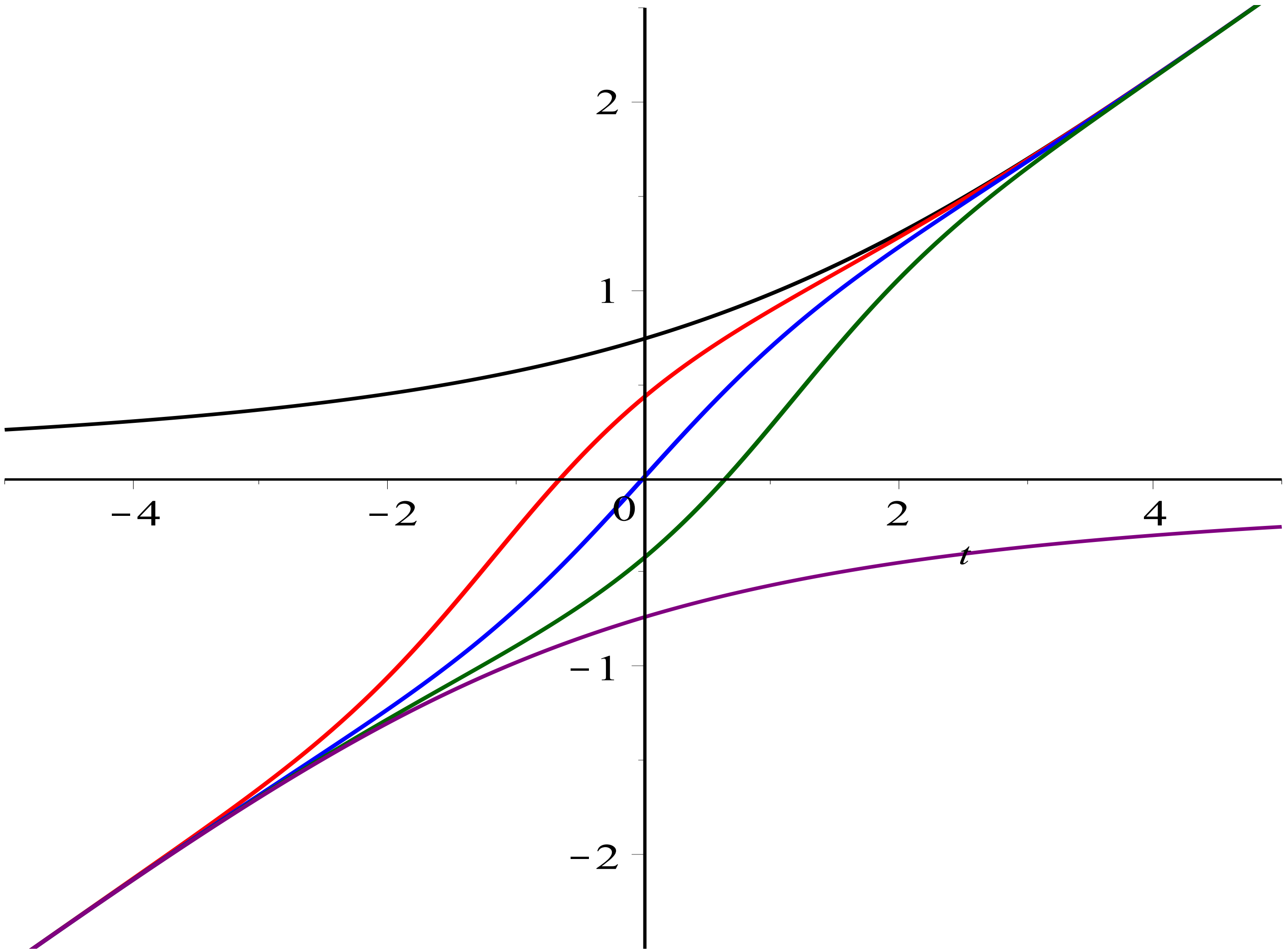
then

$$\Phi_\nu(t;0) = \begin{cases} \tfrac{1}{2}t + \mathcal{O}(t^{-1}), & \textbf{as} \quad t\to\infty, \\ -\dfrac{\nu+1}{t} + \mathcal{O}(t^{-3}), & \textbf{as} \quad t\to-\infty, \end{cases}$$

