Hanani–Tutte for approximating maps of graphs

Radoslav Fulek a Jan Kynčl

Weakly simple polygons

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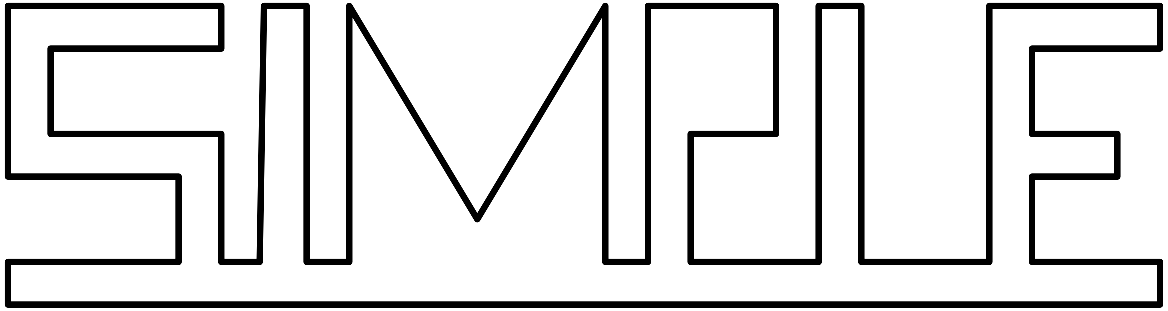

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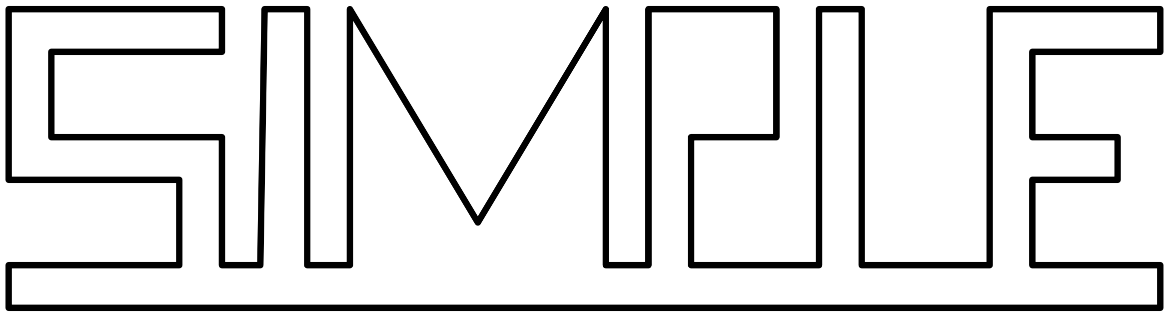

Weakly simple polygon is a PL map $\varphi : S^1 \to \mathbb{R}^2$ such that for every $\varepsilon > 0$ there exists PL $\pi : S^1 \hookrightarrow \mathbb{R}^2$ s.t. $\|\varphi - \pi\| \leq \varepsilon$.

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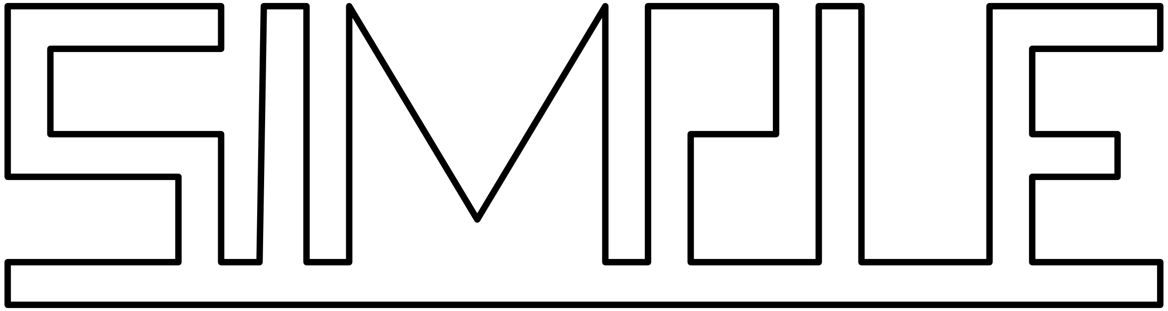

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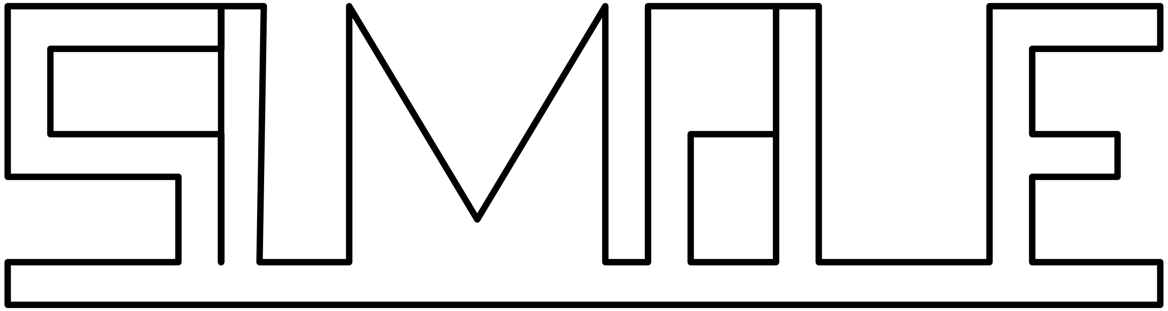

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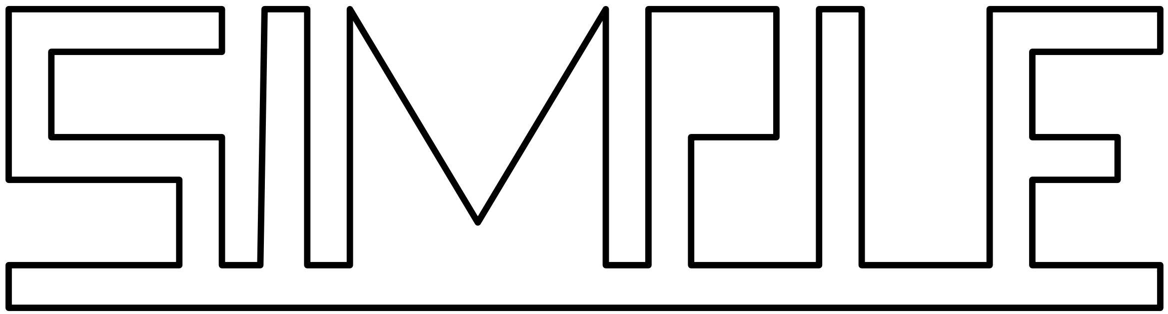

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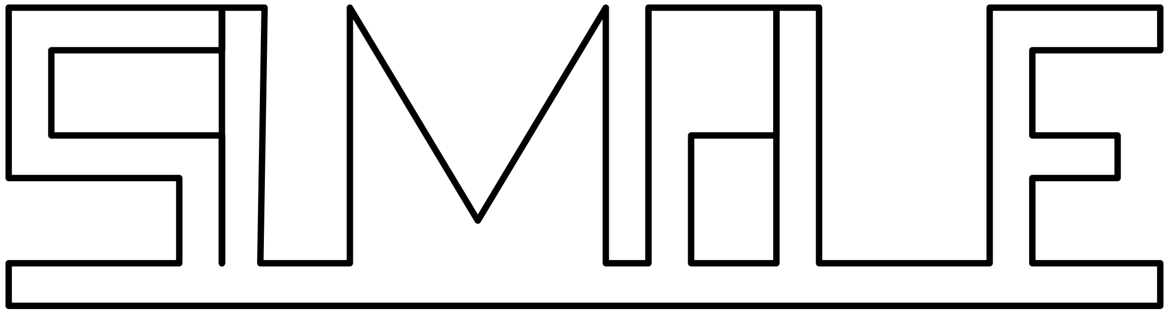

Testing whether a polygon is wekly simple is solvable in $O(n \log n)$ time (Cortese et al. 2009, Chang et al. 2015, Akitaya et al. 2016).

Detecting weakly simple polygonal path

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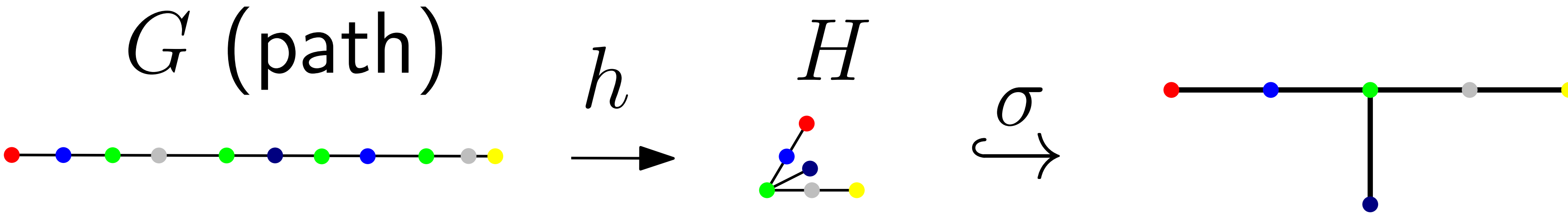

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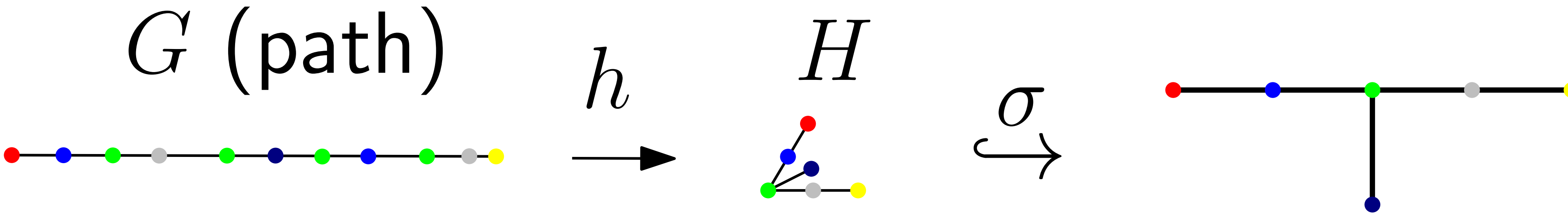
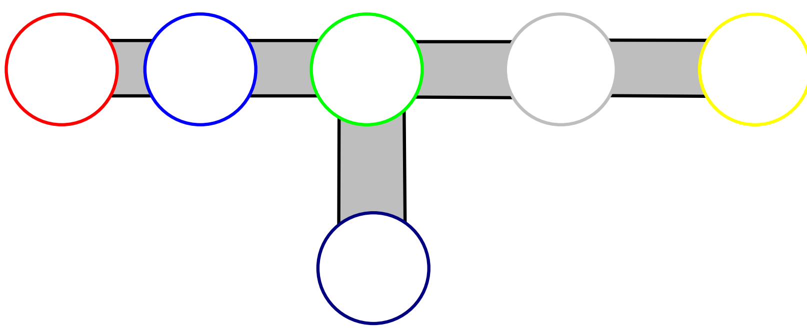


ε-neighborhood of $\sigma(H)$



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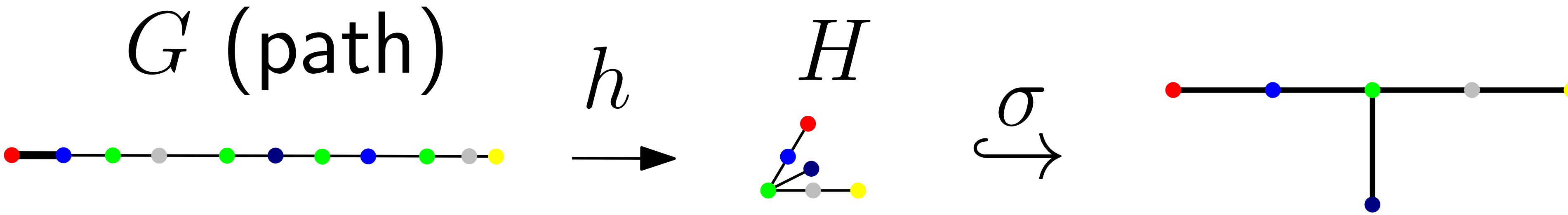

