A quick introduction to dessins d'enfants, Belyi's theorem and Galois action

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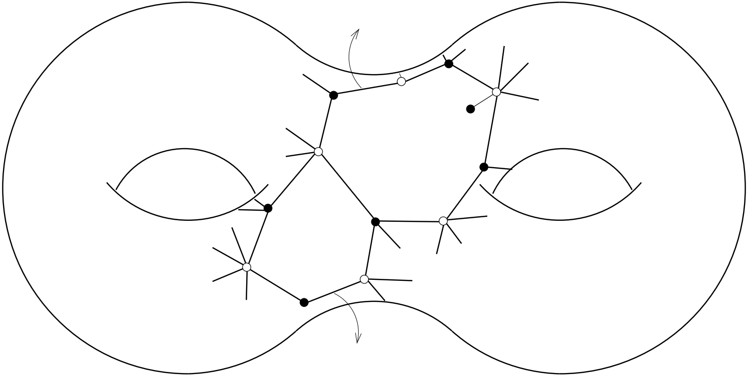
Definition of dessin d'enfant (Grothendieck, 80's)



Definition of dessin d'enfant

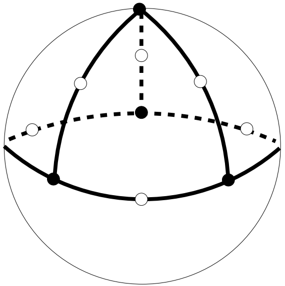
A **dessin d'enfant**, or simply a *dessin*, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite **graph** such that:

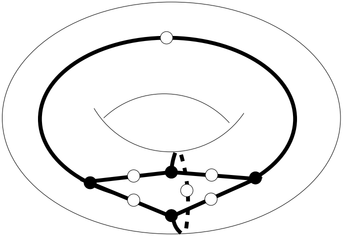
1. $\mathcal{D}$ is **connected**.
2. $\mathcal{D}$ is **bicoloured**, i.e. the vertices have been given either white or black colour and vertices connected by an edge have different colours.
3. $X \setminus \mathcal{D}$ is the union of finitely many topological discs, which we call *faces* of $\mathcal{D}$.

The **genus** of $(X, \mathcal{D})$ is simply the genus of X.

We consider two dessins $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ **equivalent** when there exists an orientation-preserving homeomorphism from X_1 to X_2 whose restriction to $\mathcal{D}_1$ induces an isomorphism between the coloured graphs $\mathcal{D}_1$ and $\mathcal{D}_2$.

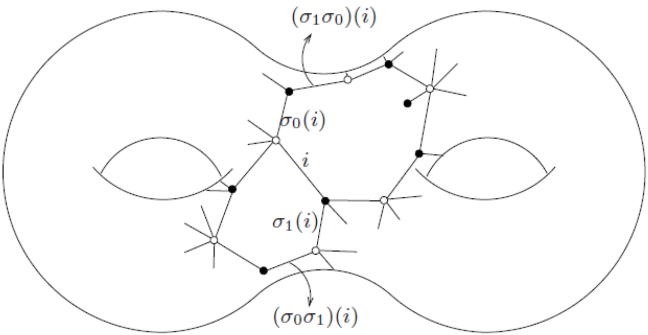
Warning: A dessin is not an abstract graph





These bicoloured graphs are equal as abstract graphs but they are different as dessins: they even have different genus!

Permutation representation pair of $\mathcal{D}$: (σ_0, σ_1)



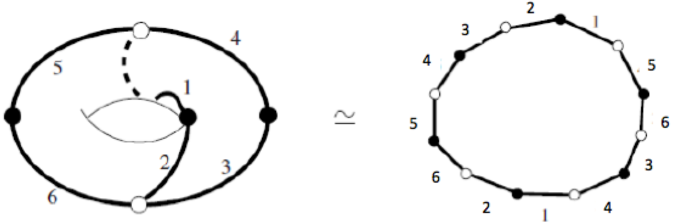
Cycles of $\sigma_0 \leftrightarrow$ white vertices of $\mathcal{D}$,
(length of each cycle = degree of the corresponding vertex).
Cycles of $\sigma_1 \leftrightarrow$ black vertices, etc.
Cycles of $\sigma_1\sigma_0 \leftrightarrow$ (1/2 of the edges of) faces of $\mathcal{D}$
Connectedness of $\mathcal{D} \Rightarrow < \sigma_0, \sigma_1 >$ is a transitive subgroup.

