Renormalized Stochastic PDE's

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The Finite Dimensional Case

$$\dot{X}_t = -\nabla V(X_t) + \sigma \xi_t, \qquad X_0 = x$$

- $V : \mathbb{R}^d \to \mathbb{R}$ is a potential, $\sigma \in \mathbb{R}$ is a scalar
- ξ_t is a white noise is time $\mathbb{E}[\xi_t \xi_s] = \delta(t - s)$
- Equivalently, $\xi = \dot{W}$, where W is a Brownian motion on $\mathbb{R}^d$
- The solution X_t^x is a stochastic process indexed by time t and the initial condition $x \in \mathbb{R}^d$
- Let μ be a stationary probability distribution on $\mathbb{R}^d$, that is for all times $t > 0$ and Borel sets $A \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \mathbb{P}(X_t^x \in A)\mu(x)dx = \int_A \mu(y)dy.$$

- Recall that the paths of $t \mapsto W_t$ are not smooth, they belong only to $C^{\frac{1}{2}-\kappa}$ for any $\kappa > 0$ (not $C^{\frac{1}{2}}$).
- This makes calculus more interesting: for a smooth $\psi : \mathbb{R}^d \to \mathbb{R}$ it holds

$$\frac{d}{dt}\mathbb{E}^x[\psi(X_t^x)] = \mathbb{E}^x\big[-\nabla V(X_t^x)\cdot\nabla\psi(X_t^x) + \frac{1}{2}\sigma^2\Delta\psi(X_t^x)\big].$$

Averaging over $x \in \mathbb{R}^d$ with respect to μ gives

$$\begin{aligned} 0 &= \sum_i \int_{\mathbb{R}^d} \big[-\partial_i V(y)\partial_i\psi(y) + \partial_i^2\psi(y)\big]\mu(y)dy \\ &= \sum_i \int_{\mathbb{R}^d} \psi(y)\partial_i\big[\partial_i V(y)\mu(y) + \frac{\sigma^2}{2}\partial_i\mu(y)\big]dy \end{aligned}$$

This leads to the familiar $\mu(y) = Cexp(-\frac{1}{2\sigma^2}V(y))$.

Stochastic Quantization of Φ_4

$$\partial_t u = \Delta u - u^3 + \xi$$

- ξ is a space/time white noise

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

- Note that $\Delta u - u^3 = -\nabla V(u)$, where

$$V(u) = \int |\nabla u|^2 + \frac{1}{4}u^4 \, dx.$$

- Formally, the invariant measure for the (infinite dimensional) stochastic dynamics is again $Cexp(-V(u))du$.
- (Parisi-Wu) Try to understand the measure $Cexp(-V(u))du$ through the (infinite dimensional) stochastic dynamics.

A Fundamental Difficulty

$$\partial_t u = \Delta u - u^3 + \xi, \qquad V(u) = \int |\nabla u|^2 + \frac{1}{4}u^4 \; dx$$

- If the noise ξ is not too rough, the PDE above is well-posed (and well understood in the deterministic literature). For instance, if $\xi = 0$ one has

$$\frac{d}{dt}|u|_{L^2}^2 + V(u) \leq 0.$$

- However, space-time white noise ξ is very singular, it is not a function, it belongs to $C^{-\frac{d+2}{2}-\kappa}$ for all $\kappa > 0$. .
- Consider the solution v to the linearized problem

$$(\partial_t - \Delta_x)v = \xi,$$

By Schauder theory for parabolic equations, v belongs to $C^{1-\frac{d}{2}-\kappa}$, so at best u belongs to this space, and there is no canonical meaning of u^3.

2d Case: Da-Prato/Debussche

$$\partial_t u = \Delta u - :u^3: + \xi \quad \text{on} \quad \mathbb{T}^2 \times \mathbb{R}_+$$

1. Earlier work by Jona Lasinio/Mitter, Albeverio/Rockner, and Mikulevicius/Rozovsky with different methods
2. Da-Prato/Debussche gave the first pathwise approach, dividing the problem into a probabilistic step and an analytic step.