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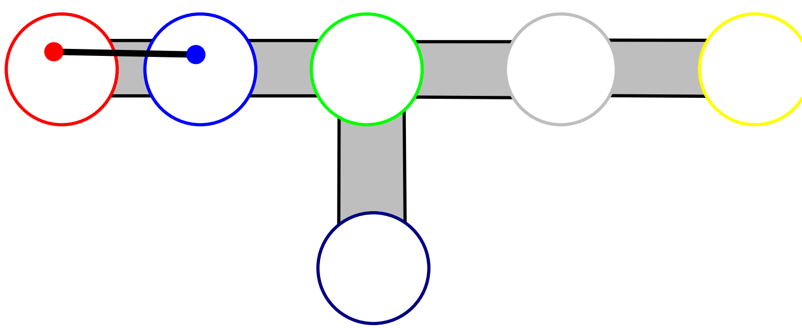

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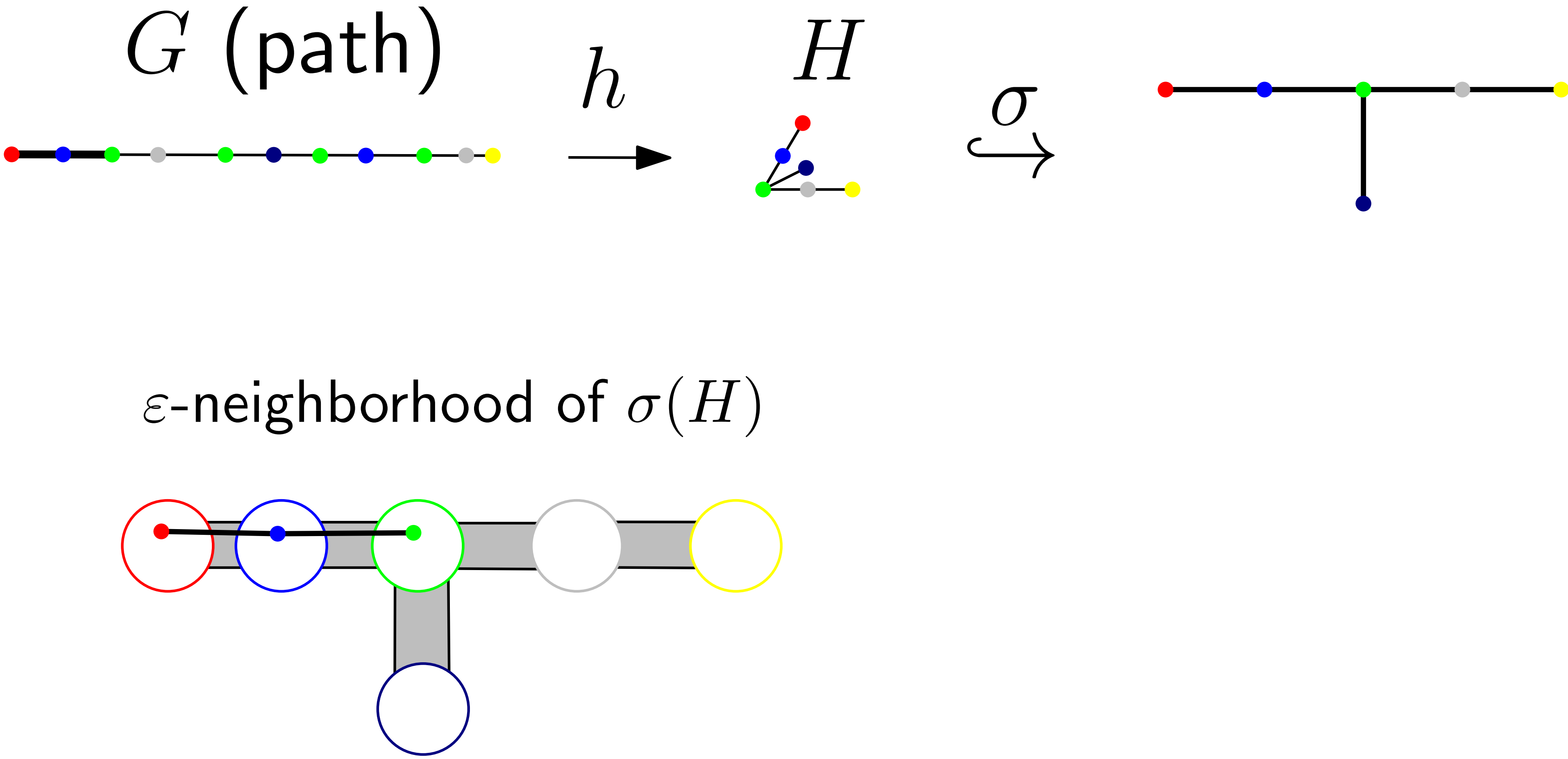

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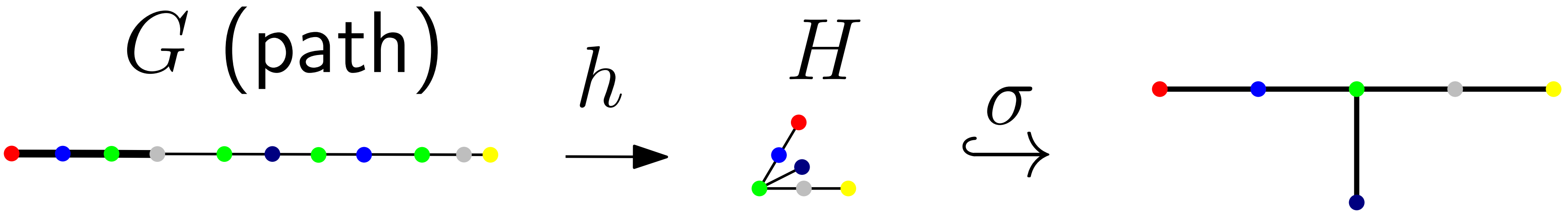

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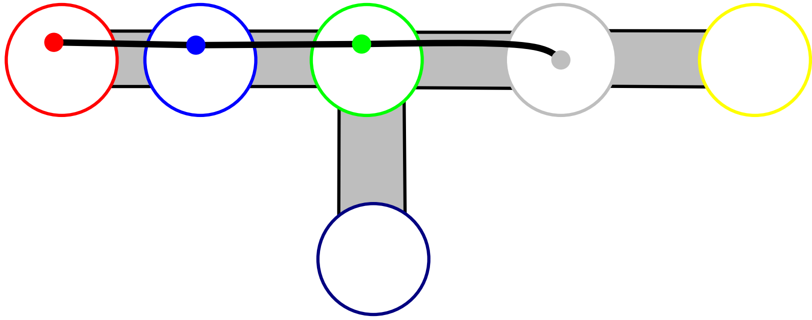

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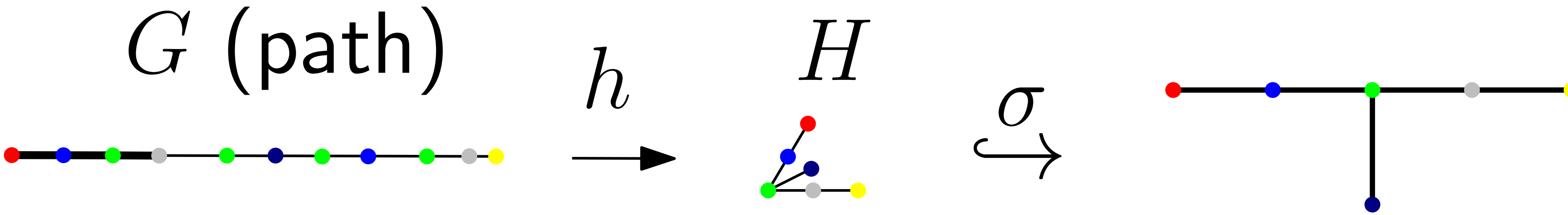

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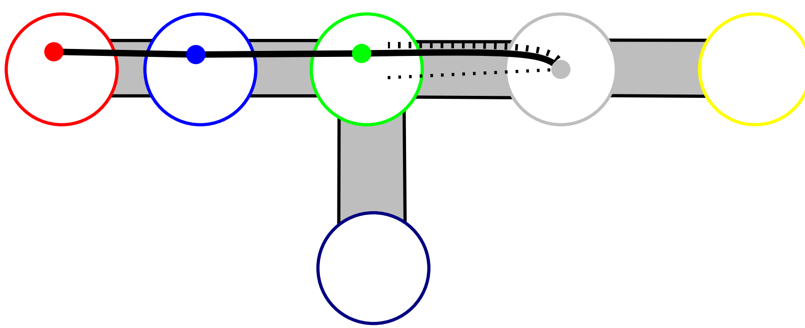

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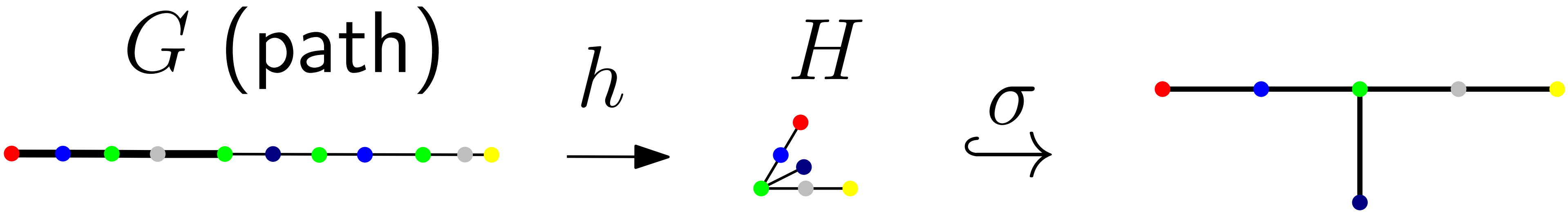

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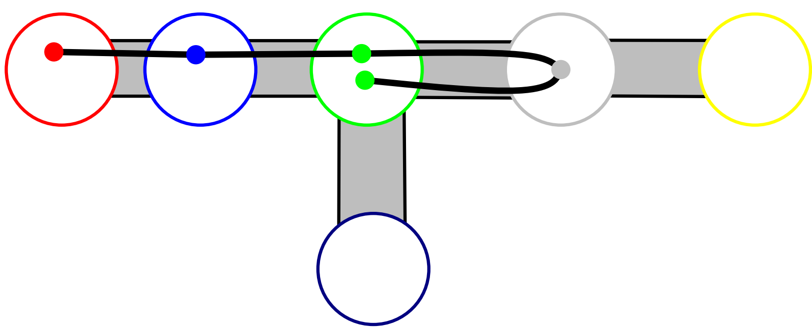

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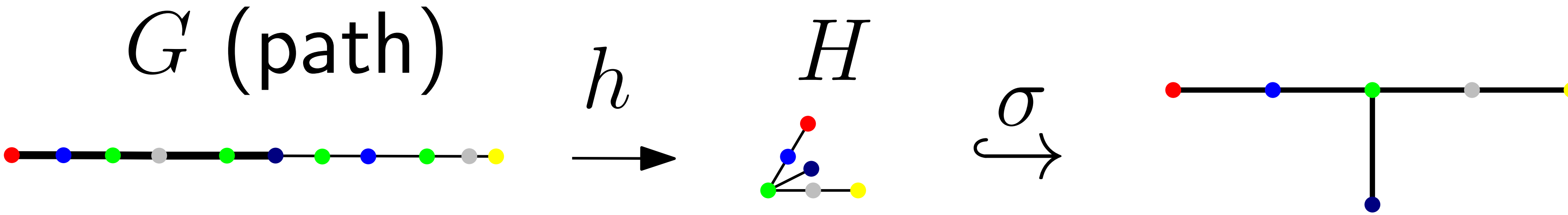

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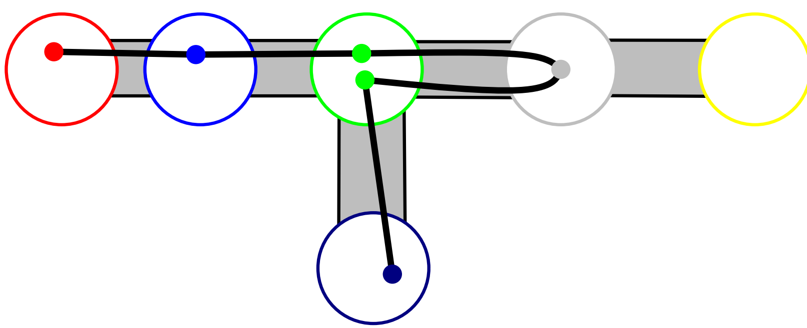

