Nilpotent Primer Hypermaps with Hypervertices of Valency a prime

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1. Definitions

Topological Map $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph X is called the *underlying graph* of the map $\mathcal{M}$.

Automorphism of a map $\mathcal{M}$: a self-homeomorphism of the surface, any automorphism of the map must be an automorphism of the underlying graph X

Orientation-Preserving Automorphism of an orientable map $\mathcal{M}$: an automorphism of preserving orientation of the map

Automorphism group $\mathrm{Aut}(\mathcal{M})$:

Orientation-preserving automorphisms group $\mathrm{Aut}^{+}\mathcal{M}$

Hypermap:

A *hypermap* $\mathcal{H}$ is a 2-cell embedding of a connected bipartite graph $\mathcal{G}$ into a compact and connected surface $\mathcal{S}$ without border

The vertices of $\mathcal{G}$ in two biparts are respectively called the *hypervertices* and *hyperedges* of the hypermap, and the connected regions of $\mathcal{G}\backslash\mathcal{S}$ are called *hyperfaces*.

Choose a certer for each hyperface and subdivide the hypermap by adjoining the hyperface centers to its adjacent hypervertices and hyperedges.

Get a triangular subdivision whose triangles are the *flags* of this hypermap, which are represented by little triangle around hypervertices

Define three involuntary permutations γ_0, γ_1 and γ_2 on the flag set F:
γ_0 exchanges two flags adjacent to the same hyperedge and center but distinct hypervertices;
γ_1 exchanges two flags adjacent to the same hypervertex and center but distinct hyperedges;
γ_2 exchanges two flags adjacent to the same hypervertex and hyperedge but distinct centers.

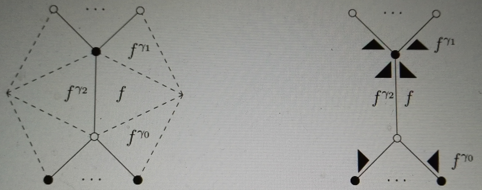

Figure 1: *Flags are drew in the left figure and represented by little triangles around hypervertices in the right figure.*

The subgroup $\langle \gamma_0, \gamma_1, \gamma_2 \rangle$ of S^F acts transitively on F.

In the orientable case, the even word subgroup $\langle \gamma_0\gamma_1, \gamma_1\gamma_2 \rangle$ of $\langle \gamma_0, \gamma_1, \gamma_2 \rangle$ acts on F with two orbits.

Each orbit determines an orientation described by the action of the even word subgroup. Fixing an orientation we get an oriented hypermap.

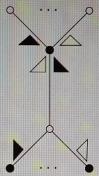

Figure 2: *Two orbits of flags are represented by black little triangles and white little triangles.*.

Given one of two orbits, say F_1, every orbit of $\langle \gamma_0\gamma_1 \rangle$, $\langle \gamma_1\gamma_2 \rangle$ and $\langle \gamma_0\gamma_2 \rangle$ on F_1 is respectively the flags contained in one hyperface, the flags around one hypervertex and the flags around one hyperedge.

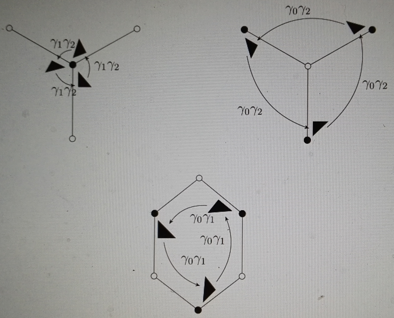

Figure 3: *Three figures show the orbit of* $\langle\gamma_1\gamma_2\rangle$, $\langle\gamma_0\gamma_2\rangle$ *and* $\langle\gamma_0\gamma_1\rangle$ *on* F_1, *respectively.*

Algebraically, given a finite set D and a transitive group $\langle R, L \rangle$ on D, define an oriented hypermap $\mathcal{H} = (D; R, L)$:

the orbits of $\langle R \rangle$, $\langle L \rangle$ and $\langle RL \rangle$ on D are called hyperfaces, hypervertices and hyperedges, respectively, with incidence given by non-empty intersection.