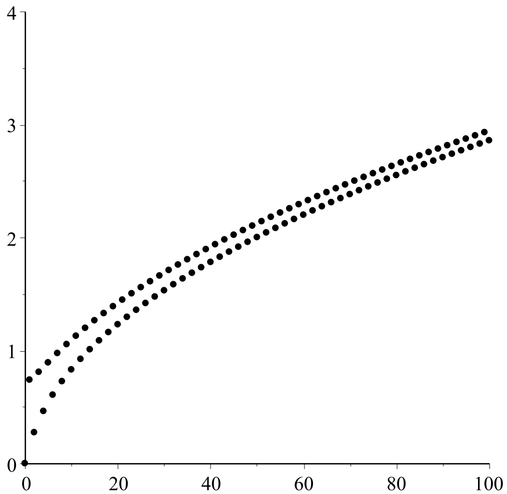
$$\Phi_\nu(t;\theta) = \tfrac{1}{2}t + \mathcal{O}(t^{-1}), \qquad \textbf{as} \quad t\to\pm\infty, \quad \textbf{if} \quad 0<\theta<\tfrac{1}{2}\pi,$$

$$\Phi_\nu(t;\tfrac{1}{2}\pi) = \begin{cases} -\dfrac{\nu+1}{t} + \mathcal{O}(t^{-3}), & \textbf{as} \quad t\to\infty, \\ \tfrac{1}{2}t + \mathcal{O}(t^{-1}), & \textbf{as} \quad t\to-\infty, \end{cases}$$

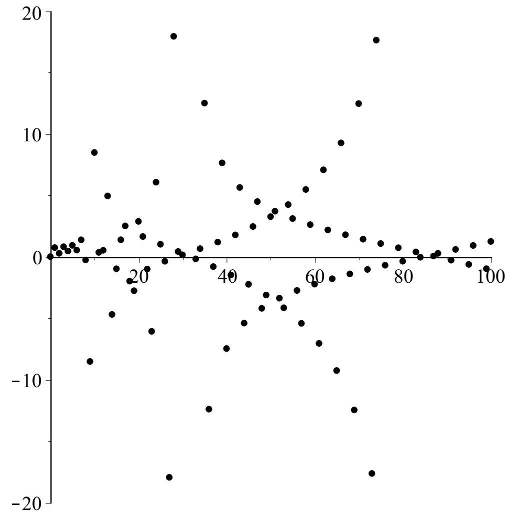
$$\Phi_\nu(t;\theta) = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\left[\frac{\cos(\theta)D_{-\nu}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) - \sin(\theta)D_{-\nu}\big(\tfrac{1}{2}\sqrt{2}\,t\big)}{\cos(\theta)D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + \sin(\theta)D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)}\right]$$



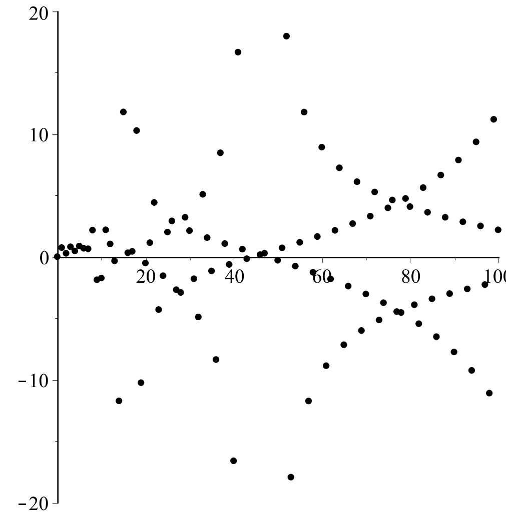
$\theta = 0$ **(black)**, $\theta = \frac{1}{12}\pi$ **(red)**, $\theta = \frac{1}{4}\pi$ **(blue)**, $\theta = \frac{5}{12}\pi$ **(green)**, $\theta = \frac{1}{2}\pi$ **(purple)**



