A database of group actions

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- Circa 2000, Thomas Breuer devised an algorithm to determine all automorphism groups of Riemann surfaces for a fixed genus, assuming a complete classification of groups of sufficiently large order.
- I am putting this data, along with additional mathematical information, into an easily searchable database.

For many more details, see Breuer's book "Characters and automorphism groups of compact Riemann surfaces".

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Given a compact Riemann surface (curve) X and finite group $G=$Aut(X), let $Y = X/G$ be the set of orbits of X under the action of G. The genus of Y is the **quotient genus (or orbit genus)**, denoted g_Y.

Y is equivalent to the quotient of the upper half plane by a Fuchsian group:

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_Y}, \beta_{g_Y}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_Y} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle.$$

The values $[g_Y; m_1, \ldots, m_r]$ are called the **signature.**

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_Y}, \beta_{g_Y}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_Y} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle.$$

Classification of G is equivalent to finding surjections $\eta : \Gamma \to G$ for each possible Γ.

We define η by describing the images of α_i, β_i, and γ_j under η. These $2g_Y + r$ values in G are called the **generating vector.** Notice there can be more than one generating vector for each pair of a group G and a signature.

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_Y}, \beta_{g_Y}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_Y} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle.$$

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This is exactly what Breuer did: his code uses other results to limit the possible G for any genus, and then searches through those remaining groups to see if such generators exist.

Breuer only recorded a list of group and signature pairs for every genus, not the generating vectors.

I needed the generating vectors in my own research, so I modified his code to also output all generating vectors up to simultaneous conjugation. I posted the results of this code on my webpage.

Similar data may be found using the **MapClass** package in GAP, and on Marston Conder's webpage.

The *L-functions and modular forms database* (LMFDB)

The *L-functions and modular forms database* (LMFDB)

Not to be confused with LMFAO:

Photo by: Jordan von Netzer/ MTV News

lmfdb.org

LMFDB

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Introduction and more

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L-functions

Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2): Classical Maass Hilbert

GL(3): Maass

Other: Siegel

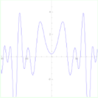
Varieties

Curves:
Elliptic: /Q /NumberFields
Genus 2: /Q
Higher genus: Families

A Database

The LMFDB is an extensive database of mathematical objects arising in Number Theory.

Sample lists: L-functions, Elliptic curves, Maass forms, Tables of zeros, Number fields



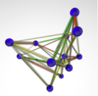
Hall of Fame

Riemann zeta function
Ramanujan Δ function and its L-function
C277 and its L-function
Gauss elliptic curve and its L-function
Grand Canyon L-function

Search and Browse

Search for objects with specific properties, or browse categories.

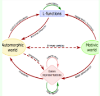
Browse: L-functions, Modular forms, Elliptic curves, Number fields



Visualize Data

Explore individual plots or view distributions of various objects.

Examples: GL(4) Level one Maass forms, Isogeny graph of elliptic curve 102.c



Explore and Learn

The LMFDB makes visible the connections predicted by the Langlands program. Knowls offer background information when you need it.

LMFDB universe

```
sage E.conductor().factor()
N    =  2 · 117223
sage E.discriminant().facto
Δ    =  2^2 · 117223
sage E.j_invariant().factor
j    =  2^-2 · 3^3 · 7^3 · 181
End(E) = Z
```

Code and Open Software

Download the data, download the code, or see how the data were generated.

GitHub SageMath Pari/GP Magma Python

- If you see yourself using this data, what additional information would be helpful for you? Other improvements that might help?
- Do you have other data which might fit in the LMFDB world?

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- Do you have other data which might fit in the LMFDB world?
- http://www.lmfdb.org/HigherGenus/C/Aut/
- http://www.github.com/jenpaulhus

lmfdb.org

LMFDB

⌂ → Higher Genus → C → Aut

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Families of Higher Genus Curves with Automorphisms

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L-functions

Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2): Classical Maass Hilbert

GL(3): Maass

Other: Siegel

Varieties

Curves:
Elliptic: /Q /NumberFields
Genus 2: /Q
Higher genus: Families

Currently the database contains all groups G acting as automorphisms of curves X from genus 2 up to genus 15 so that the quotient space X/G is the Riemann sphere (X/G has genus 0).