Eg: $\sigma_0 = (1,5,4)(2,6,3), \quad \sigma_1 = (1,2)(3,4)(5,6)$

$$\sigma_1\sigma_0 = (1,6,4,2,5,3)$$

$$\begin{aligned} 2-2g &= (\#\{\text{cycles of } \sigma_0\} + \#\{\text{cycles of } \sigma_1\}) - \#\{\text{edges}\} \\ &\quad + \#\{\text{cycles of } \sigma_1\sigma_0\} = 2+3-6+1 = 0 \end{aligned}$$

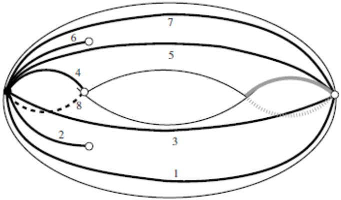
Of course this is nothing but the Euler–Poincaré characteristic of X corresponding to the polygonal decomposition induced by the dessin.

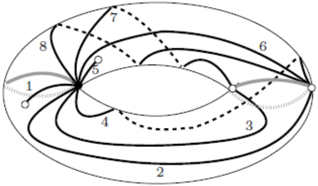


Main theorem of Grothendieck's theory of dessins

There is a correspondence between dessins d'enfants and algebraic curves defined over number fields, i.e. curves $C : F(x,y) = 0$ **with** $F(X,Y) \in \overline{\mathbb{Q}}[X,Y]$. E.G.:

$$\sigma_0 = (1,7,5,3) \qquad \tilde{\sigma}_0 = (4,6,8,2)$$
$$\sigma_1 = (1,2,3,4,5,6,7) \qquad \tilde{\sigma}_1 = (1,2,3,4,6,7)$$





represent the algebraic curves

$$y^2 = x(x-1)(x+\sqrt{2}) \qquad \text{and} \qquad y^2 = x(x-1)(x-\sqrt{2})$$

Sketch of the proof of the main result

PART I: To any dessin one can associate a compact Riemann surface (= an algebraic curve) together with a (Belyi) function on it.

PART II: The curves so obtained account for all algebraic curves defined over number fields.

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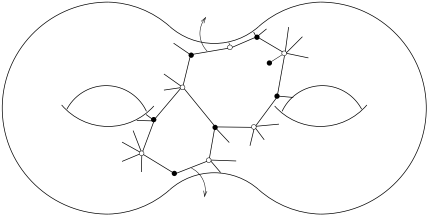
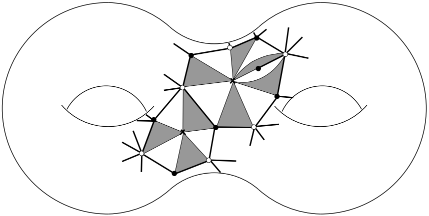
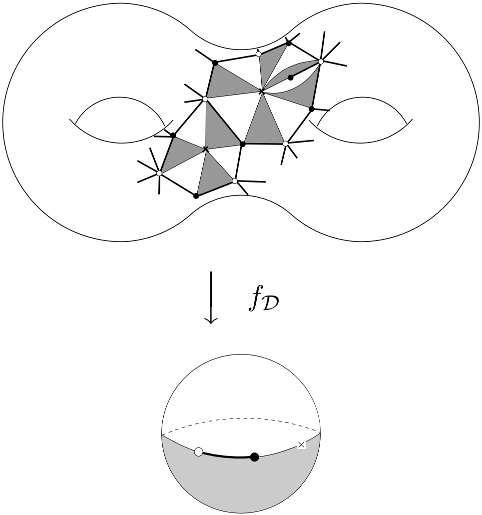
Then we will discuss the action of the absolute Galois group on dessins with special attention to the case of regular ones.