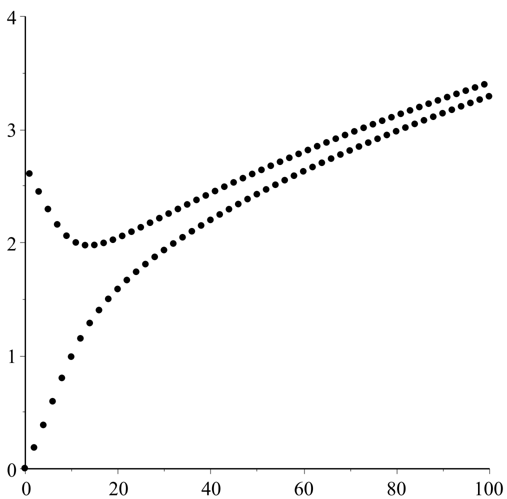
$\beta_0 = 0, \quad \beta_1 = \Phi(0;0)$



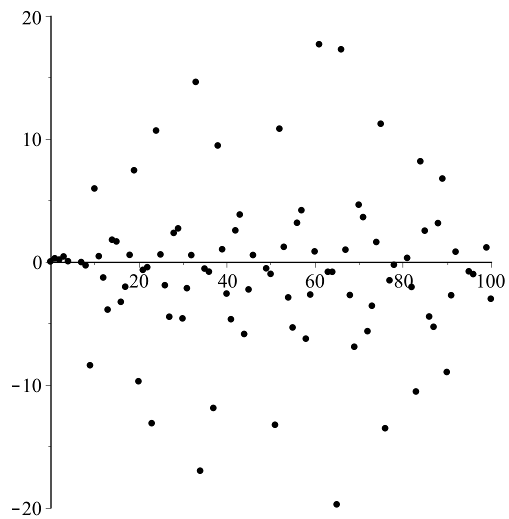
$\beta_1 = \Phi(0;0) + 10^{-4}$



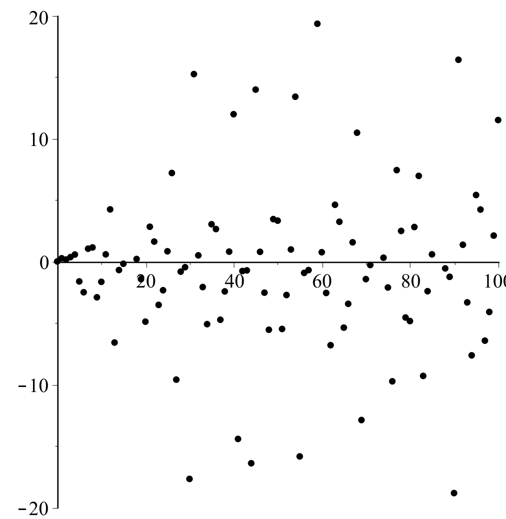
$\beta_1 = \Phi(0;0) - 10^{-4}$



$\beta_0 = 0, \quad \beta_1 = \Phi(5;0)$

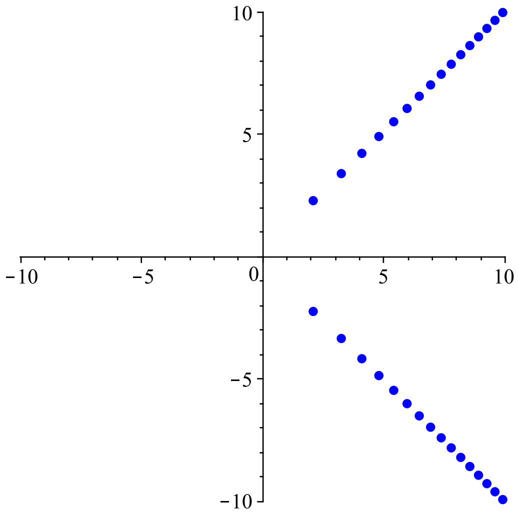


