Impact of Women in Number Theory

Allysa Lumley

Impact of Women Mathematicians on Research and Education in Mathematics
BIRS 2018

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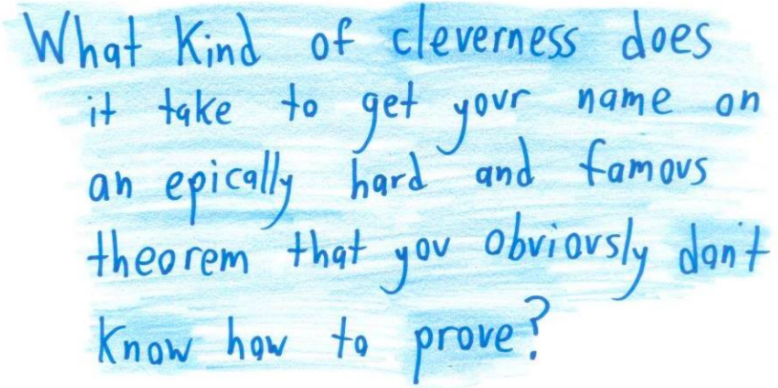
Introduction



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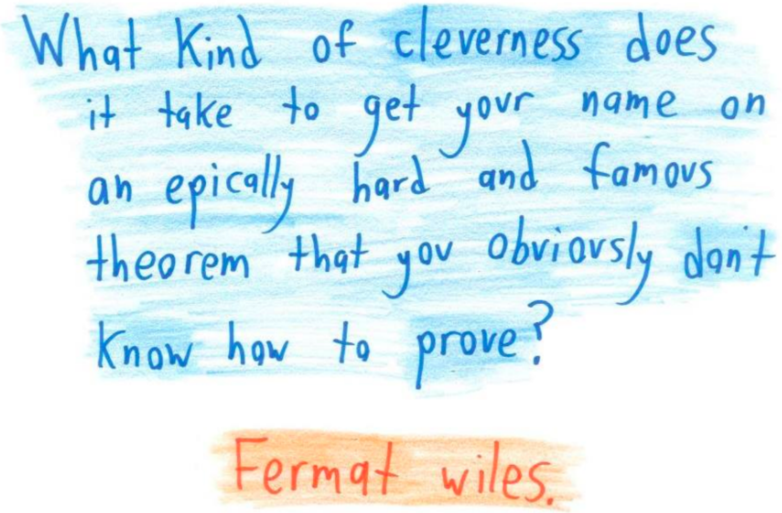

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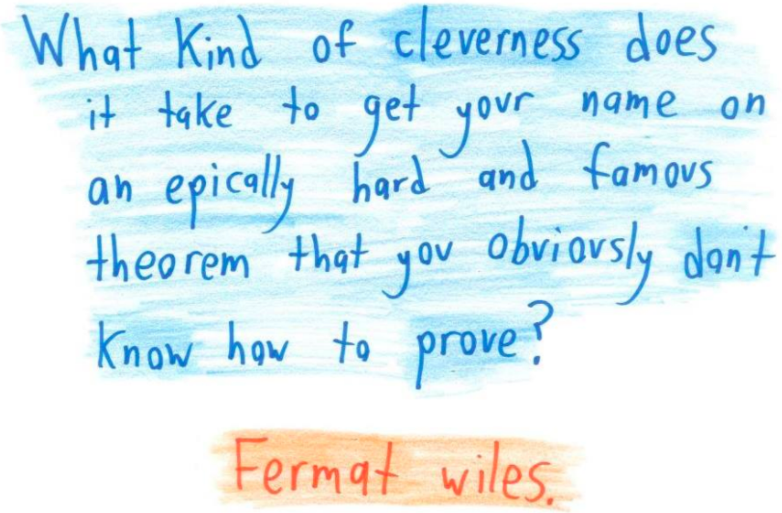

Credit: Math With Bad Drawings

Fermat's Last Theorem

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'Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.'

Pierre de Fermat, ~1630

translated: " It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in general any power higher than the second into two powers of like degree. I have discovered a truly remarkable proof which this margin is too small to contain."

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In 1630's, Fermat himself did prove this for the case $n = 4$.

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Theorem (Germain, 1800's)

If p is an odd prime and there exists an auxiliary prime $q = 2pn + 1$ which satisfies

- *there are no consecutive p^{th} power residues modulo q*
- *p is not a p^{th} power reside modulo q,*

then in any solution to $x^p + y^p = z^p$ we have p^2 must divide one of x, y or z. Thus, Case 1 of FLT is true.

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Legendre used this idea
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Primes p satisfying that $2p + 1$ is also prime are called Sophie Germain primes.

FLT - a new attack

Theorem

The positive integers x, y, z satisfying $x^2 + y^2 = z^2$ are described exactly by the following form:

$$x = r(s^2 - t^2), y = 2rst \text{ and } z = r(s^2 + t^2)$$

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What is a prime? Can we factor uniquely? How do you define the notion of a common factor?
Unfortunately, if we consider $n = p > 19$, then for ξ a p^{th} root of unity we have $\mathbb{Z}[\xi]$ the *elements* **do not** have unique factorizations.

Emmy Noether

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We have a simple algorithm to check if a solution exists: check if $\gcd(a, b) | c$.

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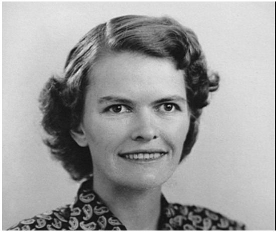

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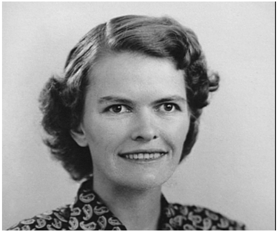

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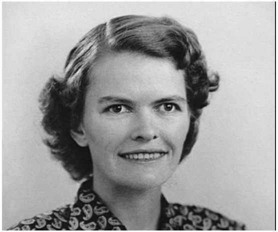

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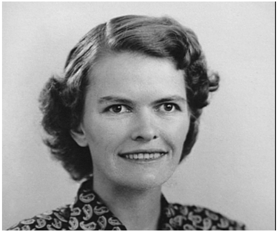

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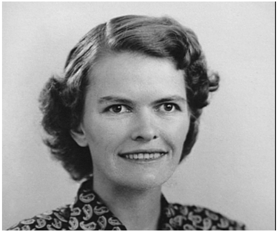

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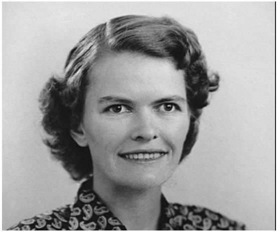

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Robinson went on to solve many other problems about decidability. Recent work of Alexandra Shlapentokh and co-authors generalize Hilbert's 10th problem to rings of integers in special algebraic number fields.

But of course, the primes

Consider

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

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$$\sum_{n \le x} \mu(n) = o(x)$$

is equivalent to the prime number theorem:

$$\sum_{n \le x} \Lambda(n) = x + o(x),$$

where $\Lambda(n) = \log p$ if $n = p^k$ and 0 otherwise.

Chowla's conjecture and twin primes

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where $\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots$ implies twin primes and if we have good control on the error term then we obtain the answer to the special case of Chowla's conjecture.

Rising stars

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In a second paper Matomäki, Radziwiłł and Tao have also made significant progress to a different specialization of Chowla's conjecture. "[. . .] the prize notes, that Matomäki and Radziwiłł, through their impressive array of deep results and the powerful new techniques they have introduced, will strongly influence the development of analytic number theory in the future."

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Thanks for Listening !