Quantum Jacobi and quantum modular forms

Amanda Folsom (Amherst College)

Quantum Jacobi and quantum modular forms

Let

$f : \mathbb{H} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}),\ \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$

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Modular transformation:

$$f(\tau) - \epsilon^{-1}(\gamma)(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = 0$$

Quantum modular forms

Let $f : \mathbb{Q} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}), \quad x \in \mathbb{Q}.$

Modular transformation:

$$f(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right) = ?$$

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k ($k \in \frac{1}{2}\mathbb{Z}$) is function $f : \mathbb{Q} \setminus S \to \mathbb{C}$, for some discrete subset S, such that

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$$h_\gamma(x) = h_{f,\gamma}(x) := f(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

extend to suitably continuous or analytic functions in $\mathbb{R}$ (or $\mathbb{R} \setminus T$).

Quantum modular forms

Example. Kontsevich's "strange" function ($x \in \mathbb{Q}$):

$$\phi(x) := e^{\frac{\pi i x}{12}} \sum_{n=0}^{\infty} \left(e^{2\pi i x}; e^{2\pi i x}\right)_n$$

$$= e^{\frac{\pi i x}{12}} \left(1 + (1 - e^{2\pi i x}) + (1 - e^{2\pi i x})(1 - e^{4\pi i x}) + \cdots\right)$$

Quantum modular forms

Here, for $n \in \mathbb{N}_0$,

$$(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j) = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}).$$

is the q-Pochhammer symbol.

Quantum modular forms

Theorem (Zagier)

The function ϕ is a quantum modular form of weight $3/2$,

Quantum modular forms

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The function ϕ *is a quantum modular form of weight* $3/2$, *i.e.*

$$\phi(x) - \zeta_{24}^{-1}\phi(x+1) = 0, \quad \phi(x) \mp \zeta_8 |x|^{-\frac{3}{2}} \phi(-1/x) = h(x),$$

where h *is a real analytic function (except at* 0*).*

Quantum modular forms

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$$\text{Let} \quad \widetilde{\eta}(\tau) := \sum_{n=0}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2}{24}}, \qquad \eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

Quantum modular forms

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Proposition (Zagier)

We have that

$$\sum_{n=0}^{\infty} \left(q^{\frac{1}{24}} (q;q)_n - \eta(\tau) \right) = -\frac{1}{2}\widetilde{\eta}(\tau) + \eta(\tau) \left(\frac{1}{2} - \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \right).$$

Quantum modular forms

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Quantum modular forms

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$$\begin{aligned} \sigma(q) &:= \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} &=: \sum_{n\geq 0} T(n) q^{\frac{n-1}{24}}, \\ \sigma^*(q) &:= 2\sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} &=: \sum_{n<0} T(n) q^{\frac{|n|-1}{24}}. \end{aligned}$$

Quantum modular forms

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(Andrews, Cohen, Ramanujan)

Quantum modular forms

For $x \in \mathbb{Q}$, define

$$f(x) := e^{\frac{\pi i x}{12}} \sigma\left(e^{2\pi i x}\right) = -e^{\frac{\pi i x}{12}} \sigma^*(e^{-2\pi i x}).$$

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Ex. $$f\left(\frac{1}{k}\right) = \zeta_{24k} \cdot \sigma(\zeta_k) = -\zeta_{24k} \cdot \sigma^*(\zeta_k^{-1}).$$

Quantum modular forms

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The function f is a quantum modular form of weight 1, i.e.

$$f(x+1) = \zeta_{24} f(x), \quad f(x) - \zeta_{24}^{-1}(2x+1)^{-1} f\left(\frac{x}{2x+1}\right) = h(x),$$

where h is a real analytic function (except at $-1/2$).

Quantum modular forms

Proof ingredient:

Theorem (Cohen)

Let $\tau = x + iy \in \mathbb{H}$. The function

$$u(\tau) := y^{\frac{1}{2}} \sum_{n \in 1+24\mathbb{Z}} T(n) K_0\left(\frac{2\pi|n|y}{24}\right) e^{\frac{\pi i n x}{12}}$$

is a Maass wave form.