$\beta_1 = \Phi(5;0) + 10^{-4}$

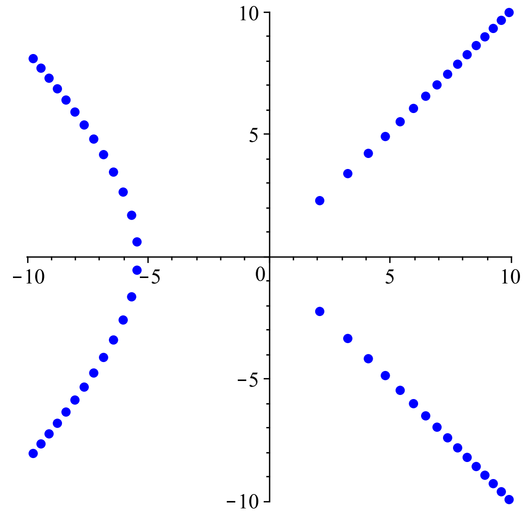


$\beta_1 = \Phi(5;0) - 10^{-4}$

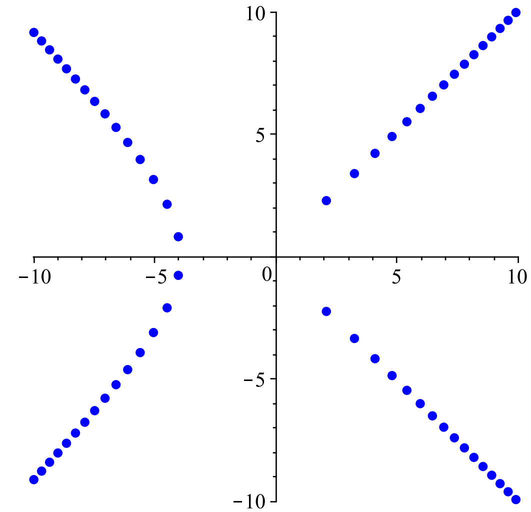
Complex roots of $\cos(\theta)D_{-\nu-1}\left(-\frac{1}{2}\sqrt{2}\,t\right)+\sin(\theta)D_{-\nu-1}\left(\frac{1}{2}\sqrt{2}\,t\right)$



