Emmy Noether and her contributions to commutative algebra

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March 17, 2018

Timeline

- March 23, 1882: Born in Erlangen, Germany
- 1907: Received her PhD from the Mathematical Institute of Erlangen
- 1908 - 1915: Worked without pay or title in Erlangen
- 1915: Joined the Mathematical Institute in Göttingen (in an unofficial capacity)
- 1919: Gained permission to lecture (without salary)
- 1922: Became an associate professor (without tenure)
- April, 1933: Denied permission to teach by the Nazi Government
- September, 1933: Accepted a professorship at Bryn Mawr College and lectured at the IAS in Princeton
- April 14, 1935: Died in Bryn Mawr from complications after surgery

From the mouths of famous men

"Emmy Noether's general mathematical insights were not confined to her specialty— algebra— but affected anyone who came in touch with her work."

—Preface to Topologie I., by P. Alexandrov and H. Hopf

"She was superior to me in many respects."

—Hermann Weyl

"... it is surely not much of an exaggeration to call her the mother of modern algebra."

—Irving Kaplansky

"Noether was the most significant creative mathematical genius thus far produced since the higher education of women began.

—Albert Einstein

"Her strength lay in her ability to operate abstractly with concepts. It was not necessary for her to be led to new results on the leading strings of concrete examples. She possessed a most vivid imagination, with the aid of which she could visualize remote connections; she constantly strove for unification. In this, she sought out the essentials in the known facts, brought them into order by means of appropriate general concepts, espied the vantage point from which the whole could best be surveyed, cleansed the object under consideration of superfluous dross, and thereby won through to so simple and distinct a form that the venture into new territory could be undertaken with the greatest prospect of success."

—Hermann Weyl

Mathematicians directly impacted by Noether

- Rudolf Hölzer
- Nikolai Chebotarev
- Werner Weber
- Werner Schmeidler
- Fritz Seidelmann
- Jakob Levitzki
- Richard Courant
- B. L. van der Waerden
- W.W. Stepanow
- Grete Hermann
- Kenjiro Shoda
- Chiungtze Tsen
- Joichi Suetsuna
- Marie Weiss
- Heinrich Kapferer
- Öystein Ore
- Ernst Witt

- Felix Klein

- Ernst Fischer
- Helmut Hasse
- Claude Chevalley
- Alexander Ostrowski
- Heinz Hopf
- Albert Einstein
- Hermann Weyl
- Edmund Laundau
- Oswald Veblen
- H.S. Vanidver
- Andre Weil
- Emil Artin
- Vladimir Kořínek
- Reinhold Baer
- A.A. Albert
- F.K. Schmidt
- Max Deuring

- Solomon Lefschetz

- Wolfgang Krull
- Gottfried Köthe
- Richard Brauer
- Arnold Scholz
- Otto Schilling
- Pavel Alexandrov
- Robert Fricke
- Olga Taussky Todd
- Heinrich Grell
- Teiji Takagi
- Ruth Staffer
- Jacques Herbrand
- Max Deuring
- Karl Dörges
- Ann Pell Wheeler
- Hans Fitting
- Hans Falckenberg

- David Hilbert

Some areas impacted by her work

Wikipedia: "As one of the leading mathematicians of her time, she developed the theoreies of rings, fields, and algebras."

- Linear Algebra
- Representation Theory
- Topology
- Galois cohomology
- Algebraic Number Theory
- Class Field Theory
- Algebraic Geometry
- Arithmetic Geometry
- Physics (through the study of differential invariants)

Does anyone here know any Physics?

About the same time she mentioned to Seidelmann that a team in Göttingen, to which she belonged, was carrying out calculations of the most difficult kind for Einstein—"although," she chuckled, "none of us understands what they are for." Felix Klein says in a letter to Hilbert, "You know that Frl. Noether is continually advising me in my projects and that it is really through her that I have become competent in the subject . . ." Hilbert's response to this letter contains this fragment, "Emmy Noether, whom I called upon to help me with such questions as my theorem on the conservation of energy . . ."

Excerpt from *Emmy Noether, 1882 - 1935*, by Auguste Dick

Mathematical objects/results named after her

- Noetherian...
  - ... group
  - ... ring
  - ... module
  - ... topological space
  - ... variety
  - ... scheme
  - ... category
- Noether normalization lemma (Commutative algebra)
- Noether's theorem (Physics)
- Noether's second theorem (Physics)
- Lasker-Noether theorem (Generalization of the fundamental theorem of arithmetic)
- Skolem-Noether theorem (Central Simple Algebras)
- Noether's equations (Galois cohomology)
- Albert-Brauer-Hasse-Noether theorem (CSA's)

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- In 1911 she began a fruitful correspondence with Ernst Fischer, who inspired her to move from the computational, algorithmic approach, to a more abstract approach characteristic of Hilbert.

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- While her lecture style was controversial and not to everyone's taste, she garnered a large audience, and had many devoted students with whom she would talk mathematics endlessly, often on long walks or at her home.
- During this time her methods and ideas became more formalized, and visitors came from around the world to learn the "Noether method" and bring it back to their universities.

The Noether School

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- She died two years later, shortly after undergoing surgery.

Influence on Princeton

In February 1934, Emmy Noether also began to give weekly lectures in nearby Princeton—not, as she wrote, at the "men's university where nothing female is admitted," but at the Flexner Institute which had only shortly before been established in 1930. . . . Her own influence in determining the mathematical activity at Princeton is indicated in a letter to H. Hasse of March 6, 1934, "I have started with representation modules, groups with operators . . .; Princeton will receive its first algebraic treatment this winter, and a thorough one at that."

Abstraction and Axiomatization

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A *ring* is a set R equipped with two binary operations $+$ and $\times$ satisfying

- R is an abelian group under addition
- R is associative with respect to multiplication and contains a multiplicative identity, 1
- Multiplication is distributive with respect to addition

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The fundamental theorem of arithmetic states that any positive $n \in \mathbf{Z}$ can be decomposed uniquely into a product of (positive) prime powers:

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

p_i distinct primes, $e_i \geq 1$.

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The second condition is also related to work of Noether, and has to do with the difference between being "irreducible" and being "prime."

Ideals and Principal Ideals

Let R be a commutative ring. An ideal I of R is a subset of R which is a subgroup under addition and such that for every $a \in I$ and $r \in R$ we have that $ra \in I$.

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In general, define $aR = \{ar : r \in R\}$ for any $a \in R$. This is the *principal ideal generated by a.*

Ascending Chain Conditions

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Theorem

Let R be an integral domain. Then factorization terminates in R if and only if R satisfies the ascending chain condition for principal ideals.

Proof Sketch

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- (The other direction is proven similarly)

The big takeaway here is that while divisibility is a condition on elements, the ACC is a condition on ideals. In particular, it can hold for more general ideals, not just principal ones (in $\mathbf{Z}$ every ideal is principal).

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For example, if $R = \mathbf{Z}[x_1, \ldots, x_n]$, the ring of polynomials in n variables with integer coefficients, then not every ideal is principal. However, Hilbert proved that every ideal is *finitely generated* and Noether was able to reframe that argument in terms of an ACC.

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- Her mathematics and her mere presence touched countless mathematicians during a crucial period in the development of modern algebra.
- Even if her name is not attached to a result in algebra during that time, her ideas certainly were. In fact, van der Waerden, who wrote arguably the first modern algebra book, was essentially writing up a treatment of what he learned from her lectures.