- Key Idea: Split the solution into a rough peice v and a more regular peice $u - v = w$ and re-write the equation as

$$\partial_t w = \Delta w - :(w+v)^3:$$

A Detour into Rough Paths

Return to the finite dimensional case and consider instead the (driftless) ODE with multiplicative noise

$$\dot{X} = \sigma(X)\dot{W},$$

where σ is some smooth function.

- Similar power counting issue, $\sigma(X) \in C^{\frac{1}{2}-\kappa}$ for all κ, while $\dot{W} \in C^{-\frac{1}{2}-\kappa}$, so no canonical meaning for $\sigma(X)\dot{W}$.
- Classical approach of Ito is probabilistic, searches for a solution in a class of (adapted), random processes.
- Approach by Terry Lyons and further developed by Massimilliano Gubinelli is pathwise. Efficient splitting into a probabilistic step and an analytic step.

$$\dot{X} = \sigma(X)\dot{W},$$

- (Analytic Step) Define a class of "controlled rough paths" X for which there exists a σ such that

$$X_t = X_s + \sigma_s(W_t - W_s) + O(|s-t|^{1-2\kappa})$$

  1. (Reconstruction) Given X controlled and a meaning for $W \diamond \dot{W}$, give a meaning to $\sigma(X)\diamond\dot{W}$.
  2. (Integration) Given a meaning for $\sigma(X) \diamond \dot{W}$ and a solution X to the ODE, show that X is controlled.
- (Probabilistic Step) Use stochastic analysis to define $W \diamond \dot{W}$.
- Better to think of the solution as a linear form acting on vectors $(1, W) \in \mathbb{R}^2$ via

$$t \mapsto X_t 1 + \sigma_t W.$$

$d = 3$ and beyond

Theorem (Hairer)

There exists a choice of constants C_ϵ such that the sequence of solutions to

$$\partial_t u_\epsilon = \Delta u_\epsilon + C_\epsilon u_\epsilon + \xi_\epsilon \quad \textit{on} \quad \mathbb{T}^3 \times \mathbb{R}_+$$

converges to a limit (for small times).

- Alternative proof using Paracontrolled Calculus by Catellier/Chouk.
- Global a priori bounds derived by Mourrat and Weber (should yield an invariant measure). Alternative, direct construction of an invariant measure by Albeverio/Kusuoka.
- (Highly non-trivial) Extension to $\Phi_4^{4-\delta}$ as a culmination of work by Bruned, Chandra, Chevrev, Hairer, Zambotti.

Singular SPDE Philosophy

Heightened interest in Singular SPDE:

- Regularity Structures: Hairer
- Paracontrolled Distributions: Gubinelli/Imkeller/Perkowski, Bailleul/Bernicot
- Renormalization group: Kupainen

General themes

- Models: building blocks
- Modelled Distributions: “regularity” in quotation marks
- Integration: gaining “regularity” through the PDE
- Reconstruction: using “regularity” + off-line inputs to give a meaning to the non-linearity

Quasi-linear SPDE

$$\partial_t u - A(u)\partial_x^2 u = f.$$

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Motivation: Toy model for PDE's of a more geometric nature, intrinsic interest in PDE and probability, Feynmac-Kac representation (for regular enough f).

For notational convenience, one space dimension. For simplicity, periodic in space/time, re-label coordinates

$$\partial_2 u - A(u)\partial_1^2 u = f,$$

for $x = (x_1, x_2) \in [0, 1]^2$ where x_1 is for space, x_2 is for time.

A Fundamental Difficulty

$$\partial_2 u - A(u)\partial_1^2 u = f,$$

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1. (Integration) For $f \in C^{\alpha-2}$, expect $u \in C^{\alpha}$.
2. (Reconstruction) For $A(u) \in C^{\alpha}$ and $\partial_1^2 u \in C^{\alpha-2}$, meaning of $A(u)\partial_1^2 u$?

