AQFT and VOAs

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Introduction

- Vertex operator algebras (VOAs) and conformal nets on S^1 give two different mathematically rigorous frameworks for chiral conformal quantum field theories (chiral CFTs) .
- In this talk I will give an overview of the present status of understanding in relation to the connections between these two approaches.

Chiral CFT

- Two-dimensional CFT $\equiv$ scaling invariant quantum field theories on the two-dimensional Minkowski space-time admitting conformal symmetry. Certain relevant fields (the chiral fields) depend only on $x - t$ (right-moving fields) or on $x + t$ (left-moving fields).
- Chiral CFT $\equiv$ CFT generated by left-moving (or right-moving) fields only. Chiral CFTs can be considered as QFTs on $\mathbb{R}$ and by conformal symmetry on its compactification $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Hence we can consider quantum fields on the unit circle $\Phi(z)$, $z \in S^1$ and the corresponding smeared field operators $\Phi(f)$, $f \in C^\infty(S^1)$.

Conformal nets on S^1

- Conformal nets are the chiral CFT version of algebraic quantum field theory (AQFT).
- A (local) conformal net $\mathcal{A}$ on S^1 is an inclusion preserving map $I \mapsto \mathcal{A}(I)$ from the set of (proper) intervals of S^1 into the set of von Neumann algebras acting on a fixed Hilbert space $\mathcal{H}_{\mathcal{A}}$ (the vacuum sector).
- The map is assumed to satisfy certain natural (and physically motivated) conditions: locality; conformal covariance; energy bounded from below; existence of the vacuum $\Omega \in \mathcal{H}_{\mathcal{A}}$.

Vertex operator algebras

- In the vertex operator algebra approach to CFT the theory is formulated in terms of fields i.e. operator valued formal distributions (equivalently formal power series with operator coefficients) with some additional requirements.
- A vertex operator algebra (VOA) is a vector space V (the vacuum sector) together with a linear map (the state-field correspondence)

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V)$$

  from V into the set of fields acting on V.
- The map $a \mapsto Y(a,z)$ is assumed to satisfy certain natural (and physically motivated) conditions: locality; conformal covariance; energy bounded from below; vacuum. The fields $Y(a,z)$ are called vertex operators.

- In order to make contact with the theory of conformal nets we need a unitary structure on $V \Rightarrow$ unitary VOAs. In this case the uniqueness of the vacuum for conformal nets (irreducibility) corresponds to the assumption that V is a simple VOA.
- It is useful to define the endomorphisms $a_n \in \mathrm{End}(V)$ by

$$Y(z^{L_0}a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}.$$

  Here L_0 is the conformal energy operator. If $L_0 a = da$ then $a_n = a_{(n+d-1)}$.
- From now on V will be a simple unitary VOA.

From VOAs to conformal nets

- The general problem of constructing conformal nets from VOAs has been recently considered by S. C., Y. Kawahigashi, R. Longo and M. Weiner: (Memoirs of the AMS 2018), [CKLW2018].
- We assume that V is energy-bounded i.e. that for every $a \in V$ there exist positive integers s, j and a constant $K > 0$ such that

$$\|a_n b\| \leq K(|n|+1)^s \|(L_0 + 1_V)^j b\| \ \forall n \in \mathbb{Z}, \ \forall b \in V.$$

- Let $\mathcal{H}_V$ be the Hilbert space completion of V and let $f \in C^\infty(S^1)$ with Fourier coefficients $\hat{f}_n$. For every $a \in V$ we define the operator $Y^0(a, f)$ on $\mathcal{H}_V$ with domain V by

$$Y^0(a,f)b = \sum_{n\in\mathbb{Z}} a_n \hat{f}_n b \quad \text{for } b \in V.$$

It is a closable operator and we denote its closure by $Y(a, f)$ (smeared vertex operator).

- We define a map $\mathcal{A}_V$ from the set of intervals of S^1 into the the set of von Neumann algebras on $\mathcal{H}_V$ by

$$\mathcal{A}_V(I) = \text{von Neumann algebra generated by}$$

$$\{Y(a, f) : a \in V, f \in C_c^\infty(I)\}.$$

- It is clear that the map $I \mapsto \mathcal{A}_V(I)$ is inclusion preserving.
- Definition [CKLW2018]: V is strongly local if $\mathcal{A}_V$ satisfies locality.

For a strongly local V we have the following results [CKLW2018]:

- $\mathcal{A}_V$ is a conformal net on S^1.
- The map $V \mapsto \mathcal{A}_V$ is "well behaved". Natural constructions in the VOA setting (subVOAs, tensor products) preserve strong locality.
- Many examples of unitary VOAs are known to be strongly local: unitary VOAs generated affine Lie algebras, the corresponding coset and orbifold subalgebras; unitary Virasoro VOAs; unitary VOAs with central charge $c = 1$; the moonshine VOA $V^\natural$ whose automorphism group is the monster group $\mathbb{M}$, the even shorter moonshine VOA $VB^\natural_{(0)}$ whose automorphism group is the baby monster group $\mathbb{B}$.