The triangle decomposition $\mathcal{T}(\mathcal{D})$ associated to $\mathcal{D}$





The topological covering of $\mathbb{S}^2$ associated to $\mathcal{T}(\mathcal{D})$



The Riemann surface structure associated to $\mathcal{D}$

Let $X^* = X \setminus \{\circ, \bullet, \times\}$.

By construction the map $f^*_{\mathcal{D}} : X^* \to \widehat{\mathbb{C}}$ is a local homeo; in fact, a topological cover of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Therefore X^* **inherits the Riemann surface structure** of $\widehat{\mathbb{C}}$; charts being defined by restriction of $f^*_{\mathcal{D}}$ to small neighborhoods.

By definition, $f^*_{\mathcal{D}} : X^*_{\mathcal{D}} \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ **is a holomorphic map**.

Moreover, by standard extension arguments **this complex structure extends to a complex structure on the compact surface $X_{\mathcal{D}}$ and the function $f^*_{\mathcal{D}}$ to a meromorphic function** $f_{\mathcal{D}} : X_{\mathcal{D}} \dashrightarrow \mathbb{C}$ (i.e. a holomorphic map $f_{\mathcal{D}} : X_{\mathcal{D}} \to \widehat{\mathbb{C}}$).

Observation/Definition: $f_{\mathcal{D}} : X_{\mathcal{D}} \to \widehat{\mathbb{C}}$ is a **Belyi function** meaning that it has ≤ 3 branching values, say $0, 1, \infty$.

1) Dessins $\mathcal{D} \longmapsto$ Belyi pairs $(X_{\mathcal{D}}, f_{\mathcal{D}}) \equiv (C_{\mathcal{D}}, R_{\mathcal{D}})$

Due to the correspondence between meromorphic functions of compact Riemann surfaces and rational functions of algebraic curves we will sometimes denote $(X_{\mathcal{D}}, f_{\mathcal{D}})$ by $(C_{\mathcal{D}}, R_{\mathcal{D}})$ to emphasize **the algebraic nature of the Belyi pair.**

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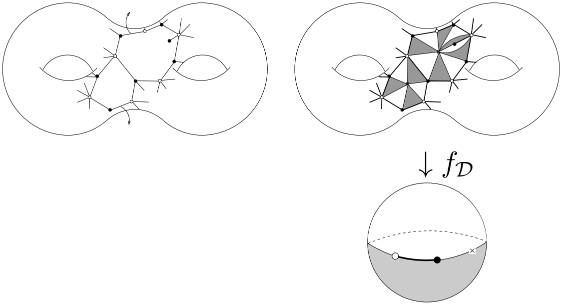
Due to the correspondence between meromorphic functions of compact Riemann surfaces and rational functions of algebraic curves we will sometimes denote $(X_{\mathcal{D}}, f_{\mathcal{D}})$ by $(C_{\mathcal{D}}, R_{\mathcal{D}})$ to emphasize **the algebraic nature of the Belyi pair.**
This classical correspondence can be established as follows: If (h, g) is a pair of generating functions there is an irreducible polynomial $F(X, Y) \in \mathbb{C}[X, Y]$ satisfying $F(h, g) \equiv 0$ and then $X_{\mathcal{D}}$ can be identified to the curve

$$C_{\mathcal{D}} : F(x, y) = 0$$

via the map
$$\phi : X_{\mathcal{D}} \longrightarrow C_{\mathcal{D}}$$
$$P \longmapsto (h(P), g(P))$$

Under ϕ the pair (h, g) is identified to the coordinate functions (x, y) and the Belyi function $f_{\mathcal{D}} = R(h, g)$, for some $R(X, Y) \in \mathbb{C}(X, Y)$, with the rational function $R(x, y)$ on C.