Classical view: regularity α of $A(u)$ must over-compensate for the irregularity $\alpha - 2$ of $\partial_1^2 u$.

- $\alpha \in (2, \infty)$: Pointwise solutions
- $\alpha \in (1, 2)$: Distributional solutions
  $(\alpha - 2 < 0, \quad \alpha + \alpha - 2 > 0)$

Modern view: need an enhanced definition of regularity of u and additional off-line structure of f

- $\alpha \in (0, 1)$: Active area, focus of the remainder of the talk
  $(\alpha + \alpha - 2 < 0)$

Related Singular SPDE Literature

Quasi-linear

1. Otto-Weber ($\alpha \in (\frac{2}{3}, 1)$, parametric model)

$$(\partial_2 - a_0\partial_1^2)v_\alpha(\cdot, a_0) = f.$$

$$u(y) - u(x) \overset{2\alpha}{\approx} \delta_{A(u(x))\cdot}(v_\alpha(y) - v_\alpha(x))$$

$$[(\cdot)_T, A(u)] \diamond \partial_1^2 u \overset{3\alpha-2}{\approx} A'(u)\delta_{A(u)\cdot}[(\cdot)_T, v_\alpha]\diamond\partial_1^2 v_\alpha$$

2. Furlan-Gubinelli , Bailleul-Debussche-Hofmanova ($\alpha \in (\frac{2}{3}, 1)$ paracontrolled approach with and without a parametric ansatz, respectively)
3. Gerenscer-Hairer (parametric model, regularity structures: big progress for $\alpha > 0$, one (crucial) step working only for $\alpha > \frac{1}{2}$) Transformation method
4. Otto-Sauer-Smith-Weber (parametric, regularity structures (with some twists); abstract tools work for $\alpha > 0$, concrete results for $\alpha > \frac{1}{2}$ posted, $\alpha > \frac{2}{5}$ also checked) Direct approach

Flexible Approach

Study (linear) parabolic PDE's with $a \in C^\alpha$ and $f \in C^{\alpha-2}$

$$\partial_2 u - a \diamond \partial_1^2 u = f$$

- Off-line step: Given a (small) model f, v_α, $v_\alpha \diamond \partial_1^2 v_\alpha$, $w_{2\alpha}$, $w_{2\alpha} \diamond \partial_1^2 v_\alpha$, $v_{2\alpha} \diamond \partial_1^2 w_{2\alpha}$, $w_{3\alpha}$, ...
- Functional framework: Modelled distributions V, linear forms acting on (abstract) functions of several parameters $(a_0, a_0', \dots)$.
- The solution map: Build the (non-linear) map $V_a \mapsto V_u$ which takes a modelled distribution describing a into a modelled distribution describing u.
- Reconstruction: Given V_a and V_u, build and characterize $a \diamond \partial_1^2 u$.
- Integration: Given $a \diamond \partial_1^2 u$ and the solution u to the PDE, show that that V_u is a modelled distribution.

Non-linear problem follows via a straightforward iteration

$$V_{\tilde{u}} \mapsto V_a \mapsto V_u, \qquad a = A(\tilde{u}).$$

Abstract Integration Theorem (Local Splitting Method)

Let $\eta \in (1,2)$ and $(x,y) \mapsto U(x,y)$ be a bounded, continuous function (periodic in y) with the following properties:

1. For all base points x and length scales $T^{\frac{1}{4}}, R \leq 1$ it holds

$$\inf_{a_0 \in I} \|(\partial_2 - a_0 \partial_1^2)(\cdot)_T U(x,\cdot)\|_{B_R(x)} \leq \sum_{\beta \in \mathrm{A}} (T^{\frac{1}{4}})^{\eta-2-\beta} R^{\beta}.$$

2. For all x, y, z it holds

$$\big|U(x,y) - U(x,z) - U(y,y) + U(y,z) - \gamma(x,y)(z-y)_1\big| \leq \sum_{\beta \in \mathrm{A}} d^{\eta-\beta}(y,x) d^{\beta}(z,y)$$

   for some function $(x,y) \mapsto \gamma(x,y)$.