D is called the dart set and the group $\mathrm{Mon}(\mathcal{H}) = \langle R, L \rangle$ is called the monodromy group of the hypermap.

In the case $(RL)^2 = 1$, $\mathcal{H}$ is an oriented map.

For oriented hypermaps $\mathcal{H} = (D; R, L)$ and $\mathcal{H}' = (D'; R', L')$, a covering $\psi : \mathcal{H} \to \mathcal{H}'$ is a mapping $\psi : D \to D'$ satisfying $R\psi = \psi R'$ and $L\psi = \psi L'$.

Now, the assignment $R \mapsto R'$ and $L \mapsto L'$ extends to an epimorphism $\mathrm{Mon}(\mathcal{H}) = \langle R, L \rangle \to \mathrm{Mon}(\mathcal{H}') = \langle R', L' \rangle$ of the monodromy groups.

As usual, one may define an isomorphism, an automorphism and the automorphism group $\mathrm{Aut}(\mathcal{H})$ for hypermaps. It is straightforward that $|\mathrm{Aut}(\mathcal{H})| \leq |D|$.

An oriented hypermap is called *regular* if the action of $\mathrm{Mon}(\mathcal{H}) = \langle R, L \rangle$ on D is regular. In this case, the set D can then be replaced by $G := \mathrm{Mon}(\mathcal{H})$, so that $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$ can be viewed as the left and right regular multiplications of G, respectively.

Denote $\mathcal{H}$ by a triple $\mathcal{H} = (G; r, \ell)$, where $G = \langle r, \ell \rangle$. Then the hyperfaces (resp. hypervertices and hyperedges) correspond to right cosets of G relative to $\langle r \rangle$, (resp. $\langle \ell \rangle$ and $\langle r\ell \rangle$).

Given a group G, $(G; r_1, \ell_1) \cong (G; r_2, \ell_2)$ if and only if there exists an automorphism σ of G such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

2. (face-) Primer map

(face-) Primer map $\mathcal{H}$: $\mathrm{Aut}(\mathcal{H})$ induces a faithful action on $\mathcal{F}$

The primer hypermaps were introduced by Breda d'Azevedo and Fernandes in 2011.

For any hypermap $(G; r, \ell)$, $(G/\langle r\rangle_G; \bar{r}, \bar{\ell})$ is primer.

The first step might be to determine the primer hypermaps. Based on the knowledge of primer hypermaps, one may determine general hypermaps.

To recover a hypermap from its primer hypermap is essentially extension problem, of a group by a cyclic group.

1. A.Breda d'Azevedo, M.E.Fernandes, Classification of primer hypermaps with a prime number of hyperfaces, *Europ.J.Combin*, 32(2011), 233-242.

2. A.Breda d'Azevedo, M.E.Fernandes, Classification of the regular oriented hypermaps with prime number of hyperfaces, *Ars Math.Contemp*, 10(1)(2016), 193-209.

3. S.F. Du, X.Y. Hu, A classification of primer hypermaps with a product of two primes number of hyperfaces, *Euro J. Combin*, **62**(2017), 245-262

3. Nilpotent hypermaps

General question: determine the regular maps and regular hypermaps with given (fibre-preserving) automorphism group G.

There are some papers on the finite simple groups G. We shall focus on nilpotent groups.

1. A. Malnič, R. Nedela and M. Škoviera, Regular maps with nilpotent automorphism groups, *European J. Combin.* 33 (2012), 1974–1986.

2. M.D.E., Conder, S.,F. Du, R. Nedela, M. Škoviera, Regular maps with nilpotent automorphism group, *J. Algebraic Combin.* **44**(2016), 863-874.