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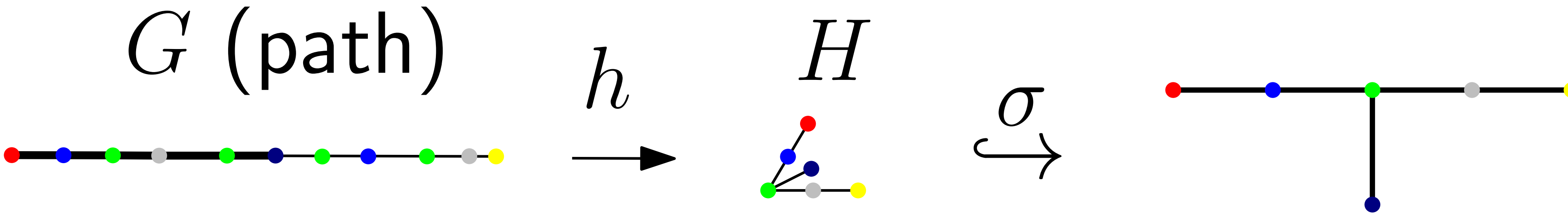

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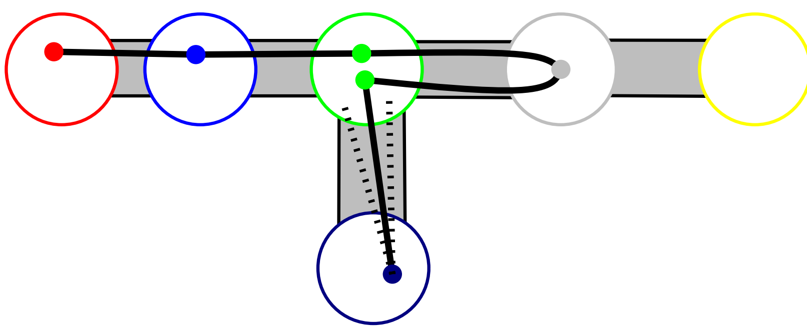

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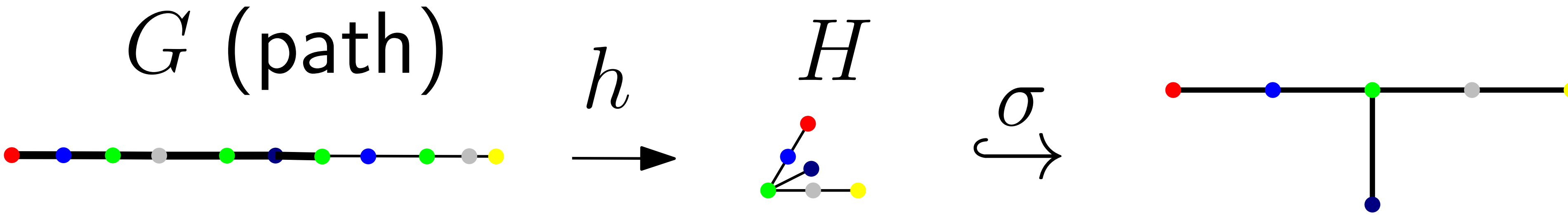

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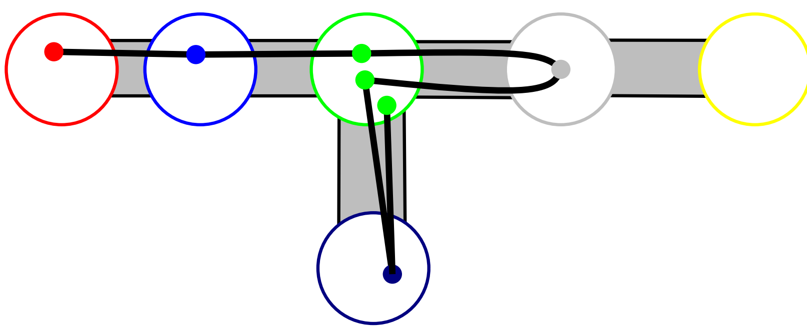

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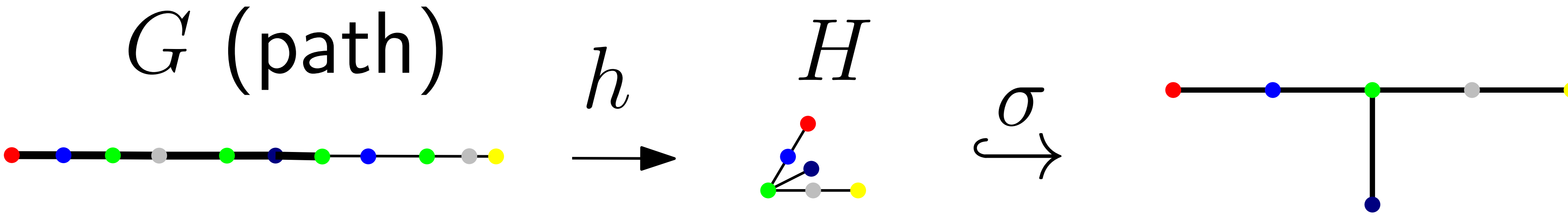

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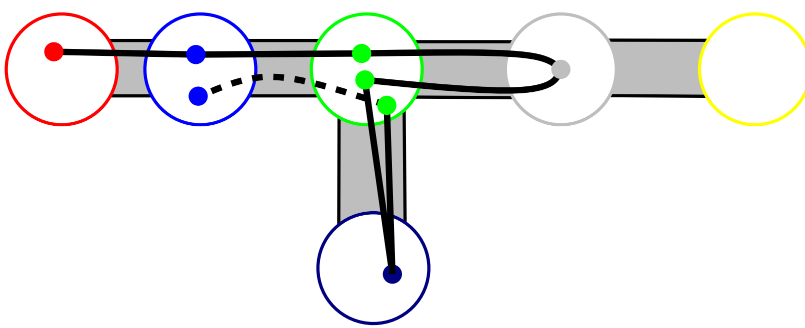

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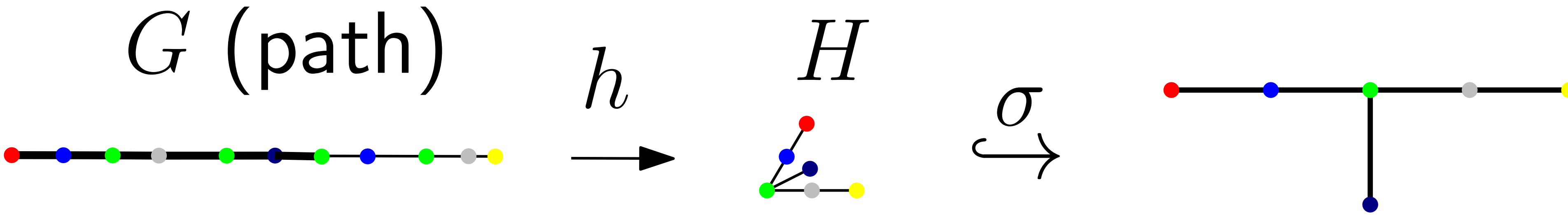

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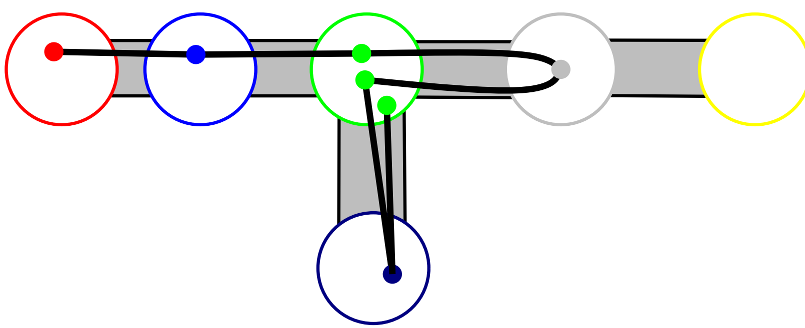

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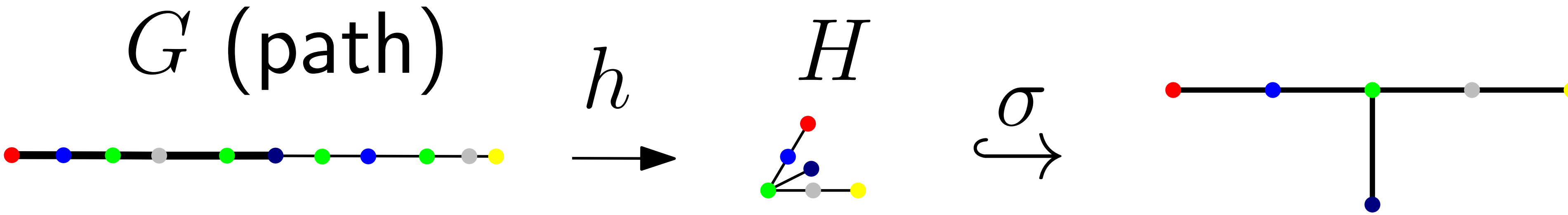

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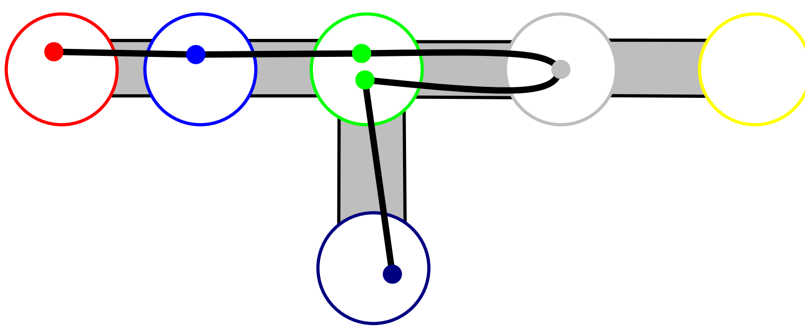

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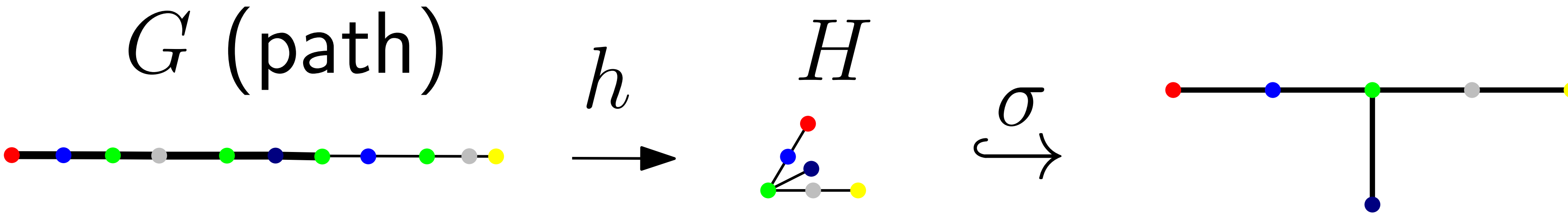

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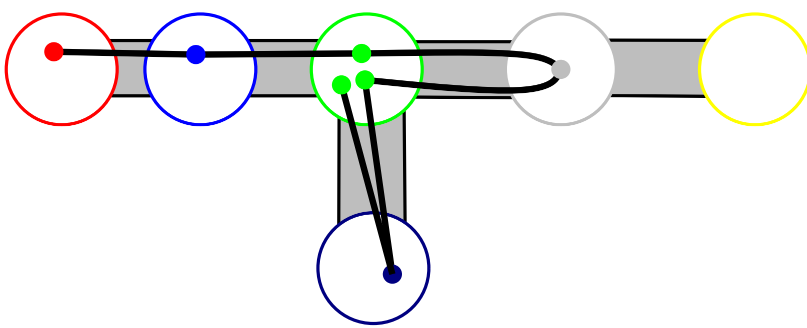

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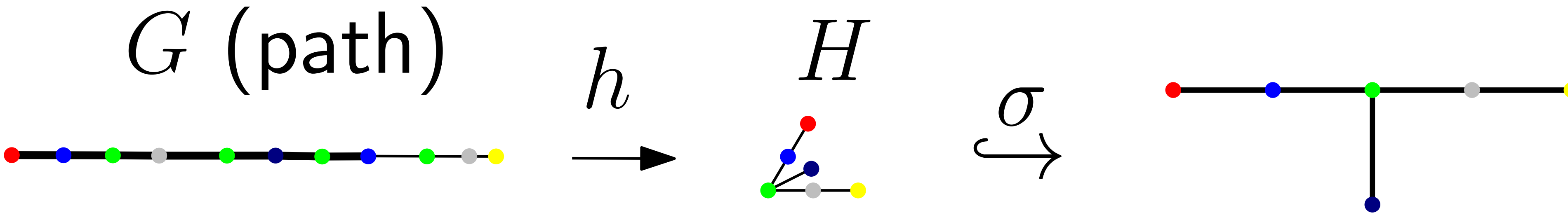