Browse automorphisms of higher genus curves

By genus: 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Hyperelliptic curves: by genus: 2 3 4 5 6 7 8 9 10 11 12 13 14 15

A random refined passport from the database.

Find specific automorphisms of higher genus curves

Search by label [2.12-4.0.2-2-2-3] Find — *family or passport label, e.g. 2.12-4.0.2-2-2-3 or 3.168-42.0.2-3-7.2*

Search

Genus:	3	*e.g. 4, or a range like 3..5*
Group:	[4,2]	*e.g. [4,2]*
Signature:	[0,2,3,3,6]	*e.g. [0,2,3,3,6] or [0;2,3,8]*
Dimension of the family:	1	*e.g. 1, or a range like 0..2*
Hyperelliptic curve(s):	include	
Cyclic trigonal curve(s):	include	
Full automorphism group:	include	
Maximum number of families to display:	20	

Search

Learn more about

Source of the data
Labeling convention

Search Example

LMFDB

⌂ → Higher Genus → C → Aut → Search results

search ...

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Families of Higher Genus Curves with Automorphisms Search Result

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L-functions

Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2): Classical Maass Hilbert

GL(3): Maass

Other: Siegel

Varieties

Curves:
Elliptic: /Q /NumberFields
Genus 2: /Q
Higher genus: Families

Genus: 10- Group: Signature:

Hyperelliptic curve(s): include Cyclic trigonal curve(s): include Full automorphism group: include

Dimension of the family: 8 Maximum number of families to display: 10

Search again

Results: (displaying matches 1-10 of 1445) Next

Refined Passport Label	Genus	Dimension	GAP/Magma group	Signature
10.4-1.0.2-2-2-2-2-2-2-4-4-4-4.1	10	8	[4,1]	[0; 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4]
10.4-1.0.2-2-2-2-2-2-2-4-4-4-4.2	10	8	[4,1]	[0; 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4]
11.4-1.0.2-2-2-2-2-4-4-4-4-4-4.1	11	8	[4,1]	[0; 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4]
11.4-1.0.2-2-2-2-2-4-4-4-4-4-4.2	11	8	[4,1]	[0; 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4]
11.4-1.0.2-2-2-2-2-4-4-4-4-4-4.3	11	8	[4,1]	[0; 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4]
11.4-1.0.2-2-2-2-2-4-4-4-4-4-4.4	11	8	[4,1]	[0; 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4]
12.4-1.0.2-2-2-4-4-4-4-4-4-4-4.1	12	8	[4,1]	[0; 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4]
12.4-1.0.2-2-2-4-4-4-4-4-4-4-4.2	12	8	[4,1]	[0; 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4]
12.4-1.0.2-2-2-4-4-4-4-4-4-4-4.3	12	8	[4,1]	[0; 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4]
12.4-1.0.2-2-2-4-4-4-4-4-4-4-4.4	12	8	[4,1]	[0; 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4]

Next

Breuer's code searches for elements in G up to **simultaneous conjugation**:

Suppose $(g_1, \ldots, g_r)$ is a generating vector with $g_i \in G$ and $g_Y = 0$. Then, conjugation of all the g_i by one $h \in G$ is also a generating vector, i.e. we are classifying generating vectors up to the action of Inn(G).

Breuer's code searches for elements in G up to **simultaneous conjugation**:

Suppose $(g_1, \ldots, g_r)$ is a generating vector with $g_i \in G$ and $g_Y = 0$. Then, conjugation of all the g_i by one $h \in G$ is also a generating vector, i.e. we are classifying generating vectors up to the action of $\mathrm{Inn}(G)$.