Properties of the Belyi pair $(X_{\mathcal{D}}, f_{\mathcal{D}})$



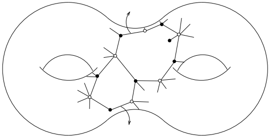
1. $deg(f_{\mathcal{D}}) = card(f_{\mathcal{D}}^{-1}(1/2)) =$ number of edges of $\mathcal{D}$.
2. **Poles** $\leftrightarrow$ $\{\times\text{'s} \equiv$ **Faces**$\}$; **Zeros**$\leftrightarrow$ $\{\circ\text{'s}\}$; **1's** $\leftrightarrow$ $\{\bullet\text{'s}\}$
3. The order of a pole $\times$ agrees with half the number of edges of the face of which $\times$ is a center (suitably counted).
4. The multiplicity of $f_{\mathcal{D}}$ at a vertex $v = \bullet, \circ$ of $\mathcal{D}$ coincides with the degree of the vertex.

2) dessins $\mathcal{D}_f \leftarrow$ Belyi pairs (S, f)

Proposition

Let $f : S \to \widehat{\mathbb{C}}$ be a Belyi function and set $\mathcal{D}_f = f^{-1}([0,1])$. Consider $\mathcal{D}_f$ as a bicoloured graph embedded in S whose set of white (resp. black) vertices is $f^{-1}(0)$ (resp. $f^{-1}(1)$). Then

- $\mathcal{D}_f$ is a dessin d'enfant and $\mathcal{D}_{f_{\mathcal{D}}} = \mathcal{D}$.



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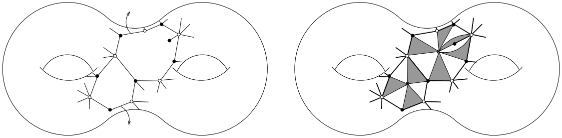

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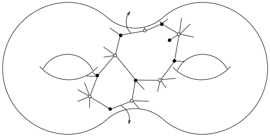
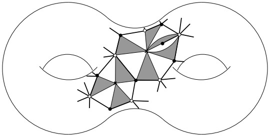
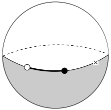




$\downarrow f_{\mathcal{D}}$



Dessins $\longleftrightarrow$ Belyi pairs

Theorem

The correspondences

$$\begin{array}{ccc} \{\textit{Equiv. classes of dessins}\} & \longrightarrow & \{\textit{Equiv. classes of Belyi pairs}\} \\ (X,\mathcal{D}) & \longmapsto & (X_{\mathcal{D}}, f_{\mathcal{D}}) = (C_{\mathcal{D}}, R_{\mathcal{D}}) \\ (S,\mathcal{D}_f) & \longleftarrow\!\shortmid & (S,f) \end{array}$$

are mutually inverse.

Equivalent ways to describe dessins

1) **Dessins** $\mathcal{D}$ of degree n.
2) **Belyi pairs** (C, R) with $deg(R) = n$.
3) **Pairs of permutations** (σ_0, σ_1) generating a transitive subgroup of S_n.
4) **Finite index subgroups** of $\Gamma(2) = \Delta(\infty, \infty, \infty)$.
5) **Finite index subgroups** of $\Delta(l, m, d)$

The link between 4) (or 5)) and 3) is established as follows:

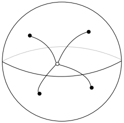
Suppose that l, m, d are the orders of $\sigma_0, \sigma_1, \sigma_0\sigma_1$ and consider the homomorphism

$$\rho : \Delta(l, m, d) \to S_n$$

determined by $x, y, z \to \sigma_0, \sigma_1, (\sigma_0\sigma_1)^{-1}$
then, the subgroup of $\Delta(l, m, d)$ in question is the preimage of the inertia subgroup of $< \sigma_0, \sigma_1 >$.