Assume that $G = \mathrm{Aut}^+(\mathcal{M})$ is nilpotent and the underlying graph of the map is simple. It is proved that

(i) the number of vertices of any such map is bounded by a function f_c of the nilpotency class of the group G, where f_c is given by applying a theorem of Labute on the ranks of the factors of the lower central series of Γ (via the associated Lie algebra),

(ii) for a fixed nilpotency class c there is exactly one such simple regular map $\mathcal{M}_c$ attaining the bound, and that this map is universal, in the sense that every simple regular map $\mathcal{M}$ for which $\mathrm{Aut}^+(\mathcal{M})$ is nilpotent of class at most c is a quotient of $\mathcal{M}_c$.

4. Nilpotent Primer Hypermaps

A *PNp hypermap* $\mathcal{H}$ means a primer hypermap such that $\mathrm{Aut}(\mathcal{H})$ is nilpotent and the hypervertex-valency is a prime p.

Theorem: Let $\mathcal{H}$ be a *PNp* hypermap. Then

(1) $\mathrm{Aut}(\mathcal{H})$ is a finite p-group;

(2) $\mathcal{H}$ has at most p^{1+f_c} hyperfaces, where c is the nilpotent class of $\mathrm{Aut}(\mathcal{H})$;

(3) For every integer $c \geq 1$, there exists a unique *PNp* hypermap $\mathcal{H}_{\mathcal{C}}$ of class c, having p^{1+f_c} hyperfaces, and type either (p, p^{m+1}, p^m) for $c = m(p-1)+1$ or (p, p^m, p^m) for $(m-1)(p-1)+1 < c < m(p-1)+1$.

(4) Every *PNp* hypermap of class at most c is a quotient of $\mathcal{H}_c$.

By $\mu(k)$ we denote the Möbius function. Given a prime p, for a positive integer n, let $\rho(n)$ be the largest integer such that $n-\rho(n)(p-1)>1$

Set $\alpha_k = -\frac{1}{k!}\frac{d^k}{dx^k}(\ln \frac{1-2x+x^{p+1}}{(1-x)^2})\mid_{x=0}$.

Set $f_1=0$ and $f_n=\sum_{i=2}^{n} R_i$ for $n \geq 2$, where

$$R_n = \sum_{i=0}^{\rho(n)-1} \sum_{\substack{k \mid (n-i(p-1)) \\ k>1}} \mu\left(\frac{n-i(p-1)}{k}\right)\frac{k}{n-i(p-1)}\alpha_k.$$

Outline of the proof

Let $\mathcal{H}$ a *PNp* hypermap with the group $G = \langle r, \ell \rangle$.

Lemma

The automorphism group $G = \langle r, \ell \rangle$ is a finite p-group, where $|r| = p^m$ for some integer m and $|\ell| = p$.

Let $\Gamma = \langle x, y \mid y^p = 1 \rangle$ so that G is a quotient of Γ, where r and ℓ are the images of x and y, respectively.

Lemma

For each $n \geq 2$, the factor Γ_n/Γ_{n+1} of the lower central series of Γ is a finite elementary abelian p-group.

Lemma

For $m \geq 2$ and $1 \leq h \leq p-1$, the factor Γ_m/Γ_{m+h} has exponent p.

Lemma

$[x^{p^m}, y] \in \Gamma_{m(p-1)+2}$ *for all* $m \geq 1$.

Lemma

$(xy)^{p^m} \in x^{p^m}\Gamma_{m(p-1)+1}$ *for all* $m \geq 1$.

The proof for above lemmas depends on Philip Hall's collection process and some related results.