Given a group G, signature, and genus, we organize generating vectors by the conjugacy classes of the images of the γ_j:

Say $\mathcal{C} = (C_1, \ldots, C_r)$ where the C_i are conjugacy classes of G and $\eta(\gamma_i) \in C_i$. The data $(g, G, \mathcal{C})$ is called a **refined passport**, or alternatively X **is of ramification type** $(g, G, \mathcal{C})$.

One Refined Passport

LMFDB

One refined passport of genus 7 with automorphism group $PSL(2,8)$

Family Information

Genus:	7
Isomorphism class:	$PSL(2,8)$
GAP/Magma notation:	[504,156]
Signature:	$[0;2,3,7]$
Conjugacy classes for this refined passport:	2, 3, 4

Jacobian variety decomposition:	E^7
Corresponding character(s):	2

Other Data

Dimension of the corresponding Shimura variety:	1
Hyperelliptic curve(s):	No
Cyclic trigonal curve(s):	No

Generating Vector(s)

Displaying the unique generating vector for this refined passport.

7.504-156.0.2-3-7.1.1

(1,33) (2,263) (3,139) (4,165) (5,493) (6,37) (7,257) (8,153) (9,368) (10,51) (11,380) (12,301) (13,228) (14,160) (15,19) (16,374) (17,252) (18,143) (20,155)
(21,364) (22,462) (23,115) (24,55) (25,194) (26,486) (27,278) (28,60) (29,209) (30,94) (31,417) (32,268) (34,68) (35,405) (36,305) (38,488) (39,274)
(40,120) (41,295) (42,46) (43,455) (44,135) (45,179) (47,290) (48,472) (49,354) (50,385) (52,320) (53,288) (54,98) (56,110) (57,175) (58,327) (59,214)
(61,77) (62,315) (63,476) (64,429) (65,389) (66,265) (67,435) (69,208) (70,419) (71,450) (72,341) (73,105) (74,335) (75,211) (76,237) (78,109) (79,329)
(80,225) (81,440) (82,123) (83,452) (84,373) (85,300) (86,232) (87,91) (88,446) (89,324) (90,215) (92,227) (93,436) (95,187) (96,127) (97,266) (99,350)
(100,132) (101,281) (102,166) (103,489) (104,340) (106,140) (107,477) (108,377) (111,346) (112,192) (113,367) (114,118) (116,207) (117,251) (119,362)
(121,426) (122,457) (124,392) (125,360) (126,170) (128,182) (129,247) (130,399) (131,286) (133,149) (134,387) (136,501) (137,461) (138,337) (141,280)
(142,491) (144,413) (145,177) (146,407) (147,283) (148,309) (150,181) (151,401) (152,297) (154,195) (156,445) (157,372) (158,304) (159,163) (161,396)
(162,287) (164,299) (167,259) (168,199) (169,338) (171,422) (172,204) (173,353) (174,238) (176,412) (178,212) (180,449) (183,418) (184,264) (185,439)
(186,190) (188,279) (189,323) (191,434) (193,498) (196,464) (197,432) (198,242) (200,254) (201,319) (202,471) (203,358) (205,221) (206,459) (210,409)
(213,352) (216,485) (217,249) (218,479) (219,355) (220,381) (222,253) (223,473) (224,369) (226,267) (229,444) (230,376) (231,235) (233,468) (234,359)