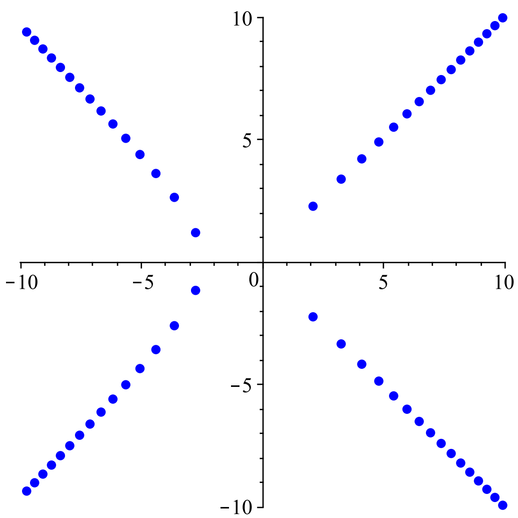
$\theta = 0$



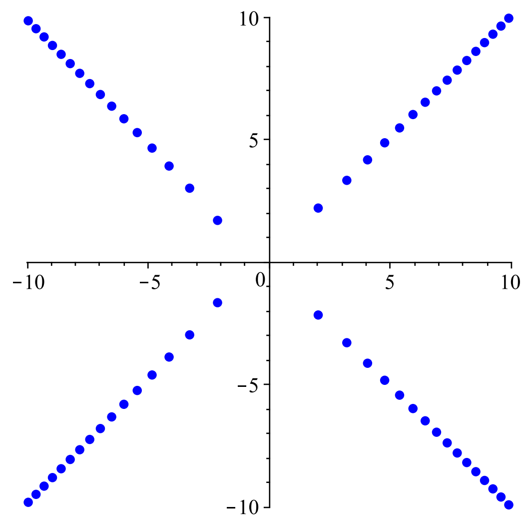
$\theta = \pi/10^7$



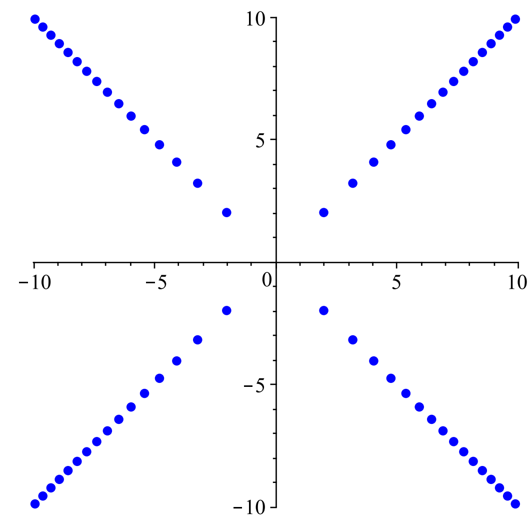
$\theta = \pi/10^4$



$\theta = \frac{1}{100}\pi$

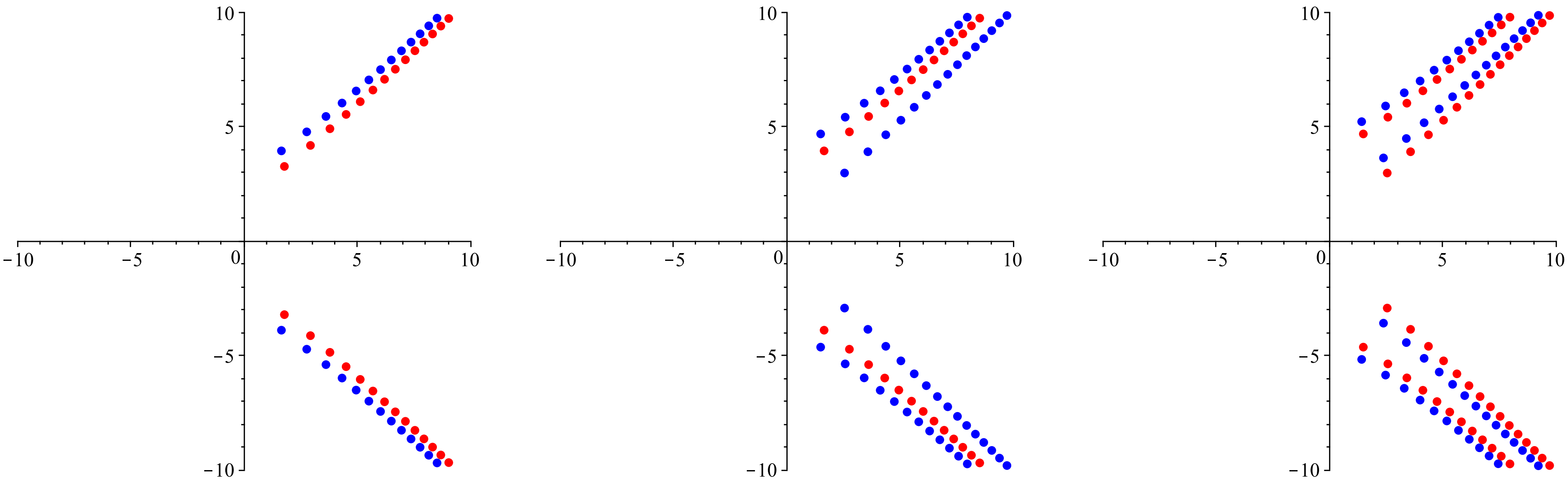


$\theta = \frac{1}{10}\pi$



$\theta = \frac{1}{4}\pi$

Complex poles of $\beta_j(t)$, $j = 2, 3, 4$



$\beta_2(t)$ $\beta_3(t)$ $\beta_4(t)$

These are **tronquée solutions** of $\mathrm{P_{IV}}$ since they have no poles in the half-plane $\Re(t) < 0$.

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Theorem

(**PAC & Jordaan [2018]**)