Quantum modular forms

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is a Maass wave form. i.e.,

$$u(\tau+1) = e^{\frac{\pi i}{12}} u(\tau),$$

$$u\left(\frac{-1}{2\tau}\right) = \overline{u(\tau)},$$

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(\tau) = \frac{1}{4} u(\tau).$$

Harmonic Maass forms

Definition (Bruinier-Funke)

A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(4N)$ is a smooth $\widehat{M} : \mathbb{H} \to \mathbb{C}$ satisfying

(1) $\forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, $\tau \in \mathbb{H}$,

$$\widehat{M}(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (c\tau + d)^k \widehat{M}(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ (c\tau + d)^k \widehat{M}(\tau), & k \in \mathbb{Z} \end{cases}$$

(2) $\Delta_k \widehat{M} = 0$, where $\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $(\tau = x + iy)$

(3) $\widehat{M}$ has linear exponential growth at all cusps.

Harmonic Maass forms

Decomposition: $\widehat{M} = M + M^{-}$

$$M := \sum_{n \geq r_M} c_M(n) q^n \quad \text{“holomorphic part”} = \text{mock modular form}$$

$$M^{-} := \sum_{n<0} c_M^{-}(n) \Gamma(1-k, 4\pi|n|y) q^n \quad \text{“non-holomorphic part”}$$

$$\Gamma(a, x) := \int_x^{\infty} t^{a-1} e^{-t} \, dt$$

Mock theta functions and mock modular forms

Zwegers: Ramanujan's mock theta functions are mock modular.

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Example:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

Quantum and mock modular forms

Question: What is the connection between quantum & mock modular forms?

Quantum modular forms

Example. Let $q = e^{2\pi i\tau}$.

$$U(\tau) := q^{-\frac{1}{24}} \sum_{n=0}^{\infty} (q;q)_n^2 q^{n+1} = q^{-\frac{1}{24}} \sum_{\substack{m\in\mathbb{Z} \\ n\geq 1}} u(m,n)(-1)^m q^n,$$

$$u(m,n) := \#\{\text{size } n \text{ strongly unimodal sequences of rank } m\}.$$

Quantum modular forms

Definition

A sequence $\{a_j\}_{j=1}^{s}$ of integers is called *strongly unimodal* of size n if

- $a_1 + a_2 + \cdots + a_s = n$,

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- $a_1 + a_2 + \cdots + a_s = n$,
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The *rank* equals $s - 2r + 1$ (difference between # terms after and before the "peak").

Quantum modular forms

Theorem (Bryson-Ono-Pittman-Rhoades)

Let $x \in \mathbb{Q}$. We have that

$$\phi(-x) = U(x).$$

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For $x \in \mathbb{H} \cup \mathbb{Q}$, the function U satisfies

$$U(x) + (-ix)^{-\frac{3}{2}} \; U(-1/x) = g(x),$$

where g is a real analytic function (except at 0).

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Quantum modular forms

Definition (Hikami, Hikami-Lovejoy)

For $t \in \mathbb{N}$, define

$$F_t(\tau) := q^t \sum_{k_t \geq \cdots k_1 \geq 0} (q;q)_{k_t} \prod_{j=1}^{t-1} q^{k_j(k_j+1)} \begin{bmatrix} k_{j+1} \\ k_j \end{bmatrix}_q,$$

$$U_t(\tau) := q^{-t} \sum_{k_t \geq \cdots k_1 \geq 1} (q;q)_{k_t-1}^2 q^{k_t} \prod_{j=1}^{t-1} q^{k_j^2} \begin{bmatrix} k_{j+1} + k_j - j + 2\sum_{s=1}^{j-1} k_s \\ k_{j+1} - k_j \end{bmatrix}_q.$$

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Note.

$$e^{\frac{-23\pi i x}{12}} F_1(x) = \phi(x),$$

$$e^{\frac{23\pi i x}{12}} U_1(x) = U(x)$$

Quantum modular forms

Theorem (Hikami-Lovejoy)

Let $k \in \mathbb{N}$. We have that

$$F_t\left(-\frac{1}{k}\right) = U_t\left(\frac{1}{k}\right).$$

Moreover, F_t and U_t are quantum modular forms.