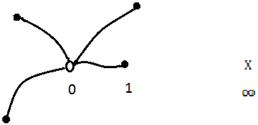
genus of $\mathcal{D}$ equals $0 \Rightarrow (C_{\mathcal{D}}, f_{\mathcal{D}}) \equiv (\widehat{\mathbb{C}}, R(z))$

$$\sigma_0 = (1,2,3,4)$$
$$\sigma_1 = (1)(2)(3)(4)$$

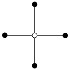
$\mathbb{S}^2$ 

$\updownarrow$

$\widehat{\mathbb{C}}$

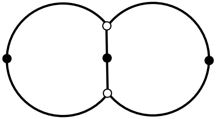


1) $\mathcal{D}$ has only one face $\Rightarrow R \equiv f_{\mathcal{D}}$ is a polynomial.
2) $\mathcal{D}$ has 4 edges $\Rightarrow deg(R) = 4$
3) $\mathcal{D}$ has a white vertex of degree 4 at $z = 0 \Rightarrow R(z) = cz^4$
4) 1 is a black vertex $\Rightarrow R(1) = 1 \Rightarrow$ $\boxed{R(z) = z^4}$



The modular function $j(\lambda) = \frac{4}{27}\frac{\left(1-\lambda+\lambda^2\right)^3}{\lambda^2(\lambda-1)^2}$

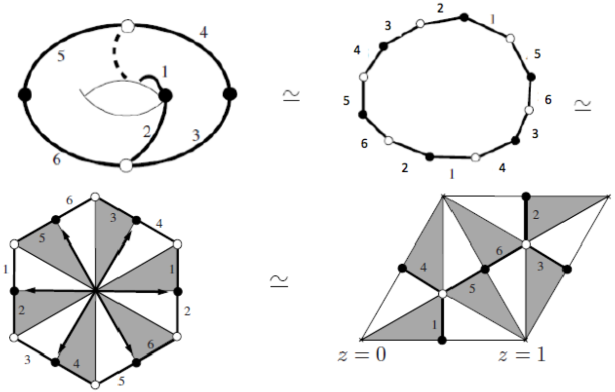
It is easy to check that this is a Belyi function and that its corresponding dessin is



- The 3 faces (of degree 4) correspond to the 3 poles $0, 1, \infty$ (of degree 4/2=2).
- The 2 white vertices (of degree 3) stand for the 2 roots of $(1-\lambda+\lambda^2)^3 = 0$ (of degree 3).
- The 3 black vertices (of order 2) correspond to the remaining three branch points (of degree 2) one encounters by computing the derivative, namely

$$j'(\lambda) = \frac{(\lambda^2-\lambda+1)^2}{\lambda^3(\lambda-1)^3}(2\lambda-1)(\lambda^2-\lambda-2) = 0$$

A genus one example



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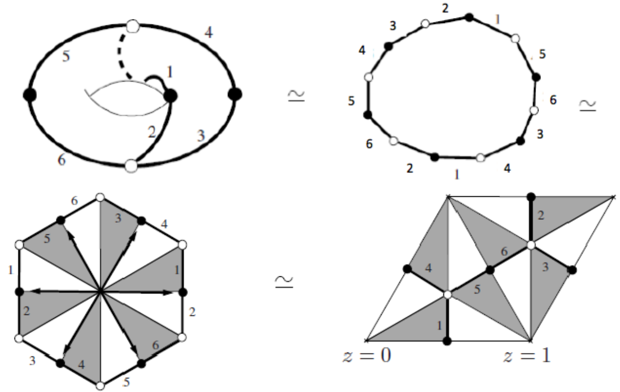


From here one can see that the corresponding Belyi pair is

$$(\mathbb{C}\mathbb{Z} \oplus \mathbb{Z}e^{2\pi i/3}, \wp^3) \cong (y^2 = x^3 - 1, x^3)$$

($\wp$=Weierstrass function)

Belyi's theorem

We now deal with the second part. We wanted to show that there is a correspondence

$$\{\text{Dessins}\} \leftrightarrow \left\{\text{Curves defined over } \overline{\mathbb{Q}} \text{ (plus a Belyi function)}\right\}$$

We already established the correspondence

$$\{\text{Equiv. classes of dessins}\} \leftrightarrow \{\text{Equiv. classes of Belyi pairs}\}$$

so it remains to show that Belyi curves are defined over $\overline{\mathbb{Q}}$.