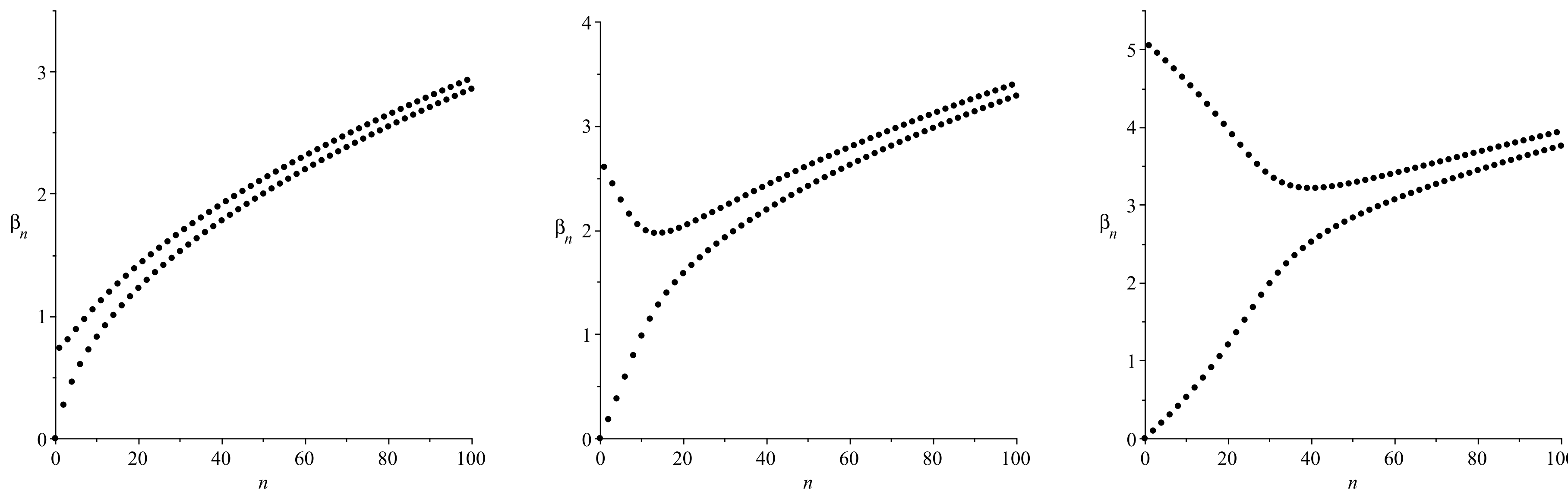
Let $t, \nu \in \mathbb{R}$, then as $n \to \infty$, the recurrence coefficient $\beta_n(t;\nu)$ associated with monic generalized Freud polynomials satisfying

$$\beta_n\left(\beta_{n+1} + \beta_n + \beta_{n-1} - \tfrac{1}{2}t\right) = \tfrac{1}{4}[n + (2\nu+1)\Delta_n],$$

where $\Delta_n = \frac{1}{2}[1-(-1)^n]$, has the asymptotic expansion

$$\beta_n(t;\nu) = \sqrt{\frac{n}{12}}\left\{1 + \frac{t}{\sqrt{12n}} + \frac{t^2 + 12(2\nu+1)\Delta_n}{24n} + \mathcal{O}(n^{-2})\right\},$$

as $n \to \infty$.



$t = 0$ $\quad$ $t = 5$ $\quad$ $t = 10$

Generalizes results of **Lew & Quarles [1983]**, **Clarke & Shizgal [1993]**

Examples Associated with the Third Painlevé Equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{aq^2+b}{z} + cq^3 + \frac{d}{q} \qquad \mathrm{P_{III}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2} - \frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 + \left[4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z^2\right]\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma\right) + 4z\vartheta_0\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}z} = (\vartheta_0^2+\vartheta_\infty^2)z^2 \qquad \mathrm{S_{III}}$$

Making the transformations

$$q(z) = t^{-1/2}u(t), \qquad \sigma(z) = 2h(t), \qquad t = \tfrac{1}{4}z^2$$

give

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{au^2}{4t^2} + \frac{b}{4t} + \frac{cu^3}{4t^2} + \frac{d}{4u} \qquad \mathrm{P_{III'}}$$

$$\left(t\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)^2 + \left\{4\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 - 1\right\}\left(t\frac{\mathrm{d}h}{\mathrm{d}t} - h\right) + \vartheta_0\vartheta_\infty\frac{\mathrm{d}h}{\mathrm{d}t} = \tfrac{1}{4}(\vartheta_0^2+\vartheta_\infty^2) \qquad \mathrm{S_{III'}}$$

Semi-classical weights with recurrence coefficients expressible in terms of solutions of S_{III} or $S_{\mathrm{III}'}$, the P_{III} and $P_{\mathrm{III}'}$ σ-equations, include