Then there exists a continuous function ν such that for all x, y

$$\big|U(x,y) - U(x,x) - \nu(x)(y-x)_1\big| \lesssim d^{\eta}(y,x).$$

Main (Concrete) Theorem

Given:

- $\alpha \in (\frac{2}{5}, 1)$ and a (small) model consisting of $C^{\alpha-2}$ distributions $v_+ \diamond v_-$ and C^α functions v_+.
- modelled distribution V_a of order η (large enough) describing a with $[a]_\alpha$ small

There exists a unique modelled distribution V_u of order η (describing to lowest order a function u) with the following properties.

- $V_{\partial_1^2 u}$ is algebraically determined by V_a.
- There exists a unique distribution $a \diamond \partial_1^2 u$ such that

$$\lim_{T \to 0} \|(a \diamond \partial_1^2 u)_T - a(\partial_1^2 u)_T - (\overline{V}_a \otimes V_{\partial_1^2 u})'.(v_+ \diamond v_-)_T\| = 0,$$

  where $\overline{V}_a = V_a - a$.
- $\partial_2 u - a \diamond \partial_1^2 u = f$ in the sense of distributions.

Moreover, the map $V_a \mapsto V_u$ is bounded. For $\alpha \in (\frac{1}{2}, 1)$, also locally Lipschitz.

The Positive Model: $\alpha \in (\frac{1}{2}, \frac{2}{3})$

$v_+ := (1, v_\alpha, x_1, w_{2\alpha})$

$$(\partial_2 - a_0 \partial_1^2) v_\alpha(\cdot, a_0) = f$$
$$(\partial_2 - a_0 \partial_1^2) w_{2\alpha}(\cdot, a_0', a_0) = (v_\alpha \diamond \partial_1^2 v_\alpha)(\cdot, a_0', a_0),$$

where f and $v_\alpha \diamond \partial_1^2 v_\alpha(a_0', a_0)$ in $C^{\alpha-2}$ satisfy

$$(T^{\frac{1}{4}})^{2-\alpha} \|f_T\| \leq N, \quad (T^{\frac{1}{4}})^{2-2\alpha} \|(v_\alpha \diamond \partial_1^2 v_\alpha)_T - v_\alpha (\partial_1^2 v_\alpha)_T\| \leq N^2.$$

Lemma

For all x, y it holds (for some function ω)

$$\Big| w_{2\alpha}(y, a_0', a_0) - w_{2\alpha}(x, a_0', a_0) - v_\alpha(x, a_0')\big(\partial_{a_0} v_\alpha(y, a_0) - \partial_{a_0} v_\alpha(x, a_0)\big) - \omega(x, a_0, a_0')(y-x)_1 \Big| \lesssim N^2 d^{2\alpha}(x, y).$$

$$(x, y) \mapsto U(x, y, a_0', a_0) := w_{2\alpha}(y, a_0', a_0) - v_\alpha(x, a_0') \partial_{a_0} v_\alpha(y, a_0).$$

Function-like Modelled Distributions

Define a graded Banach space for placeholders
$v_+ = (1, v_\alpha, x_1, w_{2\alpha})$.