Back to VOAs and two conjectures

- In 1996 K. Fredenhagen and M. Jörss proposed a construction of certain fields starting form a conformal net $\mathcal{A}$ (FJ fields).
- In our work we show that if V is strongly local then the FJ fields of $\mathcal{A}_V$ give back the vertex operators of V.
- Conjecture 1. [CKLW2018] Every simple unitary VOA is strongly local.
- Conjecture 2. [CKLW2018] For every conformal net $\mathcal{A}$ there is a strongly local VOA V such that $\mathcal{A} = \mathcal{A}_V$.

Representation theory

- Conformal nets and VOAs have very interesting representation theories (theory of superselection sectors).
- These representation theories are also very important for the construction and classification of chiral CFTs. For this reason the study of the above conjectures should also requires a direct connection between the representation theories VOAs and those of the corresponding of conformal nets.
- Connecting the representation theories in a direct way is interesting in itself and has many potential applications. Some recent progress in this direction have been made by S.C, M. Weiner and F. Xu [CWX$\geq$2018] (in preparation). Further progress has been mad by B. Gui (arXiv 2017).

Representations of conformal nets

- A representation π of a conformal net $\mathcal{A}$ is a family $\{\pi_I : I \subset S^1 \text{ is a proper interval}\}$, where each π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$, which is compatible with the net structure, i.e. $\pi_{I_2} \restriction_{\mathcal{A}(I_1)} = \pi_{I_1}$ if $I_1 \subset I_2$.
- A VOA module for the VOA V is a vector space M together with a linear map $a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$ which is compatible with the vertex algebra structure of V. In particular there is a conformal energy operator L^M_0 acting on M and diagonalizable.
- If V is unitary then the VOA module M is said to be a unitary VOA module if it is equipped with a scalar product scalar product $(\cdot|\cdot)_M$ which is compatible with the unitary structure of V.

Fom VOA modules to representations of conformal nets

- Now let V be a strongly local VOA and let M be a unitary VOA module for V.
- We assume that M is energy-bounded i.e. that for every $a \in V$ there exist positive integers s_M, j_M and a constant $K_M > 0$ such that

$$\|a_n^M b\| \leq K_M(|n|+1)^{s_M} \|(L_0^M + 1_M)^{j_M} b\| \ \forall n \in \mathbb{Z}, \ \forall b \in M.$$

- We can define smeared vertex operators $Y_M(a, f)$ acting on the Hilbert space completion $\mathcal{H}_M$ of M.

- Definition [CWX≥2018]. Let M be a unitary energy-bounded VOA module for V. We say that M is strongly integrable if there is a locally normal representation π^M of $\mathcal{A}_V$ on $\mathcal{H}_M$ such that $\pi^M_I(Y(a,f)) = Y_M(a,f)$ for all $a \in V$ and all $f \in C^\infty_c(I)$ and all intervals $I \subset S^1$.
- Let $\mathrm{Rep}^u(V)$ be the category of unitary VOA modules for V. Then the strongly integrable V-modules define a full subcategory $\mathrm{Rep}^{si}(V)$ of $\mathrm{Rep}^u(V)$ which is closed under subobjects and direct sums. Moreover, let $\mathrm{Rep}(\mathcal{A}_V)$ be the category of (locally normal) representations of $\mathcal{A}_V$.

We have the following results [CWX≥2018]

- The map $M \mapsto \pi^M$ gives rise to a linear faithful full *-functor $\mathcal{F} : \mathrm{Rep}^{si}(V) \to \mathrm{Rep}(\mathcal{A}_V)$.
- If V is type A affine VOA then $\mathrm{Rep}^{si}(V) = \mathrm{Rep}^{u}(V)$.
- Many examples of integable modules for type A coset VOAs.
- Solution to a long standing problem in the representation theory of coset VOAs by using functional analytic methods and in particular the Jones theory of subfactors.

Further results and directions

- From representations of loop group conformal nets to representations of affine VOAs (S. C. and M. Weiner – Y. Tanimoto, – A. Henriques)
- Analytic properties of VOA intertwiners operators (B. Gui)
- C*-tensor structure on $\mathrm{Rep}^u(V)$ (B. Gui – S.C., S. Ciamprone and C. Pinzari)
- Conformal nets, VOAs and Segal CFT (J. Tener)
- From conformal nets to VOAs (S.C and L. Tomassini)
- Reconstruction of C*-tensor categories from conformal nets and VOAs (D. Evans and T. Gannon – M. Bischoff)
- ....

THANK YOU VERY MUCH!