$w(x;t)$		$\mu_0(t) = \int w(x;t)\,\mathrm{d}x$
$\left(1-x^2\right)^{\nu-1/2}\mathrm{e}^{\pm tx}$	$x\in[-1,1]$	$2^{\nu}\,\Gamma(\nu+\tfrac12)\pi^{-1/2}\,t^{-\nu}I_{\nu}(t)$
$\left(x^2-1\right)^{\nu-1/2}\mathrm{e}^{-tx}$	$x\in[1,\infty)$	$2^{\nu}\,\Gamma(\nu+\tfrac12)\pi^{-1/2}\,t^{-\nu}K_{\nu}(t)$
$x^{\nu-1}\,\mathrm{e}^{-x-t/x}$	$x\in[0,\infty)$	$2t^{\nu/2}K_{\nu}\left(2\sqrt{t}\right)$

with $I_\nu(z)$ and $K_\nu(z)$ **modified Bessel functions**, which have the integral representations

$$I_\nu(z) = \frac{\left(\frac12 z\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\int_{-1}^{1}(1-x^2)^{\nu-1/2}\,\mathrm{e}^{\pm xz}\,\mathrm{d}x$$

$$K_\nu(z) = \frac{\left(\frac12 z\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\int_{1}^{\infty}(x^2-1)^{\nu-1/2}\,\mathrm{e}^{-xz}\,\mathrm{d}x$$

$$K_\nu(2\sqrt{t}) = \tfrac12 t^{-\nu/2}\int_0^\infty x^{\nu-1}\,\exp\left(-x-t/x\right)\mathrm{d}x$$

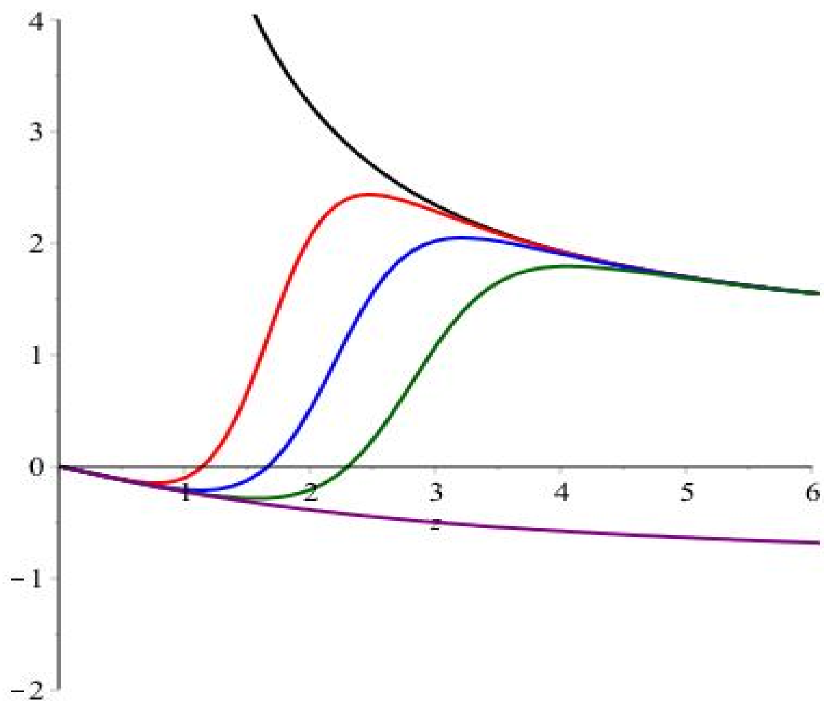
Modified Bessel function solution of $\mathrm{P_{III}}$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{aq^2+b}{z} + cq^3 + \frac{d}{q} \qquad \mathrm{P_{III}}$$

has the solution

$$q(z) = \frac{2\nu}{z} + \frac{\cos(\theta)I_{\nu+1}(z) - \sin(\theta)K_{\nu+1}(z)}{\cos(\theta)I_\nu(z) + \sin(\theta)K_\nu(z)}$$

with $I_\nu(z)$, $K_\nu(z)$ **modified Bessel functions** and θ a constant, for the parameters $a=-2\nu$, $b=2(1-\nu)$, $c=1$ and $d=-1$