Quantum modular forms

Previously, $U(\tau) := q^{-\frac{1}{24}} \sum_{n=0}^{\infty} (q;q)_n^2 q^{n+1} = q^{-\frac{1}{24}} \sum_{\substack{m \in \mathbb{Z} \\ n \geq 1}} u(m,n)(-1)^m q^n.$

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Definition

Let $w = e^{2\pi i z}, q = e^{2\pi i \tau}$. Define the two-variable function

$$U(z;\tau) := \sum_{n=0}^{\infty} (wq;q)_n (w^{-1}q;q)_n q^{n+1} = \sum_{\substack{m \in \mathbb{Z} \\ n \geq 1}} u(m,n)(-w)^m q^n.$$

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Note.

$$U(0;\tau) = q^{\frac{1}{24}} U(\tau).$$

Quantum modular forms

Definition (F-Ki-TruongVu-Yang)

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Quantum modular forms

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Note.

$$e^{\frac{-\pi i x}{12}} F(0;x) = \phi(x).$$

Quantum modular forms

Theorem (F-Ki-TruongVu-Yang)

Let $k \in \mathbb{N}$*, and* $h \in \mathbb{Z}$ *such that* $\gcd(h, k) = 1$*. For suitable* $z \in \mathbb{C}$*, we have for* $x = \frac{h}{k}$ *that*

$$F\left(z; -x\right) = U\left(z; x\right).$$

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Moreover, for suitable fixed z, these functions are quantum modular forms when viewed as functions of $x \in \mathbb{Q}$.

Quantum modular forms

Question: Can we view $U(z;x)$ as a two variable quantum modular form?

Quantum Jacobi Forms

Definition (Bringmann-F)

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **and index** $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** is function $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ such that $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$, the functions

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$$h_\gamma(z;\tau) := \phi(z;\tau) - \epsilon_1^{-1}(\gamma)(c\tau+d)^{-k} e^{\frac{-2\pi i m c z^2}{c\tau+d}} \phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right),$$

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$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \epsilon_2^{-1}(\lambda,\mu) e^{2\pi i m(\lambda^2\tau+2\lambda z)} \phi(z+\lambda\tau+\mu;\tau),$$

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$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \epsilon_2^{-1}(\lambda,\mu) e^{2\pi i m(\lambda^2\tau + 2\lambda z)} \phi(z+\lambda\tau+\mu;\tau),$$

satisfy a “suitable” property of continuity or analyticity in $\mathbb{R}^2$.

Quantum Jacobi Forms

We define using the unimodal function

$$Y^{+}(z;\tau) := -2i\sin(\pi z)q^{-\frac{1}{24}}U(z;\tau),$$

Quantum Jacobi Forms

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$$Y^{+}(z;\tau) := -2i\sin(\pi z)q^{-\frac{1}{24}}U(z;\tau),$$

and the “error” function

$$H(z;\tau) := \frac{i}{2}\frac{\vartheta(z;\tau)}{\eta(\tau)}\int_{\mathbb{R}}\frac{e^{\pi i\tau t^2-4\pi zt}}{\cosh(\pi t)}dt - \frac{i}{\sqrt{3}}\int_{\mathbb{R}}e^{\frac{\pi i\tau t^2}{3}-2\pi zt}\frac{\sinh\left(\frac{2\pi t}{3}\right)}{\cosh(\pi t)}dt.$$

Quantum Jacobi Forms

We define the following subset $\mathcal{Q}_2$ of $\mathbb{Q}^2$

$$\mathcal{Q}_2 := \left\{ \left(\frac{a}{b}, \frac{h}{k} \right) \in \mathbb{Q}^2 : b, k \in \mathbb{N},\ \gcd(a, b) = \gcd(h, k) = 1, b \mid k \right\}.$$

Quantum Jacobi Forms

Theorem (Bringmann-F)

The following transformation properties hold.

(i) *For* $(z,\tau) \in (\mathbb{C} \times \mathbb{H}) \cup \mathcal{Q}_2$, *we have that*

$$Y^+(z;\tau) - e^{\frac{\pi i}{12}} Y^+(z;\tau+1) = 0, \tag{1}$$

$$Y^+(z;\tau) + ie^{\frac{3\pi i z^2}{\tau}} (-i\tau)^{-\frac{1}{2}} Y^+\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = -H(z;\tau), \tag{2}$$

$$Y^+(z;\tau) + Y^+(z+1;\tau) = 0, \tag{3}$$

$$Y^+(z;\tau) + e^{-6\pi i z - 3\pi i\tau} Y^+(z+\tau;\tau) \tag{4}$$

$$= e^{-5\pi i z - \frac{25\pi i\tau}{12}} \left(1 - e^{4\pi i z + 2\pi i\tau}\right) - \frac{i\vartheta(z;\tau)}{\eta(\tau)} \left(e^{-2\pi i z - \frac{\pi i\tau}{4}} - e^{-6\pi i z - \frac{9\pi i\tau}{4}}\right).$$