Properties

Genus	7
Small Group	$PSL(2,8)$
Signature	$[0;2,3,7]$
Generating Vectors	1

Related objects

Family containing this refined passport

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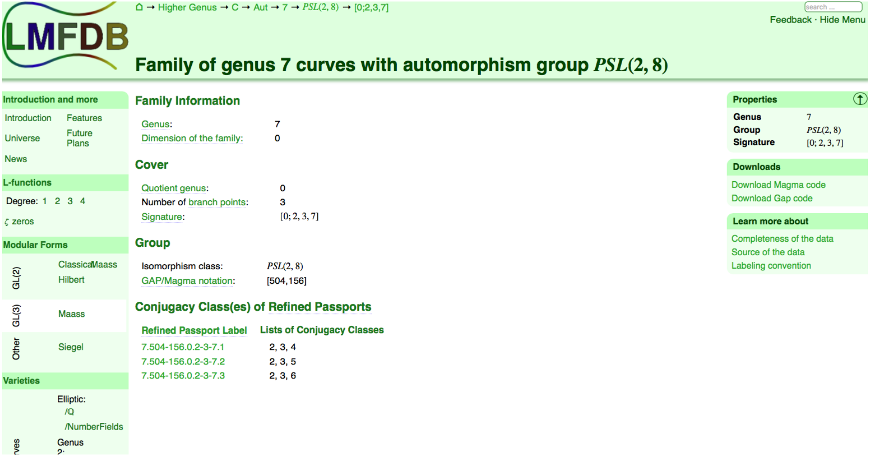
Completeness of the data
Source of the data
Labeling convention

One Family



⌂ → Higher Genus → C → Aut → 7 → $PSL(2,8)$ → [0;2,3,7]

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Family of genus 7 curves with automorphism group $PSL(2,8)$

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L-functions

Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2): Classical · Maass · Hilbert

GL(3): Maass

Other: Siegel

Varieties

Elliptic: /Q · /NumberFields

Genus 2:

Family Information

Genus:	7
Dimension of the family:	0

Cover

Quotient genus:	0
Number of branch points:	3
Signature:	[0; 2, 3, 7]

Group

Isomorphism class:	$PSL(2,8)$
GAP/Magma notation:	[504,156]

Conjugacy Class(es) of Refined Passports

Refined Passport Label	Lists of Conjugacy Classes
7.504-156.0.2-3-7.1	2, 3, 4
7.504-156.0.2-3-7.2	2, 3, 5
7.504-156.0.2-3-7.3	2, 3, 6

Properties

Genus	7
Group	$PSL(2,8)$
Signature	[0; 2, 3, 7]

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Labeling convention

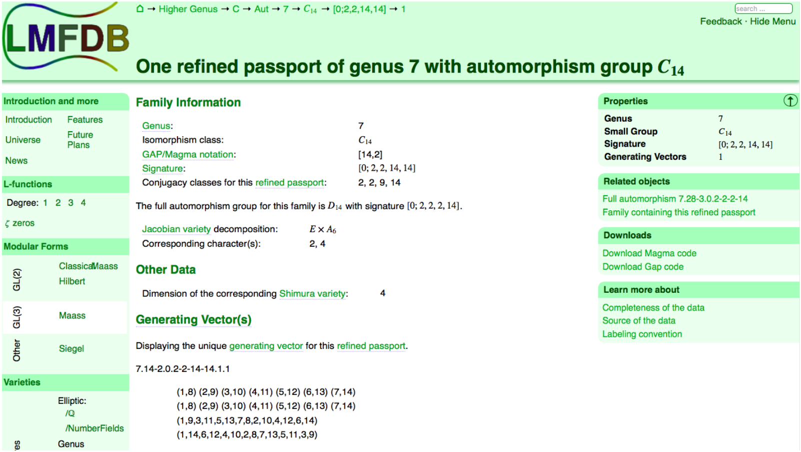
In the paper "Subvarieties of moduli space determined by finite groups acting on surfaces", John F. X. Ries lays out explicit criteria, given a generating vector, for whether the action is the full action of the generic element of the family in the moduli space of Riemann surfaces of genus g.

In the paper "Subvarieties of moduli space determined by finite groups acting on surfaces", John F. X. Ries lays out explicit criteria, given a generating vector, for whether the action is the full action of the generic element of the family in the moduli space of Riemann surfaces of genus g.

I've written code which uses his criteria to determine if a given generating vector corresponds to the full automorphism group. The database connects non-full examples with the corresponding full action.

One caveat: My code only determines the full group and signature, not which refined passport.

A refined passport which is not the full automorphism group



⌂ → Higher Genus → C → Aut → 7 → C_{14} → [0;2,2,14,14] → 1

search ...