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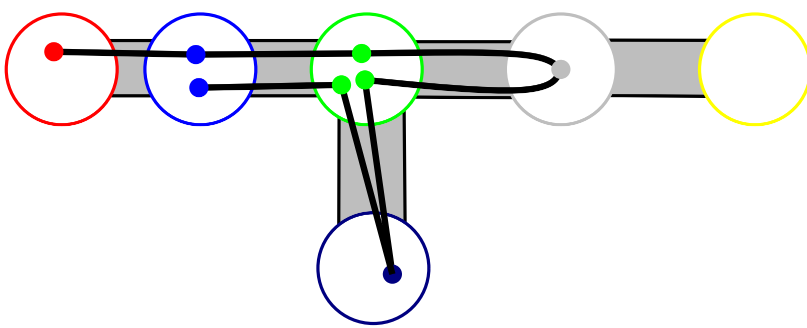

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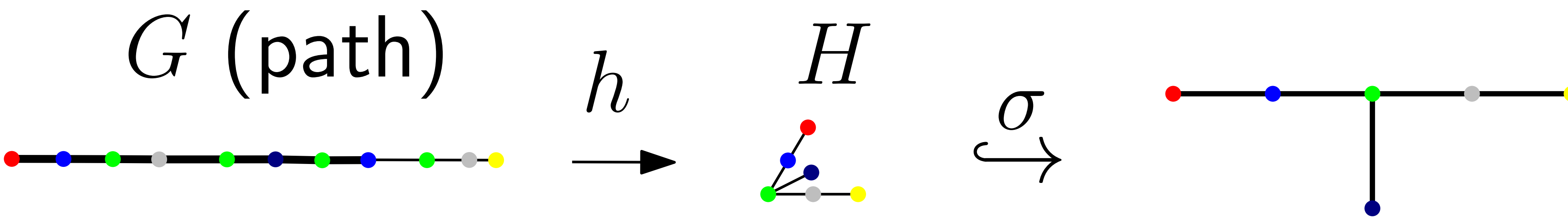

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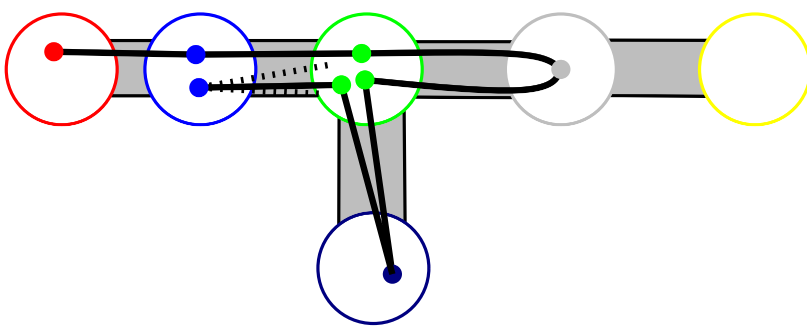

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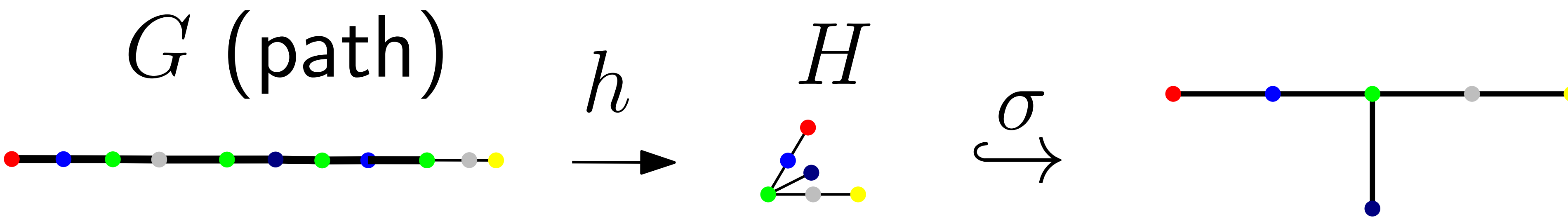

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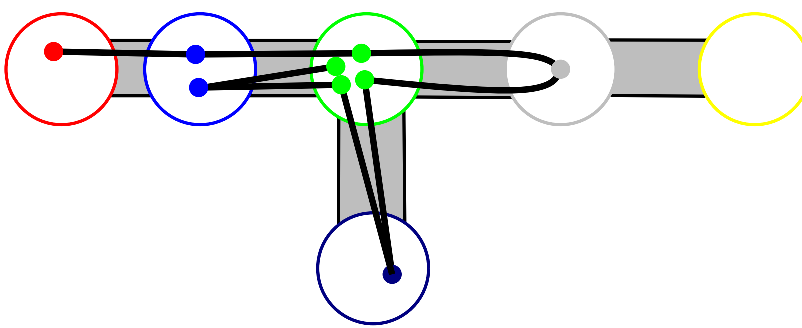

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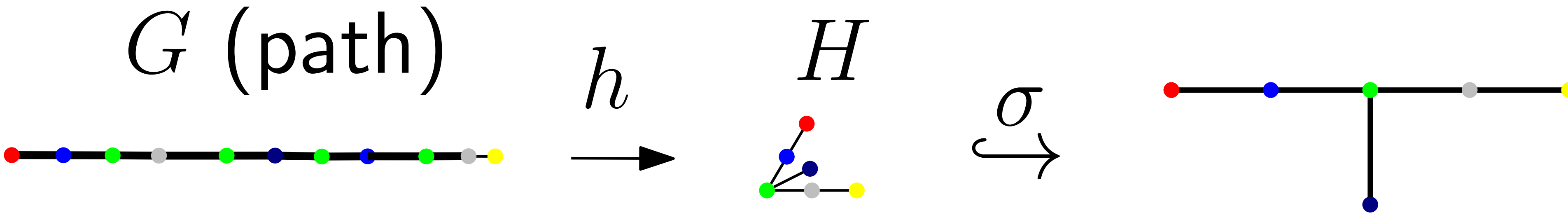

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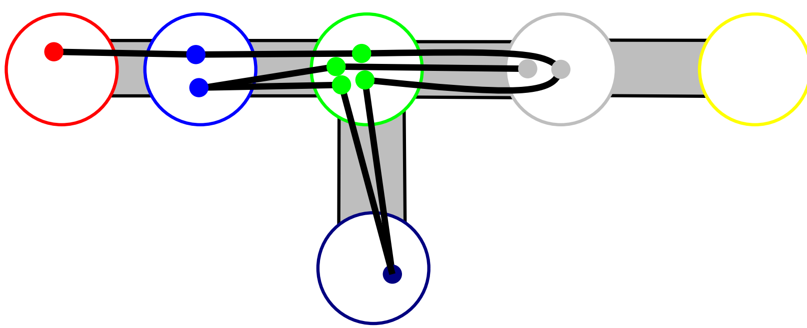

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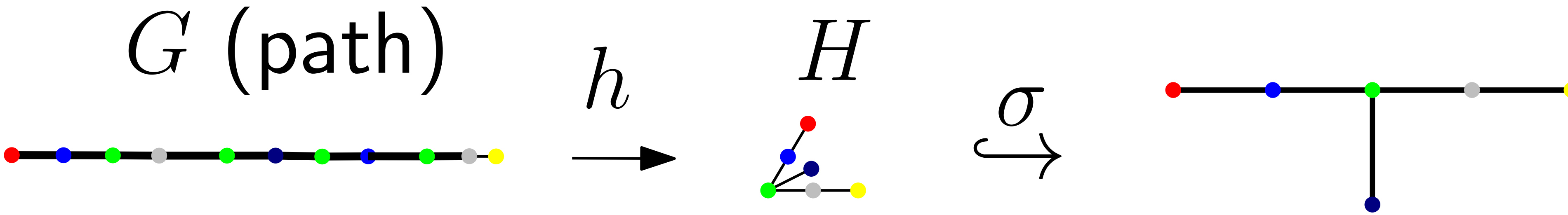

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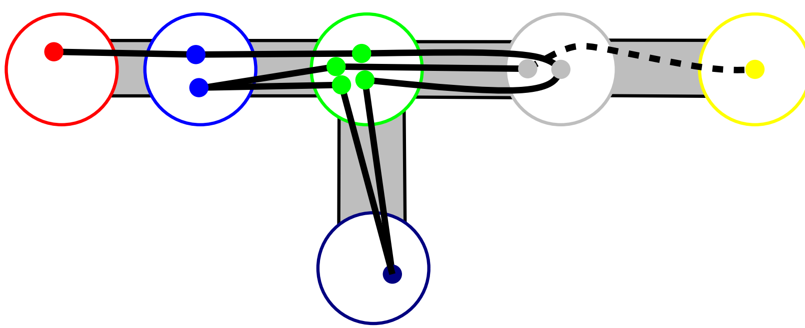
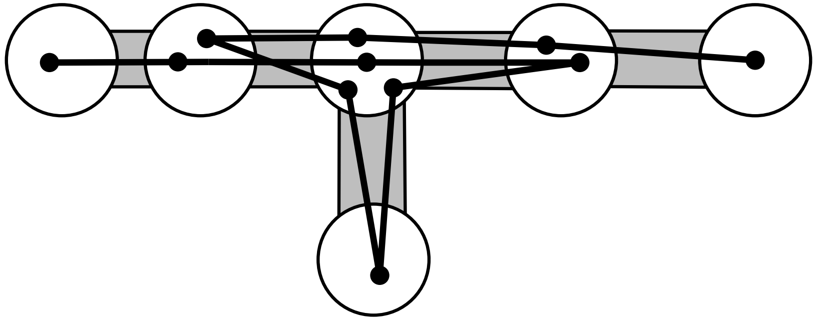


Detecting weakly simple polygonal path

$$(G, H, h, \sigma)$$



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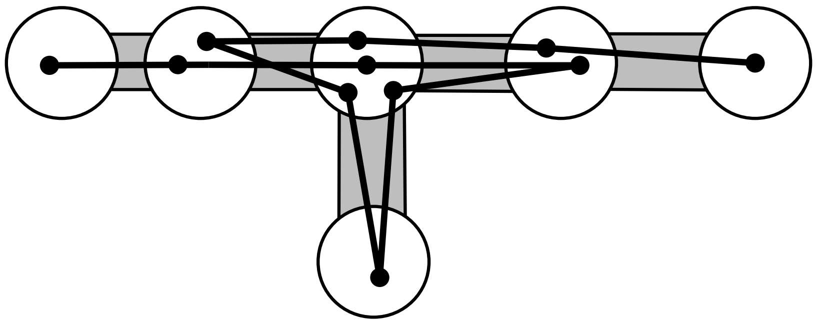
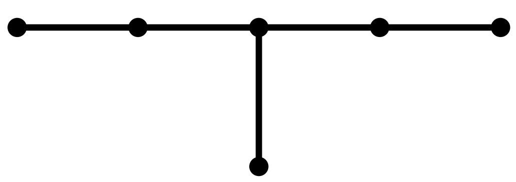


We turn H into its line graph.