Theorem (Belyi, 1979)

Let S be a compact Riemann surface. The following statements are equivalent:

(a) *S is defined over $\overline{\mathbb{Q}}$ (or, equivalently, over a number field).*

(b) *S admits a morphism $f : S \longrightarrow \mathbb{P}^1$ with at most three branching values, i.e. a Belyi function.*

(Proof of $\Rightarrow$) $\quad P(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$

$$
\begin{tikzcd}
\left\{ y^2 = x(x-1)\left(x - \sqrt{\frac{m+n}{m}}\right)\right\} \cup \{\infty\} \arrow[d, "(x{,}y)\mapsto x"] \arrow[dddd, bend right, "f"'] & \\
\widehat{\mathbb{C}} \arrow[d, "t\mapsto t^2"] & \left\{0, 1, \sqrt{\frac{m+n}{m}}, \infty\right\} \\
\widehat{\mathbb{C}} \arrow[d, "u\mapsto 1/u"] & \left\{0, 1, \frac{m+n}{m}, \infty\right\} \\
\widehat{\mathbb{C}} \arrow[d, "z\mapsto \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n"] & \left\{\infty, 1, \frac{m}{m+n}, 0\right\} \\
\widehat{\mathbb{C}} & \{\infty, 1, 0\}
\end{tikzcd}
$$

$(\approx \textit{minimal pol.})$

$(\textit{Belyi's pol.})$

Proof of $\Leftarrow$)

A complex number α lies in $\overline{\mathbb{Q}} \Leftrightarrow$ the set of complex numbers

$$G_{\mathbb{C}}(\alpha) = \{\alpha^{\sigma} := \sigma(\alpha) \text{ such that } \sigma \in G_{\mathbb{C}} := Gal(\mathbb{C})\}$$

is finite.

Examples:

- $G_{\mathbb{C}}(\alpha) = \{\alpha\} \Leftrightarrow \alpha \in \mathbb{Q}$
- $G_{\mathbb{C}}(\sqrt{2}) = \{\sqrt{2}, -\sqrt{2}\}$
- $G_{\mathbb{C}}(\pi) = \{\text{transcendental numbers}\}$

If $\boxed{C : F(x,y) = \sum a_{ij} x^i y^j = 0}$ we define

$\boxed{C^{\sigma} : F^{\sigma}(x,y) = \sum a_{ij}^{\sigma} x^i y^j = 0}$, $\boxed{G_{\mathbb{C}}(C) = \{C^{\sigma} : \sigma \in G_{\mathbb{C}}\}}$

Then, again: C is defined over $\overline{\mathbb{Q}} \Leftrightarrow G_{\mathbb{C}}(C)$/equiv. is finite.

The analogous statement holds for coverings (C, R) of $\mathbb{P}^1$.

Proof of $\Leftarrow$)

Consider the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d, "R"] & C^\sigma \arrow[d, "R^\sigma"] \\ \mathbb{P}^1 \arrow[r, "\sigma"] & \mathbb{P}^1 \end{tikzcd}$$

$$(\ \sigma(a,b) = (a^\sigma, b^\sigma)\ )$$

$$(i.e.\ \ R(a,b)^\sigma = R^\sigma(a^\sigma, b^\sigma)\ )$$

It shows that σ sends Belyi pairs of **degree** n to Belyi pairs of **same degree** n, hence there is an action of $G_{\mathbb{C}}$ on dessins:

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed] \arrow[d] & \mathcal{D}^\sigma \\ (C_{\mathcal{D}}, R_{\mathcal{D}}) \arrow[r, "\sigma"] & (C^\sigma_{\mathcal{D}}, R^\sigma_{\mathcal{D}}) \arrow[u] \end{tikzcd}$$

Because the degree and the branching is preserved, **the set**

$$G_{\mathbb{C}}(C,R) = \{(C^\sigma, R^\sigma) : \sigma \in G_{\mathbb{C}}\} \leftrightarrow G_{\mathbb{C}}(\mathcal{D}) = \{\mathcal{D}^\sigma : \sigma \in G_{\mathbb{C}}\}$$

is finite $\Rightarrow$ C and R **can be defined over a number field**.

Grothendieck's action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on dessins

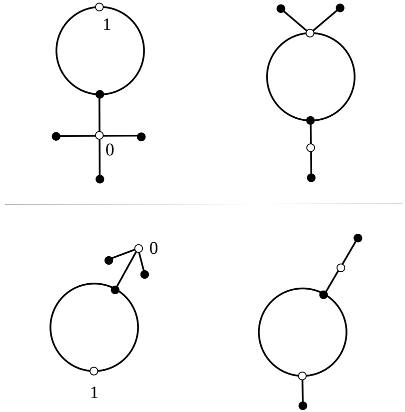
Once we know that Belyi pairs are defined over $\overline{\mathbb{Q}}$ it makes sense to "restrict" the action of $\mathrm{Gal}(\mathbb{C})$ on dessins

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed] \arrow[d] & \mathcal{D}^\sigma \\ (C_\mathcal{D}, R_\mathcal{D}) \arrow[r, "\sigma"] & (C_\mathcal{D}^\sigma, R_\mathcal{D}^\sigma) \arrow[u] \end{tikzcd}$$

to an action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ in the hope this action of the absolute Galois group on combinatorial objects might help to understand the nature of the group.

Let us look at a couple of examples

$\mathrm{Gal}(\overline{\mathbb{Q}})$-orbits of female and male symbols

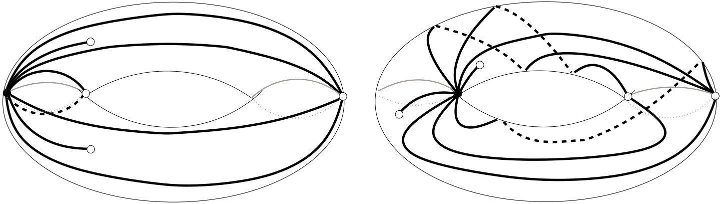


$$f(z) = \frac{z^4(z-1)^2}{(z-b)} \quad \text{where} \quad 36b^2 - 44b + 9 = 0$$

$$m(z) = \frac{z^3(z-1)^2}{(z-b)} \quad \text{where} \quad 25b^2 - 28b + 4 = 0$$

$Gal(\overline{\mathbb{Q}})$-action on the dessin corresponding to $\left(y^2 = x(x-1)(x+\sqrt{2}),\ f(x,y) = \frac{-4(x^2-1)}{(x^2-2)^2}\right)$

It can be seen that the following two dessins form a complete orbit for the action of $Gal(\overline{\mathbb{Q}})$.

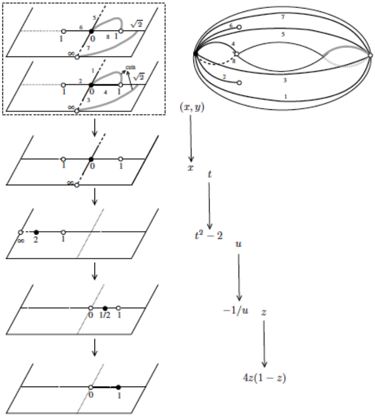


$\left(y^2 = x(x-1)(x+\sqrt{2}),\ f\right)$ $\left(y^2 = x(x-1)(x-\sqrt{2}),\ f\right)$