1. M. Hall, JR., *Combinatorial Theory*, Macmilan, New York, 1967.

2. R.R.Struik, On the nilpotent products of cyclic groups, *Canad J of Math.*, **12**(1960), 447-462.

Definition

(Hall) Let $G = \langle a_1, a_2, \cdots, a_t \rangle$ *be a group. Then the* basic commutators *of* G *are elements of* G*, defined and ordered as follows:*

1) *The basic commutators of weight 1 are the generators* $a_1 < a_2 < \cdots < a_t$ *(in order);*
2) *Inductively, the basic commutators of weight* $w > 1$ *are the elements* $[x, y]$ *where* $\omega(x) + \omega(y) = \omega([x, y])$*, such that* $x > y$ *and if* $x = [u, v]$ *for basic commutators* u *and* v*, then* $y \geq v$*;*
3) *Commutators are ordered so that* $x > y$ *if* $\omega(x) > \omega(y)$ *and for commutators of any fixed weight, let* $[x_1, y_1] < [x_2, y_2]$ *if either* $y_1 < y_2$ *or* $y_1 = y_2$ *and* $x_1 < x_2$*.*

Proposition

(Hall) Let $x_1, x_2, \cdots, x_s$ *be elements of a group. Let* $c_1, c_2, \cdots$ *be the basic commutators on* $x_1, x_2, \cdots, x_s$ *of weight at least* 2 *in order. Then*

$$(x_1 \cdots x_s)^n = x_1^n x_2^n \cdots x_s^n c_1^{f_1(n)} \cdots c_i^{f_i(n)} d_1 d_2 \cdots d_t,$$

where for $1 \leq j \leq i$, *and*

$$f_j(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + \cdots + a_{\omega_j} \binom{n}{\omega_j},$$

for a'*s are rational integers not depending on* n *but only on* c_j *and* $a_{\omega_j} = 0$ *for* $\omega_j > n$; *and* d'*s are uncollected basic commutators.*

Proposition

(Struik) Let x, y be elements of a group. Let $u_1, u_2, \cdots$ be the sequence of basic commutators of weight at least 2 on x and $[x, y]$ in order. Then

$$[x^n, y] = [x, y]^n u_1^{f_1(n)} \cdots u_i^{f_i(n)} d_1 d_2 \cdots d_t,$$

where f_i, a's and d's have the same meaning as in last proposition.

Proposition

(Struik) Let α *be a fixed integer and* G *a group such that* $G_n = 1$*. Then if* $b_j \in G$ *and* $m < n$*,*

$$[b_1, \cdots, b_{i-1}, b_i^{\alpha}, b_{i+1}, \cdots, b_m] = [b_1, \cdots, b_m]^{\alpha} v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \cdots v_t^{f_t(\alpha)}$$

where every v_k *is a (not necessarily basic) commutator on* $b_1, \cdots, b_m$ *of weight* $> m$*, every* $b_j, 1 \leq j \leq m$ *appears in each commutator* v_k*, and every* f_h *is of form with* $\omega_h = \omega(v_h) - (m-1)$ *where* $\omega(v_h)$ *is the weight of* v_h *on* $b_1, \cdots, b_m$*.*

Properties of $Z_{p^m} \wr Z_p$

Lemma

Let p be an odd prime and $k = (m-1)(p-1)+1+i$, where $1 \le i \le p-2$ and $m \ge 1$. Let $W = Z_{p^m} \wr Z_p = \langle a \rangle \wr \langle b \rangle$ and $\overline{W} = W/W_{k+1}$. Then we have

- (i) $c(W) = m(p-1)+1$;
- (ii) $\mathrm{Core}_W(\langle a_1 \rangle) = 1$.
- (iii) $|[a_1, tb^{-1}]| = |[a_1, tb]| = p^{m-s}$, where $s(p-1)+1 \le t < (s+1)(p-1)+1$ for $0 \le s \le m$;
- (iv) $|a_1 b| = p^{m+1}$;
- (v) $c(\overline{W}) = k$;
- (vi) $|\overline{a_1}| = p^m, |\overline{b}| = p$ and $|\overline{a_1 b}| = p^m$;
- (vii) $\mathrm{Core}_{\overline{W}}(\langle \overline{a_1} \rangle) = 1$.