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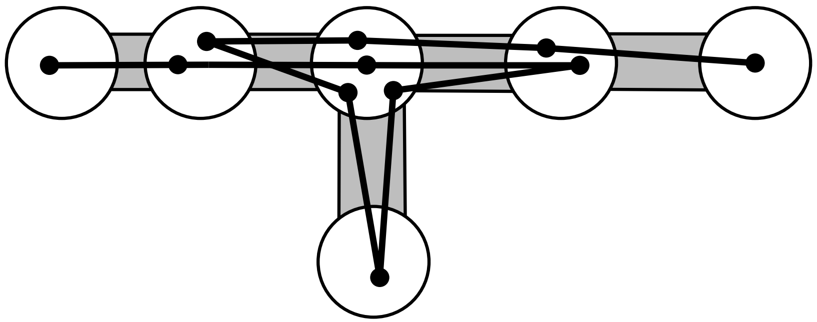
(G, H, h, σ)



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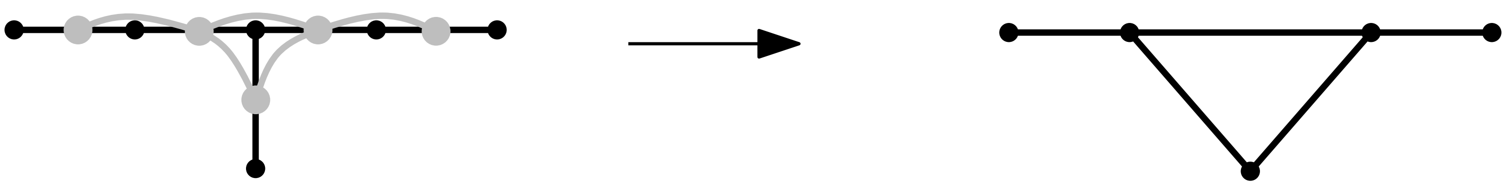

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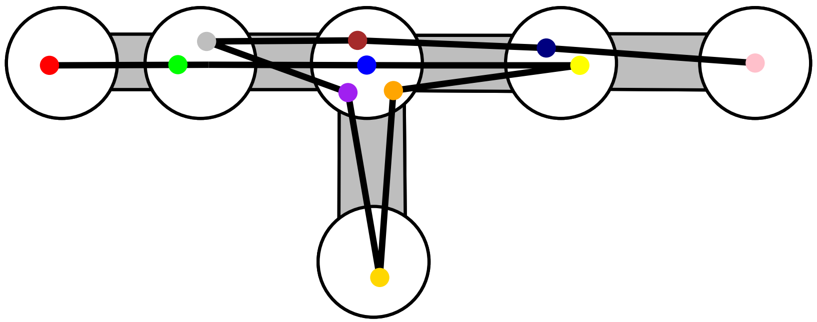

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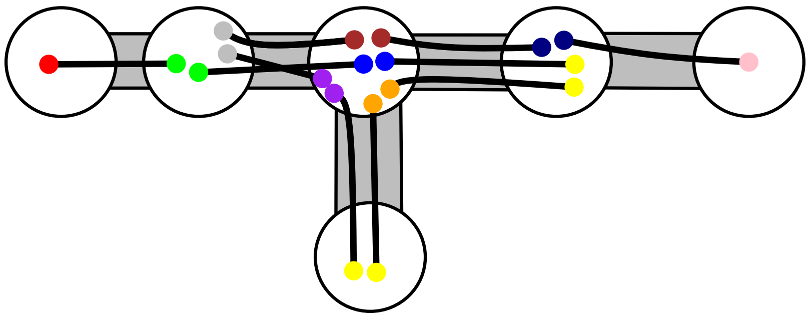

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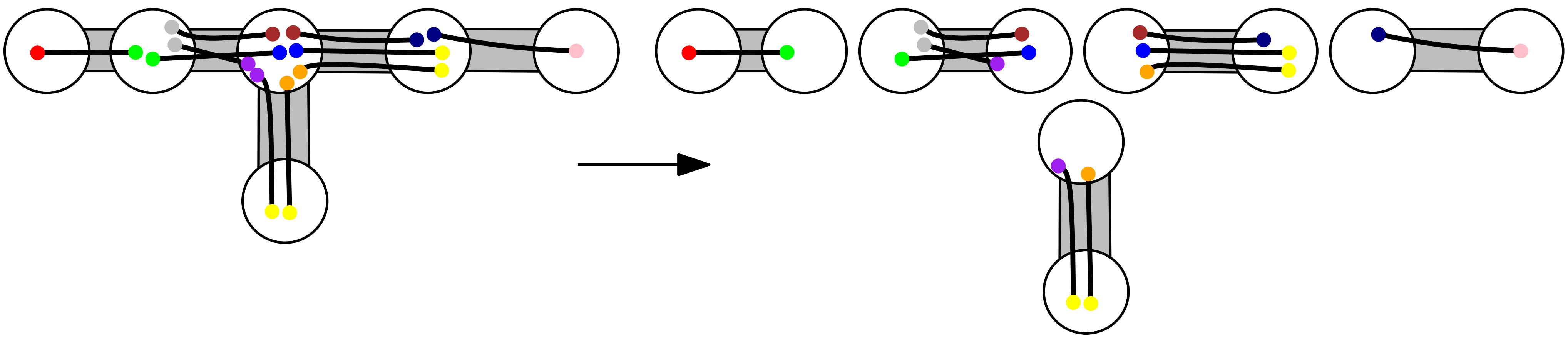

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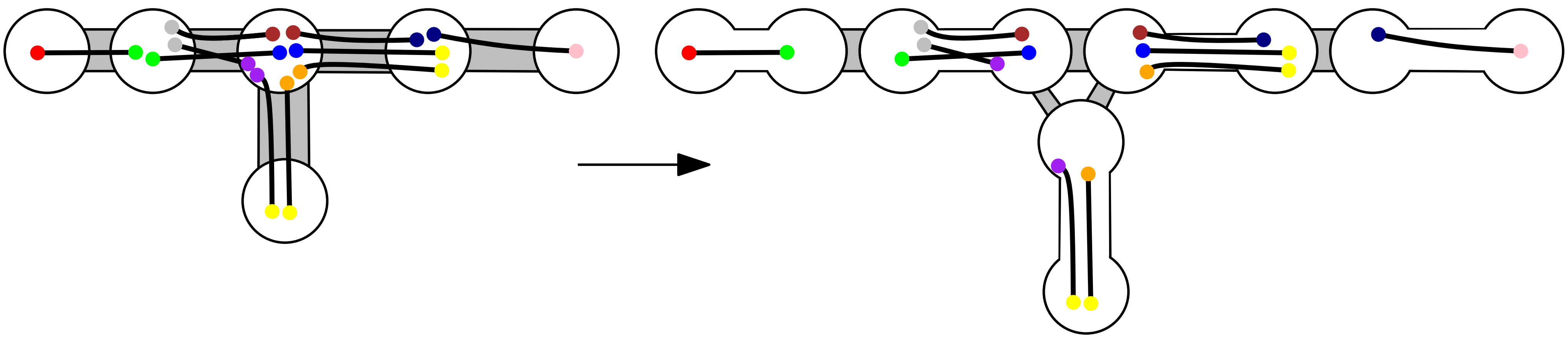

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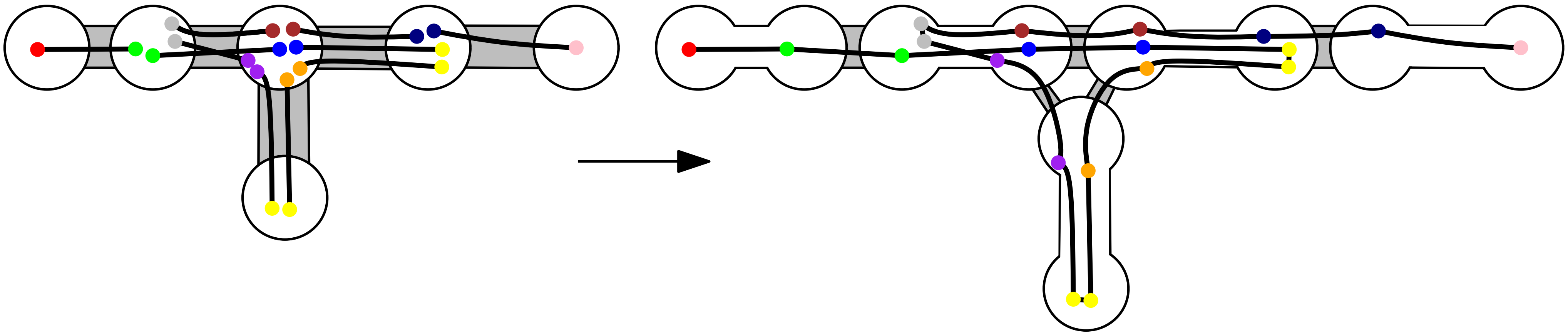

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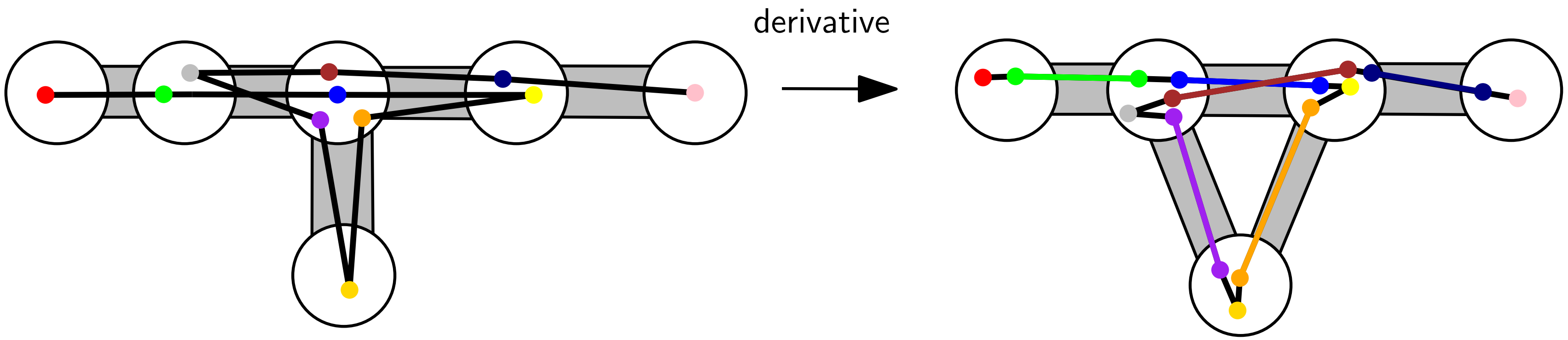
$$(G, H, h, \sigma) \qquad (G', H', h', \sigma')$$



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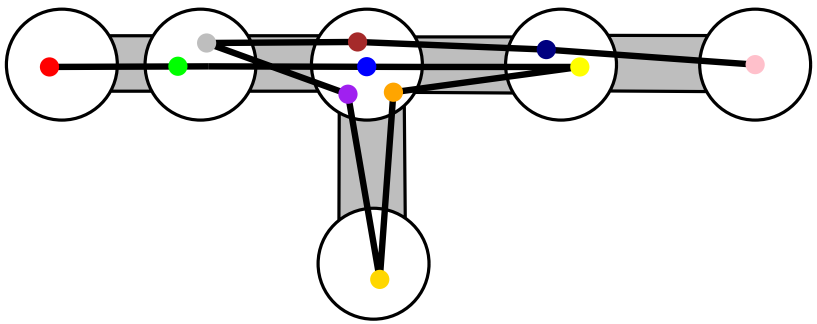