$$
\begin{aligned}
\mathsf{T}_+ &= \mathsf{T}_0 \oplus \mathsf{T}_\alpha \oplus \mathsf{T}_1 \oplus \mathsf{T}_{2\alpha} \\
&:= \mathbb{R} \oplus C^2(I) \oplus \mathbb{R} \oplus C^{2,1}(I \times I),
\end{aligned}
$$

where $C^{2,1}(I \times I) = C^2(I) \otimes C^1(I)$.
A modelled distribution V_u describing u to order η is a family of linear forms on T_+ indexed by points $x \in \mathbb{R}^2$ such that
Continuity condition: For each $x, y \in \mathbb{R}^2$ and placeholder $v_+ \in \mathsf{T}_+$

$$
\begin{aligned}
&\left|(V_u(y) - V_u(x)).v_+\right| \\
&\quad \le d^\eta(y,x)|1| + d^{\eta-\alpha}(y,x)\|v_\alpha - v_\alpha(x)1\|_{\mathsf{T}_\alpha} \\
&\qquad + d^{\eta-2\alpha}(y,x)\|w_{2\alpha} - w_{2\alpha}(x)1 \\
&\qquad\qquad - v_\alpha(x) \otimes (\partial_{a_0} v_\alpha - \partial_{a_0} v_\alpha(x)1)\|_{\mathsf{T}_{2\alpha}}.
\end{aligned}
$$

$u(y) - V_u(x).v_+(y) = O(d^\eta(x,y))$ (by prev. Lemma).

Algebraic Lemma

Given V_a on T_+, define $V_{\partial_1^2 u}$ on placeholders $v_- = (\partial_1^2 v_\alpha, \partial_1^2 w_{2\alpha})$ in $\mathsf{T}_- = \partial_1^2 \mathsf{T}_+$ by

$$V_{\partial_1^2 u}.v_- = \delta_a.\partial_1^2 v_\alpha + (\overline{V}'_a \otimes \delta_a).\begin{pmatrix} \partial_{a_0}\partial_1^2 v_\alpha \\ \partial_1^2 w_{2\alpha}, \end{pmatrix}$$

where. V'_a is the reduction of the form V_a to act only on the placeholders $(1, v_\alpha) \in \mathsf{T}_0 \oplus \mathsf{T}_\alpha$.

Lemma

Given a modelled distribution V_a of order η on T_+, the above definition of $V_{\partial_1^2 u}$ yields a modelled distribution of order $\eta - 2$ on T_-.

The Negative Model: $\alpha \in (\frac{1}{2}, \frac{2}{3})$

Recall that the rough diffusion operator is characterized by

$$\lim_{T \to 0} \|(a \diamond \partial_1^2 u)_T - a(\partial_1^2 u)_T - (\overline{V}_a \otimes V_{\partial_1^2 u})'.(v_+ \diamond v_-)_T\| = 0.$$

Need additional distributions $w_{2\alpha} \diamond \partial_1^2 v_\alpha$, $v_\alpha \diamond \partial_1^2 w_{2\alpha}$

$$v_+ \diamond v_- = \begin{pmatrix} \partial_1^2 v_\alpha & v_\alpha \diamond \partial_1^2 v_\alpha & x_1 \partial_1^2 v_\alpha & w_{2\alpha} \diamond \partial_1^2 v_\alpha \\ \partial_1^2 w_{2\alpha} & v_\alpha \diamond \partial_1^2 w_{2\alpha} & & \end{pmatrix}$$

Level $3\alpha - 2$:

$$(T^{\frac{1}{4}})^{3\alpha-2} \|[(\cdot)_T, w_{2\alpha}] \diamond \partial_1^2 v_\alpha(a_0'', a_0', a_0) - v_\alpha(a_0'')\partial_{a_0'}[(\cdot)_T, v_\alpha] \diamond \partial_1^2 v_\alpha(a_0', a_0)\| \leq N^3$$

$$(T^{\frac{1}{4}})^{3\alpha-2} \|[(\cdot)_T, v_\alpha] \diamond \partial_1^2 w_{2\alpha}(a_0'', a_0', a_0) - v_\alpha(a_0')\partial_{a_0}[(\cdot)_T, v_\alpha] \diamond \partial_1^2 v_\alpha(a_0'', a_0)\| \leq N^3$$

Thanks for your attention