Proposition

(Liebeck) Suppose that $W = A \wr B$, where A and B are two finite p-groups. Take $a \in A$ for $|a| = p^{n+1}$ and $b \in B$ for $|b| = p^h$, where $n \geq 0$ and $h \geq 1$. For any integer t, set $c_t = [a_1^{-1}, tb]$, where a_1 is a copy of the element a which is labeled by the identity of B. Then we have

(i) $c_t = 1$ for $t \geq p^h + n(p-1)p^{h-1}$;

(ii) $|c_t| = p^l$ where $l \leq n$ and

$$p^h + (n-l)(p-1)p^{h-1} \leq t < p^h + (n-l+1)(p-1)p^{h-1};$$

(iii) $|c_t| = p^{n+1}$ for $0 < t < p^h$.

Proposition

(Liebeck) If A is an Abelian p-group of exponent p^n and $B = \langle b_1 \rangle \times \cdots \times \langle b_m \rangle$ is a direct product of m cyclic groups, whose orders are $p^{\beta_1}, \cdots, p^{\beta_m}$, respectively, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$, then $W = A \wr B$ has nilpotency class

$$c = \sum_{i=1}^{m} (p^{\beta_i} - 1) + (n-1)(p-1)p^{\beta_1 - 1} + 1.$$

Lemma

For given p *and* $n \geq 2$, *let* m *be such that* $(m-1)(p-1)+1 < n \leq m(p-1)+1$. *Then the subgroup of* $H^{(n)} = \Gamma/\Gamma_{n+1}$ *generated by the image of* x^{p^m} *is normal, but the subgroup generated by the image of* $x^{p^{m-1}}$ *is not normal.*

The rank of the abelian p-group Γ_n/Γ_{n+1} for $n \geq 1$ has been determined by Gaglione:

A. M. Gaglione, Factor groups the lower certral series for special free products, *J. Alge.*, **37**(1975), 172-185

By $\mu(k)$ we denote the Möbius function. Given a prime p, for a positive integer n, let $\rho(n)$ be the largest integer such that $n - \rho(n)(p-1) > 1$
Set $\alpha_k = -\frac{1}{k!}\frac{d^k}{dx^k}(\ln \frac{1-2x+x^{p+1}}{(1-x)^2}) \big|_{x=0}$.
Set $f_1 = 0$ and $f_n = \sum_{i=2}^{n} R_i$ for $n \geq 2$, where

$$R_n = \sum_{i=0}^{\rho(n)-1} \sum_{\substack{k \mid (n-i(p-1)) \\ k>1}} \mu(\frac{n-i(p-1)}{k})\frac{k}{n-i(p-1)}\alpha_k.$$

Proposition

(Gaglione) The rank of the factor group Γ_n/Γ_{n+1} is R_n, for all $n \geq 2$, while the rank of Γ/Γ_2 is 2.

Set $f_1 = 0$ and $f_n = \sum_{i=2}^{n} R_i$ for $n \geq 2$, Immediately, we have

Corollary

The order of the quotient Γ_2/Γ_{n+1} is p^{f_n}, for all $n \geq 2$.

For $p = 3, 5$, the first $2 \leq n \leq 18$ terms of these two sequences $\{R_n\}$ and $\{f_n\}$ are given below (with help of MATLAB):

$p = 3$	R_n :	$1, 2, 3, 6, 8, 16, 23, 42, 65, 116, 186, 328, 543, 948, 1607, 2804, 4816$
	f_n :	$1, 3, 6, 12, 20, 36, 59, 101, 166, 282, 468, 796, 1339, 2287, 3894, 669$
$p = 5$	R_n :	$1, 2, 3, 6, 9, 18, 29, 54, 92, 172, 301, 558, 1004, 1858, 3399, 6316, 11$
	f_n :	$1, 3, 6, 12, 21, 39, 68, 122, 214, 386, 687, 1245, 2249, 4107, 7506, 13$

Lemma

$\langle x \rangle \cap \Gamma_2 = 1$ *and* $\langle y \rangle \cap \Gamma_2 = 1$.