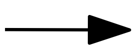

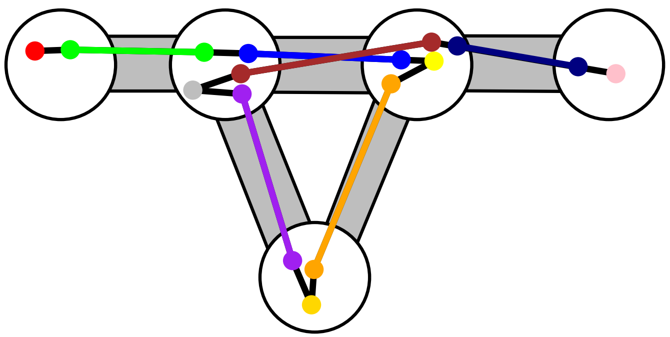

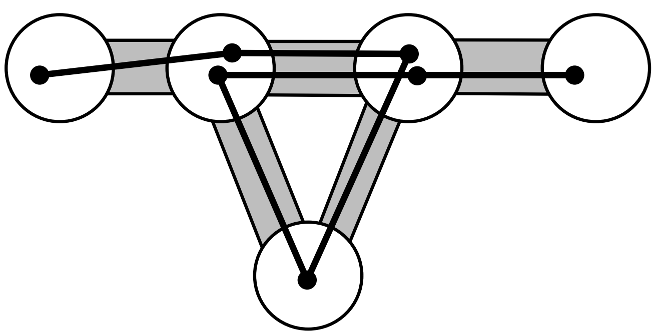

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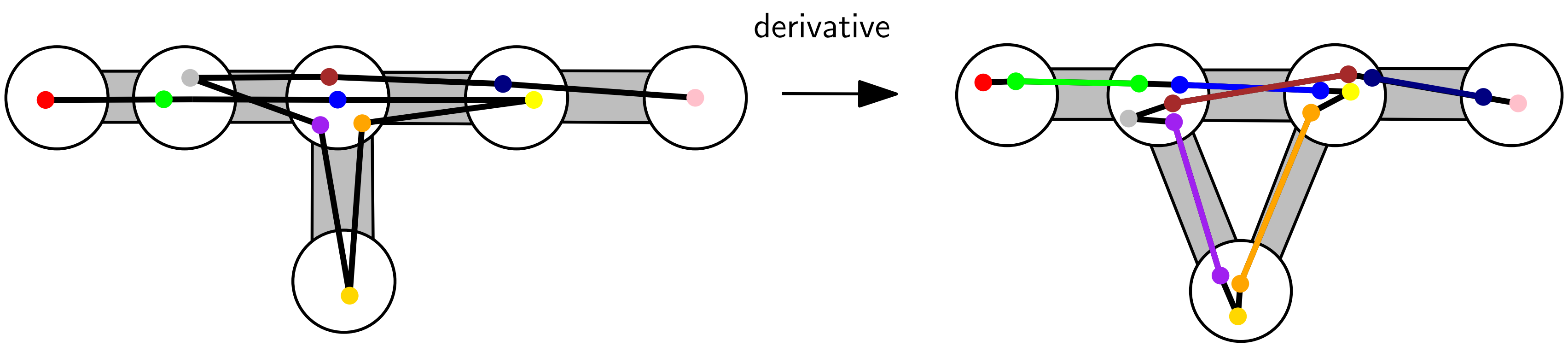
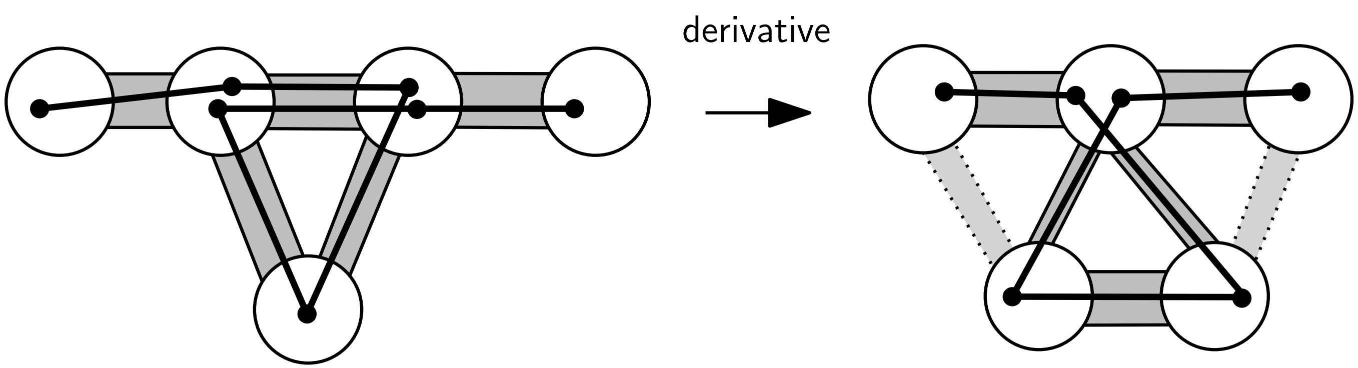
(G, H, h, σ) (G', H', h', σ')





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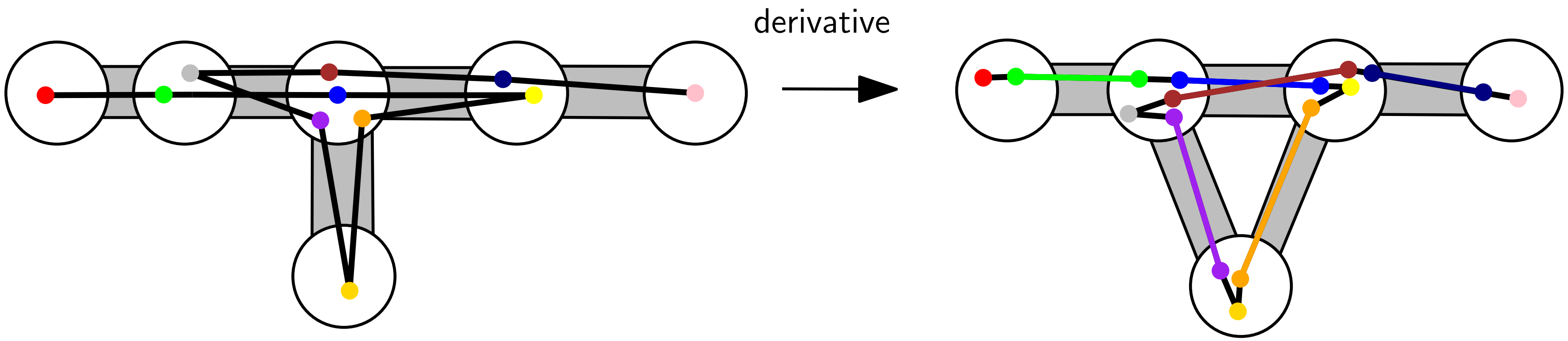

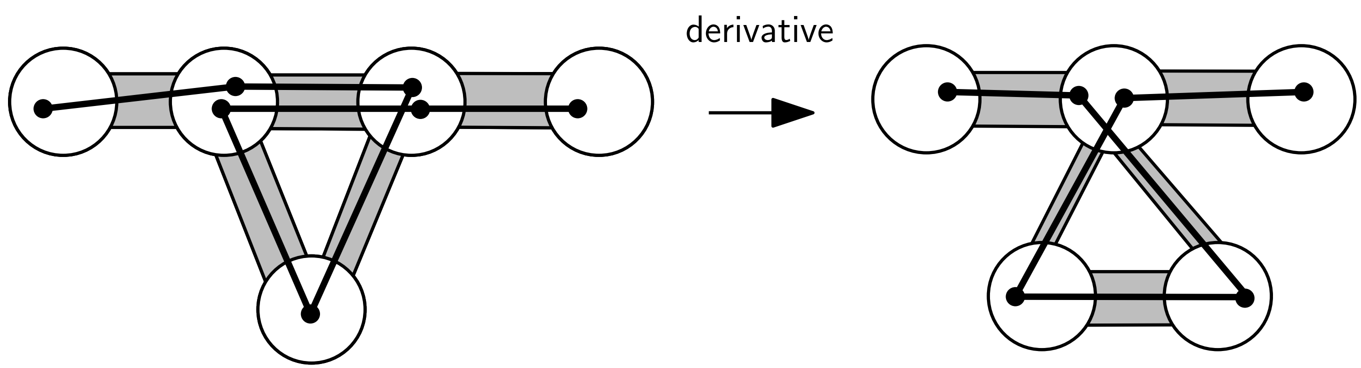

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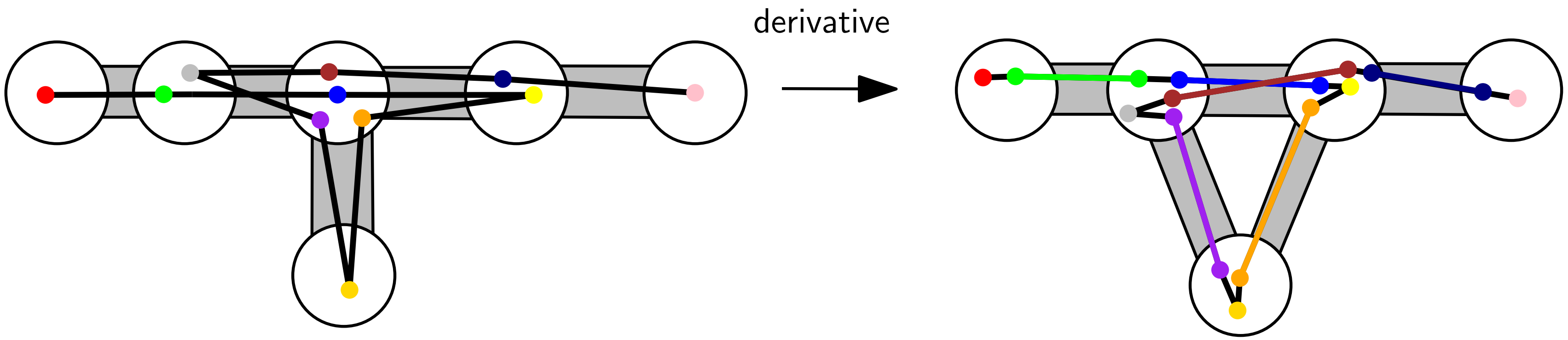

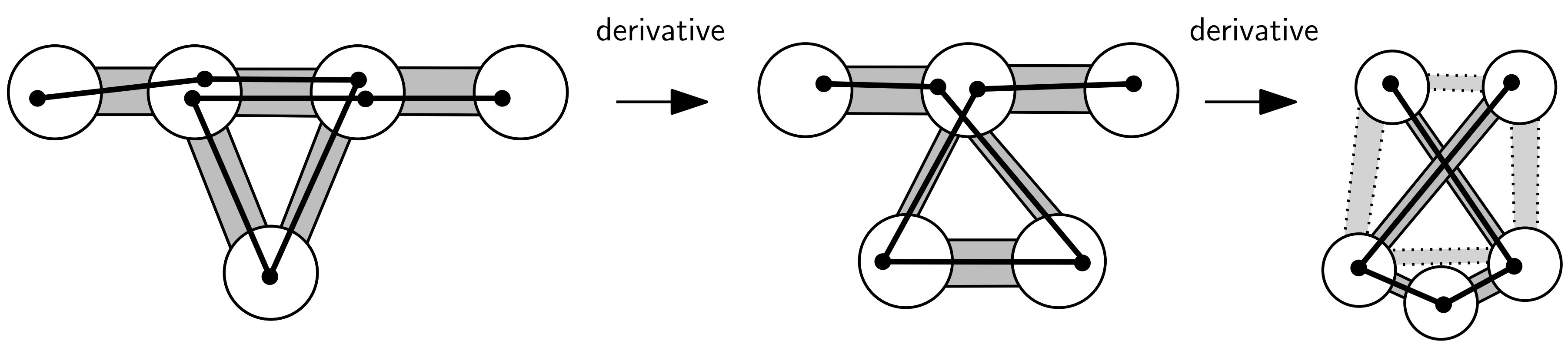

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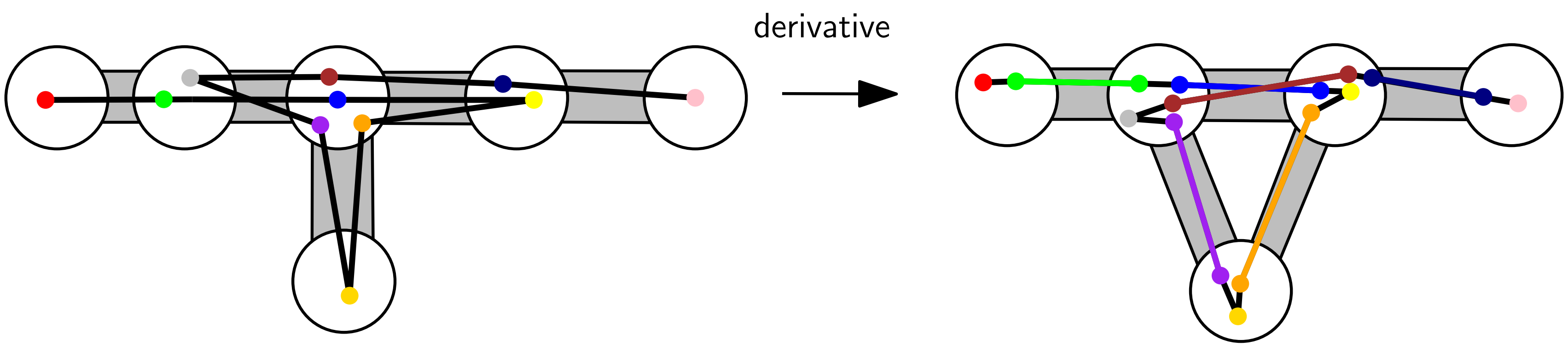

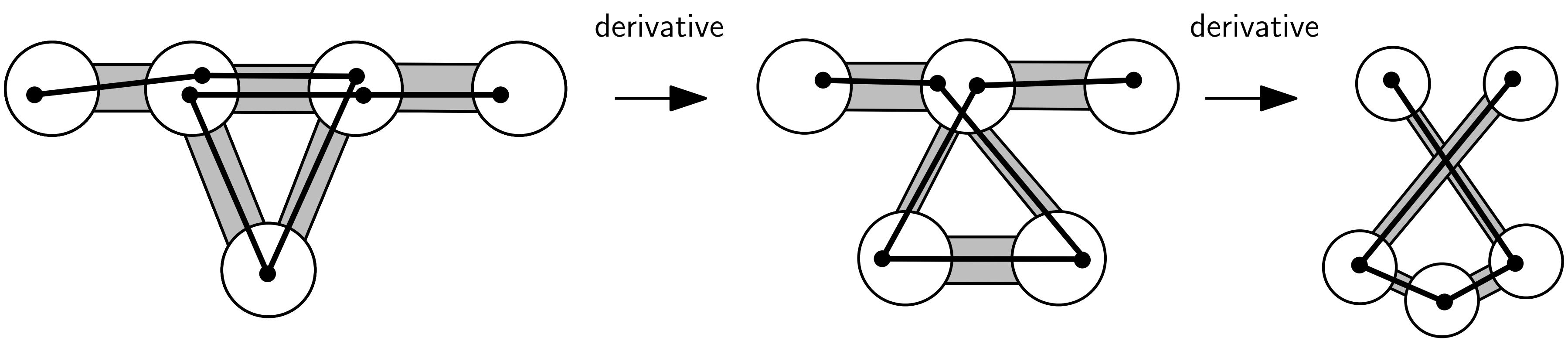

Detecting weakly simple polygonal path

Minc (1997) Let G be a path, for (G, H, h, σ) the polygonal path $\sigma \circ h$ is weakly simple if and only if for (G', H', h', σ') the composition $\sigma' \circ h'$ is weakly simple. Furthermore, by applying the derivative iteratively finitely many times we either obtain $((\emptyset, \emptyset), (\emptyset, \emptyset), h^{(i)}, \sigma^{(i)})$ or a crossing in $\sigma^{(i)}$ for some $i \in [n]$, where $n = |V(G)|$.