$\boldsymbol{\theta = 0}$

$\boldsymbol{\theta = \frac{1}{20}\pi}$

$\boldsymbol{\theta = \frac{1}{4}\pi}$

$\boldsymbol{\theta = \frac{9}{20}\pi}$

$\boldsymbol{\theta = \frac{1}{2}\pi}$

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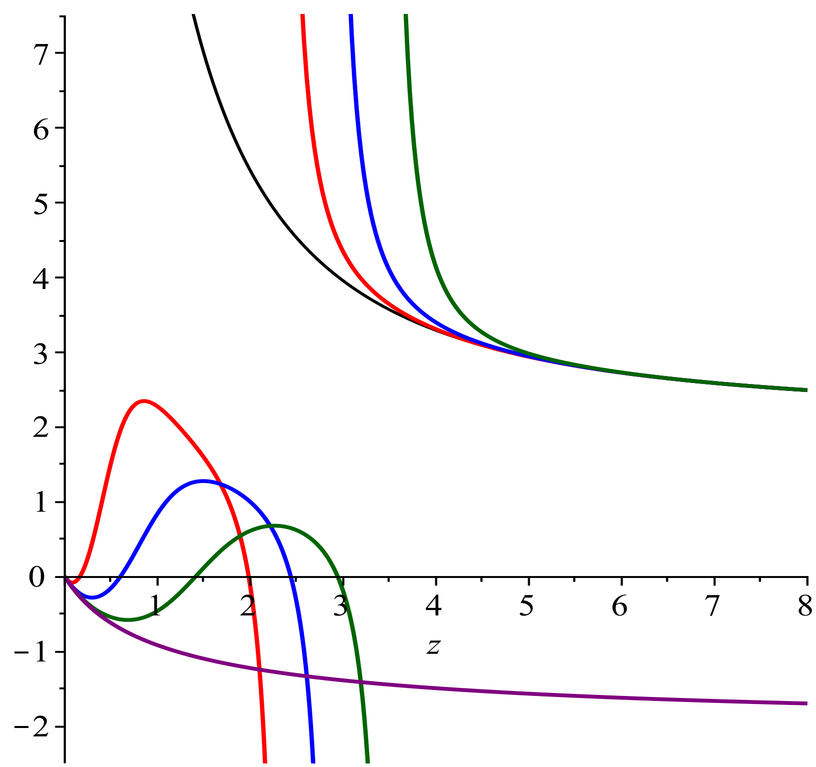
Consider the tau-function

$$\tau_n(z;\nu) = \det\left[z^{\nu+j+k}\left\{\cos(\theta)I_{\nu+j+k}(z) + \sin(\theta)K_{\nu+j+k}(z)\right\}\right]_{j,k=0}^{n-1}$$

then

$$S_n(z;\nu) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\nu)$$

satisfies an equation equivalent to $\mathrm{S_{III}}$.



$\boldsymbol{\theta = 0}$

$\boldsymbol{\theta = \frac{1}{20}\pi}$

$\boldsymbol{\theta = \frac{1}{4}\pi}$

$\boldsymbol{\theta = \frac{9}{20}\pi}$

$\boldsymbol{\theta = \frac{1}{2}\pi}$

Conclusions

- The coefficients in the three-term recurrence relations associated with semi-classical generalizations of orthogonal polynomials can often be expressed as tau-functions (Hankel determinants) which arise in the solution of the Painlevé equations and the Painlevé σ-equations.
- These solutions of the Painlevé equations are those given in terms of the classical special functions, the so-called "classical solutions", which are not transcendental.
- The moments of the semi-classical weights provide the link between the orthogonal polynomials and the associated Painlevé equation.
- These ideas can be applied to orthogonal polynomials in other contexts:
  - **discrete orthogonal polynomials**;
  - **on the unit circle**;
  - **curves in the complex plane**;
  - **discontinuous weights**.
- These results illustrate the increasing significance of integrable systems, in particular Painlevé equations, in the field of orthogonal polynomials and special functions.

Thank You!

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Remark 5.3 (Relation with Painlevé equations). Since all the Painlevé equations arise as equations of isomonodromic deformation of certain 2×2 ODEs with prescribed pole locations, it is clear that choosing appropriately the set of contours and the semiclassical measure one can generate special solutions of the Painlevé equations in terms of such orthogonal polynomials.