For a quotient of Γ to be the automorphism group of a *PNp* hypermap, we need the stabilizer of a hyperface to be core-free. It follows that the largest nilpotent quotient of class c that is admissible is the quotient $U^{(c)} = H^{(c)}/K$, where K is the core of the subgroup generated by the image of x in $H^{(c)} = \Gamma/\Gamma_{c+1}$.

Lemma

For any $c \geq 1$, take m such that $(m-1)(p-1)+1 < c \leq m(p-1)+1$. Then the group $U^{(c)} = H^{(c)}/K$ has order p^{m+1+f_c}. The corresponding PNp hypermap $\mathcal{H}_c$ has type either (p, p^{m+1}, p^m) for $c = m(p-1)+1$ or (p, p^m, p^m) for $(m-1)(p-1)+1 < c < m(p-1)+1$.

Lemma

For any integer, $c \geq 1$, there exists a unique PNp hypermap $\mathcal{H}_c$ of class c, having p^{1+f_c} hyperfaces and type either (p, p^{m+1}, p^m) for $c = m(p-1)+1$ or (p, p^m, p^m) for $(m-1)(p-1)+1 < c < m(p-1)+1$. Furthermore, every PNp hypermap of class at most c is a quotient of $\mathcal{H}_c$.

The main theorem is proved !

PNp hypermaps of small class

Table 1: ***PNp*** **hypermaps of class** $1, 2, 3$

p	Defining relators for G	$\lvert G\rvert$	c	(r, ℓ)	type	g
	$x, y^p, [y, x]$	p	1	(x, y)	(p,p,1)	0
$p = 2$	$x^p, y^p, [y, x, x], [y, x, y]$	p^3	2	(x, y)	$(2, 4, 2)$	0
$p \geq 3$					(p, p, p)	$1 + \frac{p^2(p-}{2}$
2	$x^2, y^2, [y, x, x, y], [y, x, x, x]$, $[y, x, y, x], [y, x, y, y]$	16	3	(x, y)	(2,8,2)	0
	$x^4, y^2, [x^2, y][y, x, x]^{-1}, [y, x, x, y]$, $[y, x, x, x], [y, x, y, x], [y, x, y, y]$	32	3	(x, y)	(2,4,4)	1
	$x^4, y^2, [x^2, y][y, x, y]^{-1}, [y, x, x, y]$, $[y, x, x, x], [y, x, y, x], [y, x, y, y]$	32	3	(x, y)	(2,8,4)	3
	$x^4, y^2, [y, x, x, y], [y, x, x, x]$, $[y, x, y, x], [y, x, y, y]$	64	3	(x, y)	(2,8,4)	5

p	Defining relators for G	$\lvert G\rvert$	c	(r,ℓ)	type	g
3	$x^3, y^3, [y,x,x], [y,x,y,x], [y,x,y,y]$	81	3	(x,y), (y,x)	(3,9,3)	10
	$x^3, y^3, [y,x,x]^{-1}[y,x,x]^2, [y,x,y,x], [y,x,y,y]$	81	3	(x,y)	(3,9,3)	10
				(x,y^2)	(3,3,3)	1
	$x^3, y^3, [y,x,y,x], [y,x,y,y], [y,x,x,x], [y,x,x,y]$	243	3	(x,y)	(3,9,3)	28
≥ 5	$x^p, y^p, [x,y,x], [x,y,y,x], [x,y,y,y]$	p^4	3	(y,x), (xy^i,y) $1 \leq i \leq p$	(p,p,p)	$1+\frac{p^3(\cdots}{\cdots}$
	$x^p, y^p, [x,y,x,x], [x,y,x,y], [x,y,y,x], [x,y,y,y]$	p^5	3	(x,y)	(p,p,p)	$1+\frac{p^4(\cdots}{\cdots}$

Thank you very much !