Detecting weakly simple polygonal path

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Corollary We can test in a polynomial time if $\sigma \circ h$ is weakly simple.

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For (G, H, h, σ), the composition $\sigma \circ h$ is $\mathbb{Z}_2$-**approximable** if for every $\varepsilon > 0$ there exists $\psi : G \hookrightarrow_{\mathbb{Z}_2} M$ such that $\|\sigma \circ h - \psi\| \leq \varepsilon$.

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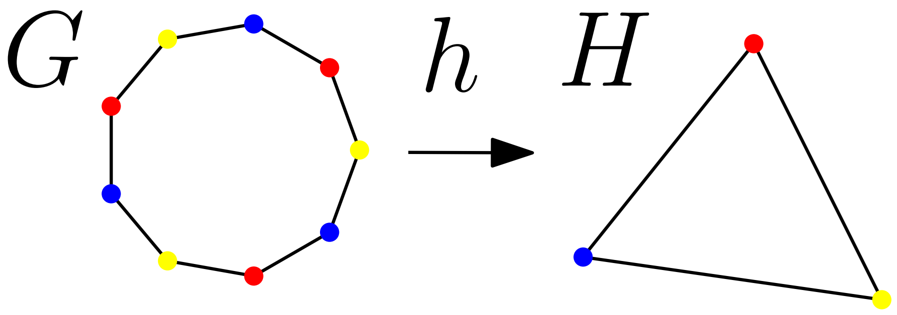

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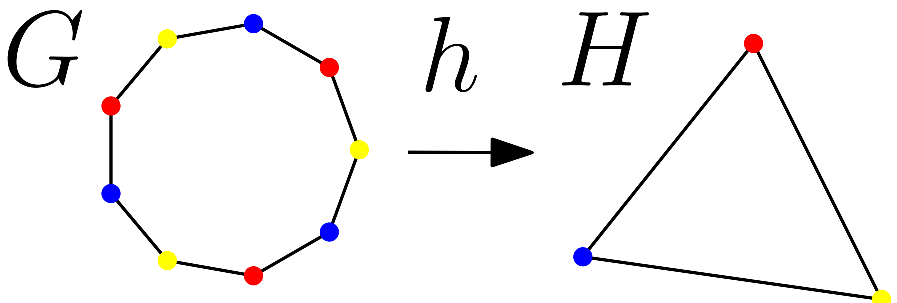

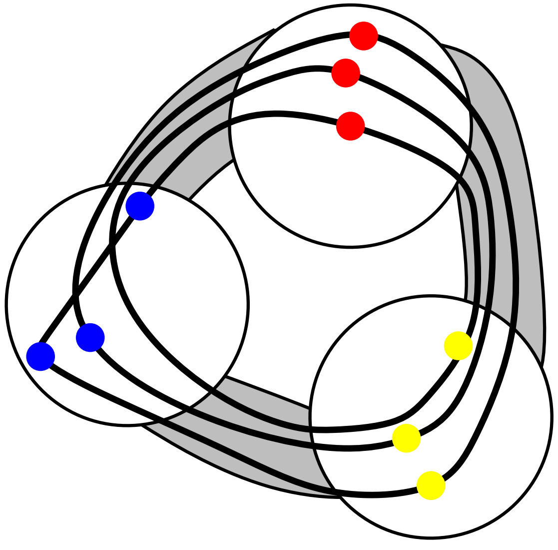

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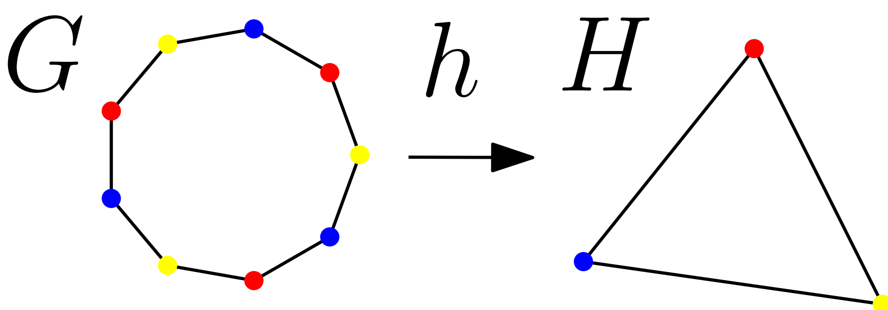

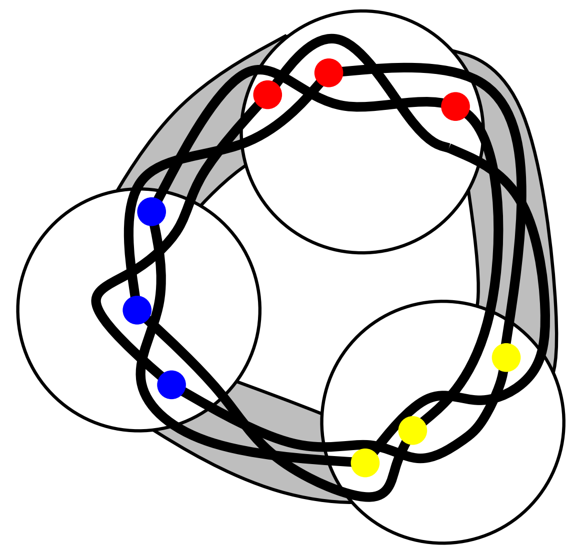

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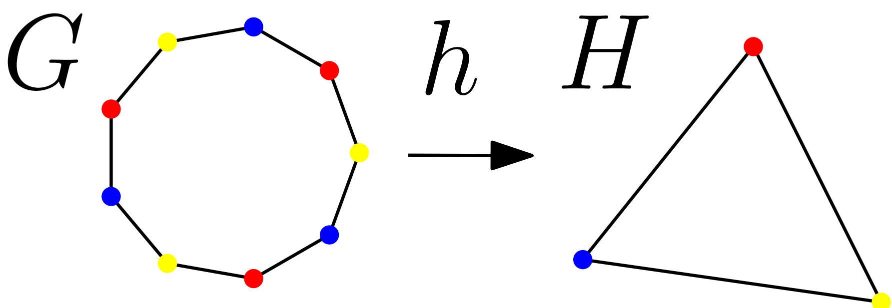

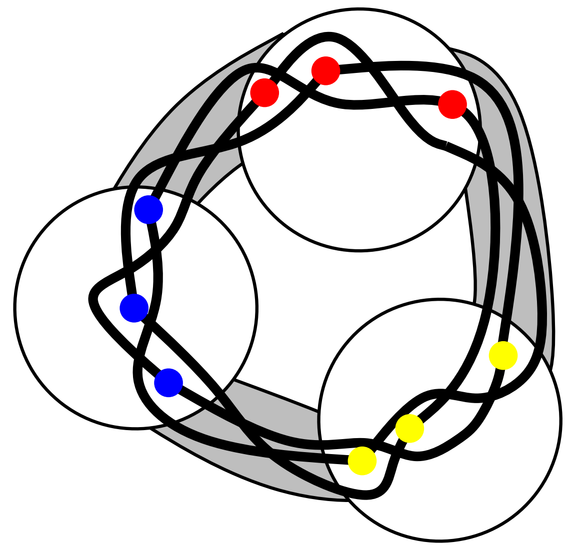

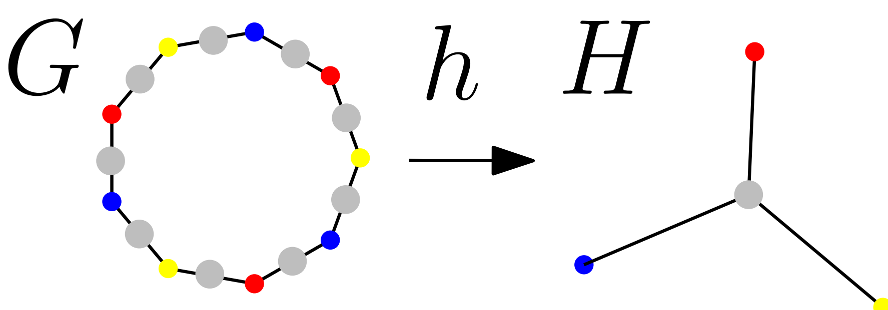

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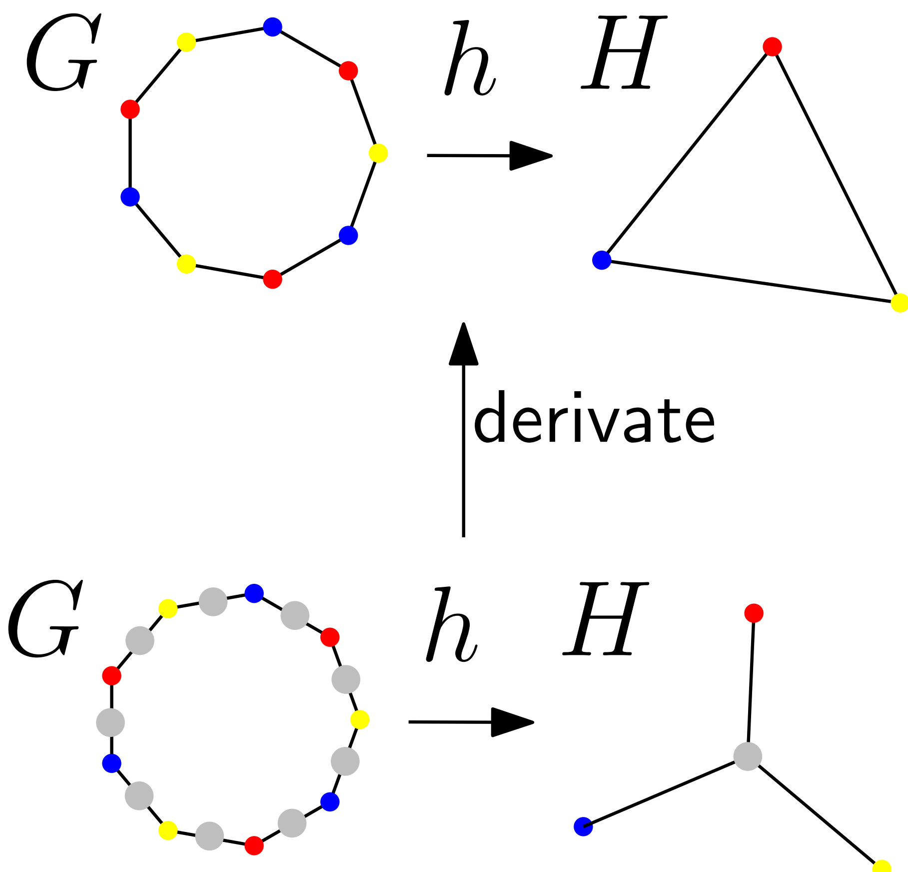

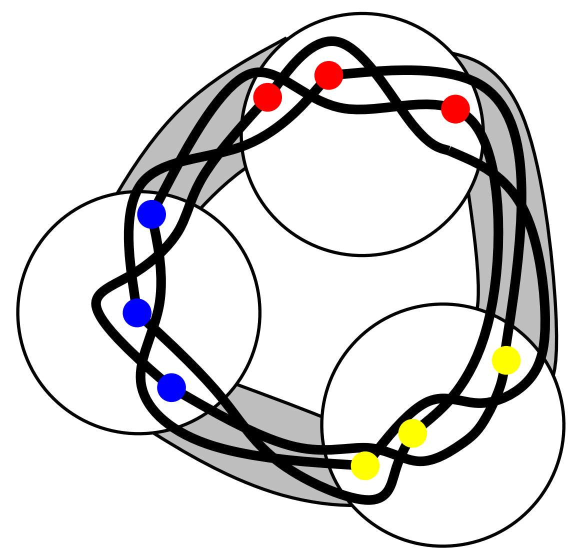

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Conjecture: **M. Skopenkov (2003)** If for (G, H, h, σ) the composition $\sigma \circ h$ is $\mathbb{Z}_2$-approximable then either it is also approximable by an embedding or G contains a cycle C as a subgraph such that for some $i \in \mathbb{N}$ the i-th derivative $(C^{(i)}, H^{(i)}, h^{(i)}, \sigma^{(i)})$ is **swe**.