They are conjugate by any σ satisfying the property

$$\sigma(\sqrt{2}) = -\sqrt{2}.$$

Dessin of $f : \{y^2 = x(x-1)(x-\sqrt{2})\} \to \widehat{\mathbb{C}}$



$$f(x,y) = \frac{-4\left(x^2-1\right)}{\left(x^2-2\right)^2}$$

Regular dessins

$\mathcal{D}$ is said to be **regular** if $G = Aut(\mathcal{D})$ acts transitively on its edges so that $(X_{\mathcal{D}}, f_{\mathcal{D}})$ is a Galois cover with covering group G.

$$\begin{tikzcd}
X_{\mathcal{D}} \arrow[d, "\pi"'] \arrow[dr, "f_{\mathcal{D}}"] & \\
X_{\mathcal{D}}/Aut(\mathcal{D}) \arrow[r, "\sim"] & \mathbb{P}^1
\end{tikzcd}$$

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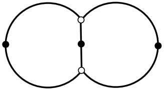
We have already seen an example:



$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[d, "\pi"'] \arrow[drr, "j(\lambda)=\frac{\left(1-\lambda+\lambda^2\right)^3}{\lambda^2(\lambda-1)^2}"] & & \\ \widehat{\mathbb{C}}/G \arrow[r, "\sim"] & & \mathbb{P}^1 \end{tikzcd}$$

$(\Rightarrow Gal(\overline{\mathbb{Q}})\ \ acts\ \ trivially)$

$(\ G =< z \to 1 - z,\ \ z \to 1/(1 - z) > \cong S_3\ )$

Faithfulness of the action of $Gal(\overline{\mathbb{Q}})$

A key point in Grothendieck's theory of dessins is that the action of $Gal(\overline{\mathbb{Q}})$ on them, or equivalently on Belyi pairs (C, f), is faithful.

Faithfulness of the action of $Gal(\overline{\mathbb{Q}})$

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In fact, except for the case $g = 0$, the action is already faithful on the set of Belyi curves C of any given genus (disregarding the Belyi function).

For instance if $\sigma \in Gal(\overline{\mathbb{Q}})$ is non-trivial and $j(\lambda)^\sigma \neq j(\lambda)$ then the elliptic curves

$C : y^2 = (x-1)(x-2)(x-\lambda)$ and

$C^\sigma : y^2 = (x-1)(x-2)(x-\lambda^\sigma)$

are non-isomorphic to each other.

Triangle curves

Q. 1: Does $\mathrm{Gal}(\overline{\mathbb{Q}})$ act faithfully on regular dessins (C, f)?

Curves arising in this way are called **triangle (o quasiplatonic) curves** i.e. C is a triangle curve if $C/Aut(C) \equiv \mathbb{P}^1$ and the projection $f : C \to C/Aut(C) \equiv \mathbb{P}^1$ ramifies over 3 values.

Q. 1': Is the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ faithful on triangle curves?
There are only finitely many triangle curves in each genus

List of triangle curves in low genus

- **Genus 2**: $y^2 = x^6 - x, \quad y^2 = x^6 - 1, \quad y^2 = x^5 - x$
- **Genus 3**: $y^2 = x^8 - x, \quad y^2 = x^7 - x, \quad y^2 = x^8 - 1,$ $y^2 = x^8 - 14x^4 + 1, \quad y^3 = x(1 - x^3),$ $y^4 = 1 - x^3, \quad y^4 = 1 - x^4, \quad y^7 = x(1 - x)^2$
- **Genus 4**: 11 curves **all of them again defined over** $\mathbb{Q}$.

$Gal(\overline{\mathbb{Q}})$ acts faithfully on triangle curves

In all previous examples of triangle curves the action of $Gal(\overline{\mathbb{Q}})$ is trivial. However:

Theorem (-, A. Jaikin-Zapirain)

$Gal(\overline{\mathbb{Q}})$ acts faithfully on triangle curves.

In fact on the subset of triangle curves which are Galois covers of a given one, e.g. Fremat's or Klein's curve, or any of the previous list.....

THANK YOU