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LMFDB

One refined passport of genus 7 with automorphism group C_{14}

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Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2) Classical Maass Hilbert

GL(3) Maass

Other Siegel

Varieties

Elliptic: /Q /NumberFields

Genus

Family Information

Genus:	7
Isomorphism class:	C_{14}
GAP/Magma notation:	[14,2]
Signature:	$[0; 2, 2, 14, 14]$
Conjugacy classes for this refined passport:	2, 2, 9, 14

The full automorphism group for this family is D_{14} with signature $[0; 2, 2, 2, 14]$.

Jacobian variety decomposition:	$E \times A_6$
Corresponding character(s):	2, 4

Other Data

Dimension of the corresponding Shimura variety: 4

Generating Vector(s)

Displaying the unique generating vector for this refined passport.

7.14-2.0.2-2-14-14.1.1

(1,8) (2,9) (3,10) (4,11) (5,12) (6,13) (7,14)
(1,8) (2,9) (3,10) (4,11) (5,12) (6,13) (7,14)
(1,9,3,11,5,13,7,8,2,10,4,12,6,14)
(1,14,6,12,4,10,2,8,7,13,5,11,3,9)

Properties

Genus	7
Small Group	C_{14}
Signature	$[0; 2, 2, 14, 14]$
Generating Vectors	1

Related objects

Full automorphism 7.28-3.0.2-2-2-14
Family containing this refined passport

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Its corresponding full automorphism group

LMFDB

⌂ → Higher Genus → C → Aut → 7 → D_{14} → [0;2,2,2,14]

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Family of genus 7 curves with automorphism group D_{14}

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L-functions

Degree: 1 2 3 4

ζ zeros

Modular Forms

GL(2): Classical Maass, Hilbert

GL(3): Maass

Other: Siegel

Varieties

Elliptic: /Q, /NumberFields

Genus

Family Information

Genus:	7
Dimension of the family:	1

Cover

Quotient genus:	0
Number of branch points:	4
Signature:	[0; 2, 2, 2, 14]

Group

Isomorphism class:	D_{14}
GAP/Magma notation:	[28,3]

Conjugacy Class(es) of Refined Passports

Refined Passport Label	Lists of Conjugacy Classes
7.28-3.0.2-2-2-14.1	2, 3, 3, 8
7.28-3.0.2-2-2-14.2	2, 3, 3, 9
7.28-3.0.2-2-2-14.3	2, 3, 3, 10
7.28-3.0.2-2-2-14.4	2, 4, 4, 8
7.28-3.0.2-2-2-14.5	2, 4, 4, 9
7.28-3.0.2-2-2-14.6	2, 4, 4, 10

Properties

Genus	7
Group	D_{14}
Signature	[0; 2, 2, 2, 14]

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Completeness of the data

Source of the data

Labeling convention

Additional data

- Which curves are hyperelliptic and cyclic trigonal (code from Swinarski), and the corresponding involution or trigonal automorphism
- Equations for hyperelliptic curves (Shaska), genus 3 curves (Magaard, Shaska, Shpectorov, and Völklein), and a few other curves up to genus 8 (Swinarski)
- Decomposition of corresponding Jacobian variety (me) and dimension of corresponding Shimura variety (code of Frediani, Ghigi, and Penegini)

There are also options to download the data (as records) to Magma or GAP readable files.

To Do List

- Quotient genus greater than 0.
- Equivalence up to action of the braid group.
- Identify topologically (or even analytically) equivalent actions.
- Identify exact refined passport for non-full/full automorphism examples.
- ??? Superelliptic curves, Riemann matrix or period matrix, field of definition of these curves.

Thanks

- Thomas Breuer
- Mike Zieve and John Voight
- LMFDB, especially: John Cremona, David Farmer, John Jones, and Drew Sutherland
- Undergraduate students: David Kraemer, Lex Martin, and David Neill Asanza
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- Grinnell College: Funding for a junior faculty research leave

The End