Quantum Jacobi Forms

Here,

$$H(z;\tau) := \frac{i}{2}\frac{\vartheta(z;\tau)}{\eta(\tau)}\int_{\mathbb{R}} \frac{e^{\pi i\tau t^2 - 2\pi zt}}{\cosh \pi t} dt - \frac{i}{\sqrt{3}}\int_{\mathbb{R}} e^{\frac{\pi i \tau t^2}{3} - 2\pi zt} \frac{\sinh\left(\frac{2\pi t}{3}\right)}{\cosh(\pi t)} dt.$$

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$$Y^+(z;\tau) + Y^+(z+1;\tau) = 0, \tag{7}$$

$$Y^+(z;\tau) + e^{-6\pi i z - 3\pi i\tau} Y^+(z+\tau;\tau) \tag{8}$$

$$= e^{-5\pi i z - \frac{25\pi i\tau}{12}} \left(1 - e^{4\pi i z + 2\pi i\tau}\right) - \frac{i\vartheta(z;\tau)}{\eta(\tau)} \left(e^{-2\pi i z - \frac{\pi i\tau}{4}} - e^{-6\pi i z - \frac{9\pi i\tau}{4}}\right).$$

Theorem (Bringmann-F, cont.)

(ii) *In particular, for $(z,\tau)\in\mathcal{Q}_2$, we have that*

$$Y^+(z;\tau)+ie^{\frac{3\pi iz^2}{\tau}}(-i\tau)^{-\frac{1}{2}}Y^+\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = \frac{i}{\sqrt{3}}\int_{\mathbb{R}} e^{\frac{\pi i\tau t^2}{3}-2\pi zt}\frac{\sinh\left(\frac{2\pi t}{3}\right)}{\cosh(\pi t)}dt, \tag{9}$$

$$Y^+(z;\tau)+e^{-6\pi iz-3\pi i\tau}Y^+(z+\tau;\tau) = e^{-5\pi iz-\frac{25\pi i\tau}{12}}\left(1-e^{4\pi iz+2\pi i\tau}\right). \tag{10}$$

Theorem (Bringmann-F, cont.)

(ii) *In particular, for* $(z,\tau) \in \mathcal{Q}_2$, *we have that*

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The function on the right-hand side of (9) extends to a C^∞ *function on* $(\mathbb{R} \setminus (\pm 1/6 + \mathbb{Z})) \times \mathbb{R}^\times$, *and the function on the right-hand-side of (10) extends to a* C^∞ *function on* $\mathbb{R}^2$.

Quantum Jacobi Forms

Theorem (Bringmann-F, cont.)

In particular, $Y^{+}(z;\tau)$ *is a quantum Jacobi form of weight* $1/2$ *and index* $-3/2$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

Quantum Jacobi Forms

Theorem (Bringmann-F)

Let $\tau = h/k$ and $z = a/b$ be such that $(z, \tau) \in \mathcal{Q}_2$. Then

$$Y^{+}(z;\tau) = -\frac{1}{2}\zeta_{2b}^{-5a}\zeta_{24k}^{-25h}\sum_{j=0}^{k-1}(-1)^{j+1}\zeta_{2k}^{-5hj}\left(1-\zeta_b^{2a}\zeta_k^{h(2j+1)}\right)\zeta_b^{-3ja}\zeta_{2k}^{-3j^2h}.$$

Quantum Jacobi Forms

Corollary (Bringmann-F)

Let $1 \leq a < b$, $1 \leq h < k$ *with* $\gcd(a,b) = \gcd(h,k) = 1$, $b|k$ *and* $h' \in \mathbb{Z}$ *with* $hh' \equiv -1 \pmod{k}$. *Then, as* $q \to \zeta_k^h$ *radially within the unit disc, we have that*

$$\lim_{q \to \zeta_k^h} \left(R\left(\zeta_b^a; q\right) - \zeta_{b^2}^{-a^2 h' k} C\left(\zeta_b^a; q\right) \right)$$

$$= \frac{1}{2}\left(1 - \zeta_b^{-a}\right) \zeta_b^{-2a} \zeta_k^{-h} \sum_{j=0}^{k-1} (-1)^{j+1} \zeta_{2k}^{-5hj} \left(1 - \zeta_b^{2a} \zeta_k^{h(2j+1)}\right) \zeta_b^{-3ja} \zeta_{2k}^{-3j^2 h}.$$

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$R(w;q)$ = partition rank g.f., $C(w;q)$ = partition crank g.f.

Quantum Jacobi Forms

Corollary (Bringmann-F)

For $\tau = h/k$ and $z = a/b$ such that $(z,\tau) \in \mathcal{Q}_2$, the Eichler integral

$$\frac{i}{\sqrt{3}} \int_{\mathbb{R}} e^{\frac{\pi i \tau t^2}{3} - 2\pi z t} \frac{\sinh\left(\frac{2\pi t}{3}\right)}{\cosh(\pi t)} dt$$

is given as an explicit 2-variable polynomial in (ζ_b^a, ζ_k^h).

Combinatorial q-series

Definitions (Kim-Lim-Lovejoy)

$$\mathcal{V}(z;\tau) := \sum_{n=0}^{\infty} \frac{(-wq;q)_n(-qw^{-1};q)_n q^n}{(q;q^2)_{n+1}}$$

$$\mathcal{W}(z;\tau) := \sum_{n=0}^{\infty} \frac{(wq;q^2)_n(w^{-1}q;q^2)_n q^{2n}}{(-q;q)_{2n+1}}$$

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Note. $\mathcal{V}$ is a rank g.f. for odd balanced unimodal sequences.
$\mathcal{W}$ is a g.f. for partitions without repeated odd parts.

Quantum modular forms

Theorem (Kim-Lim-Lovejoy)

The functions $v(x) := e^{14\pi i x}\mathcal{V}(\frac{1}{2}; 8x)$ *and* $w(x) := e^{6\pi i x}\mathcal{W}(0; x)$ *are quantum modular forms.*

Quantum Jacobi Forms

Theorem (Barnett-F-Ukogu-Wesley-Xu)

The functions

$$\mathcal{V}^+(z;\tau) = 2\cos(\pi z) q^{\frac{7}{8}} \mathcal{V}(w;q),$$

and

$$\mathcal{W}^+(z;\tau) := 2\cos(\pi z) q^{\frac{3}{8}} \mathcal{W}(w;q^{\frac{1}{2}}),$$

are quantum Jacobi forms of weight $\frac{1}{2}$ *and index* $-\frac{1}{2}$ *on* $\Gamma_0(4)$ *and* $\Gamma_0(2)$, *respectively.*

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Note. As was the case with Y^+, $\mathcal{V}^+$ and $\mathcal{W}^+$ exhibit suitable transformation properties on $(\mathbb{C} \times \mathbb{H}) \cup Q_2$, where $Q_2 \subseteq \mathbb{Q} \times \mathbb{Q}$.

Quantum Jacobi Forms

Theorem (Barnett-F-Ukogu-Wesley-Xu)

For $\left(\frac{a}{b}, \frac{h}{k}\right) \in Q_2 \subset \mathbb{Q} \times \mathbb{Q}$, *we have that*

$$\mathcal{V}^+\left(\tfrac{a}{b}; \tfrac{h}{k}\right) = -\frac{1}{2}\zeta_{2b}^{-a}\zeta_{8k}^{7h}\sum_{j=0}^{k-1}(-1)^j\zeta_{2k}^{-hj(1+j)}\zeta_b^{a(k-j)},$$

$$\mathcal{W}^+\left(\tfrac{a}{b}; \tfrac{h}{k}\right) = \sum_{j=1}^{k}(-1)^{j+1}\zeta_{2b}^{a(2j-1)}\zeta_{8k}^{h(2j-1)(4k-2j+1)},$$

where $\zeta_N := e^{\frac{2\pi i}{N}}$.

Quantum Jacobi Forms

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where $\zeta_N := e^{\frac{2\pi i}{N}}$.

Note. As was the case with Y^+, we obtain explicit evaluations of (2-variable) Eichler integrals at pairs of rationals as (2-variable) polynomials in roots of unity.

Quantum Jacobi Forms

Proof ingredients.

- Re-write functions in terms of Appell-lerch sums A_ℓ (Zwegers, Mortenson).

Quantum Jacobi Forms

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- Re-write functions in terms of Appell-lerch sums A_ℓ (Zwegers, Mortenson).
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Quantum Jacobi Forms

Proof ingredients.

- Re-write functions in terms of Appell-lerch sums A_ℓ (Zwegers, Mortenson).
- Analytic continuation: transformation properties $\mathbb{C} \times \mathbb{H} \to \mathbb{Q} \times \mathbb{Q}$.
- Elliptic Quantum Jacobi transformation properties: $\Rightarrow$ functional equation $\Rightarrow$ polynomial evaluations.

Quantum Jacobi Forms

Thank you