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Resolves also the strip planarity problem by Angelini et al. (2013).

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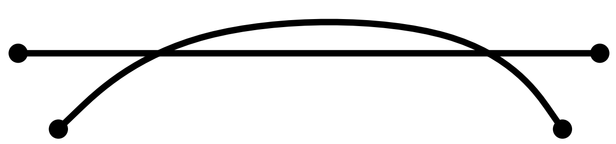

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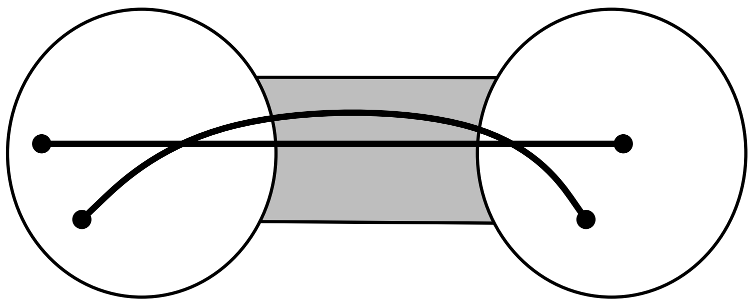

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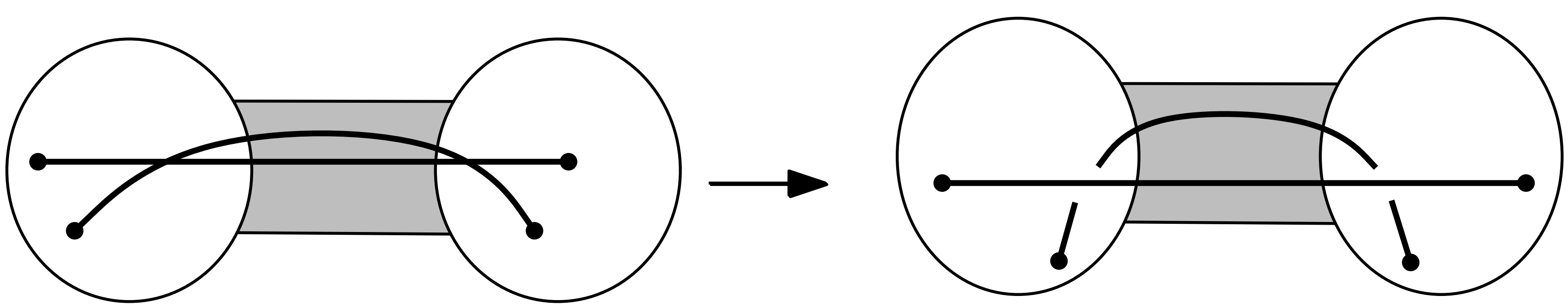

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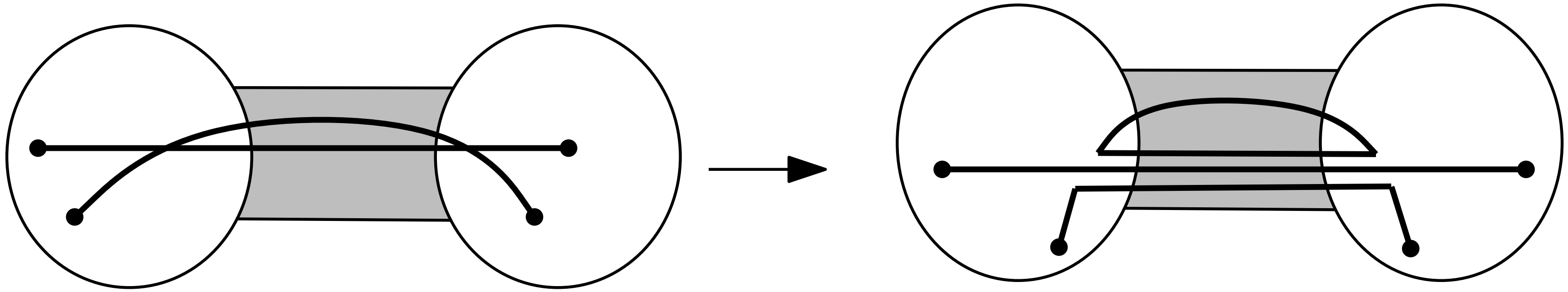

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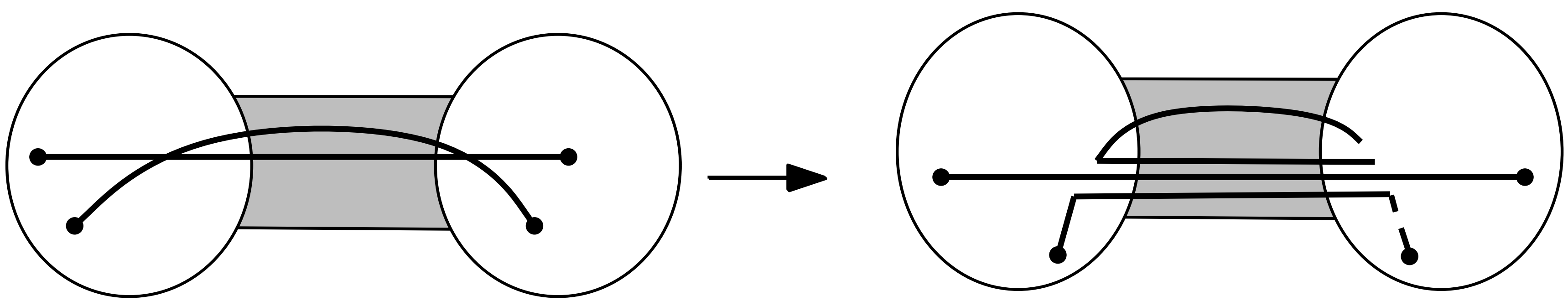

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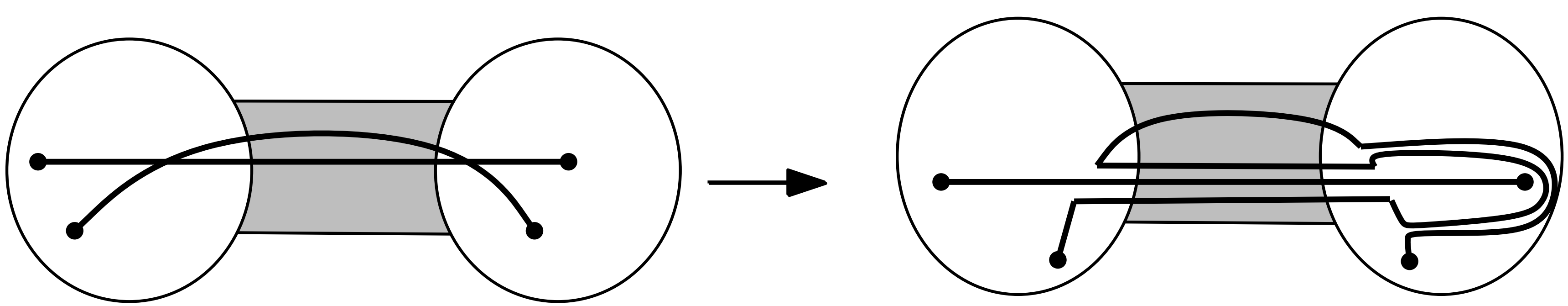

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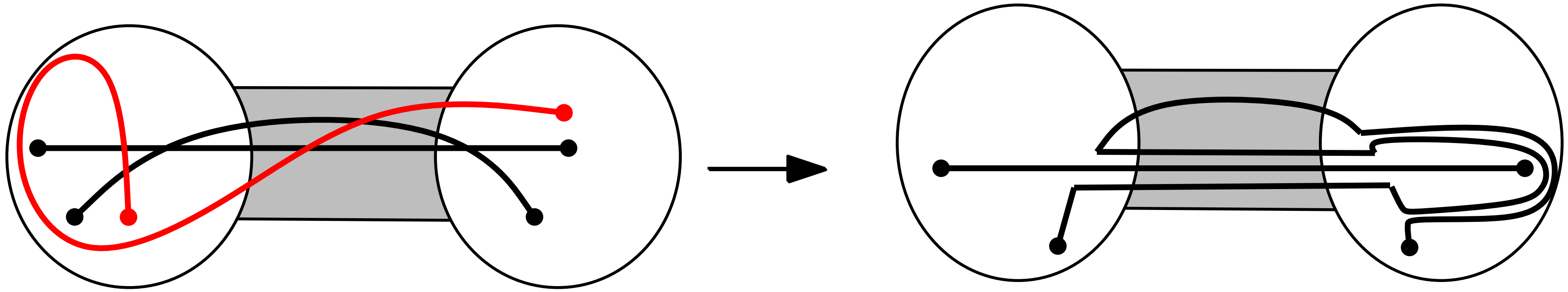

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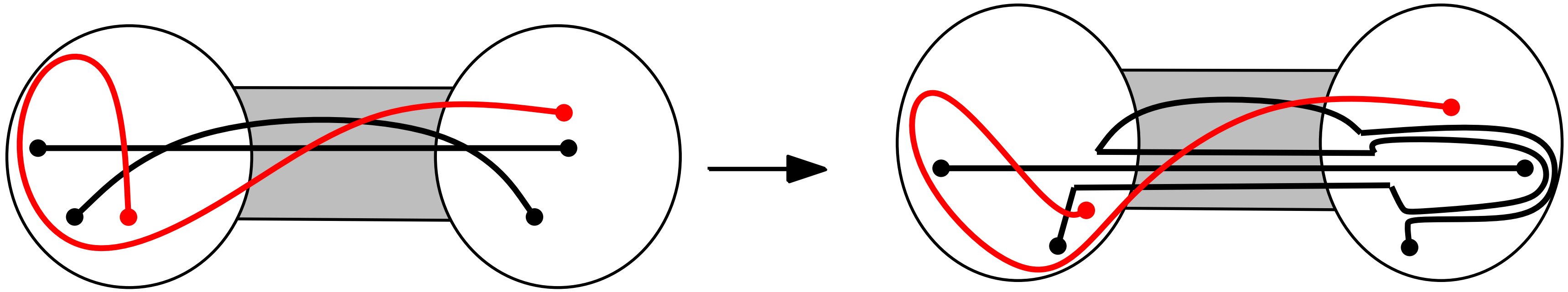

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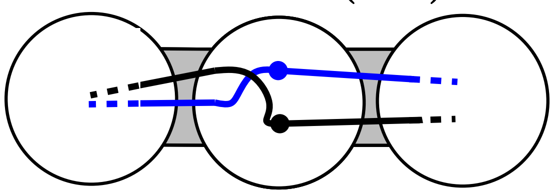

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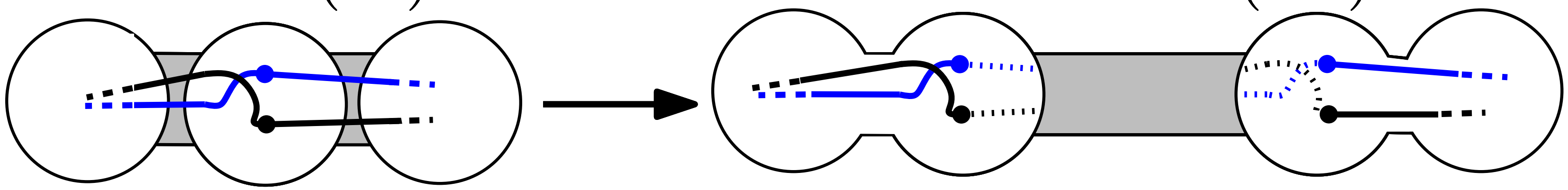
$v \in V(H)$ $\quad$ $v^* \in E(H')$



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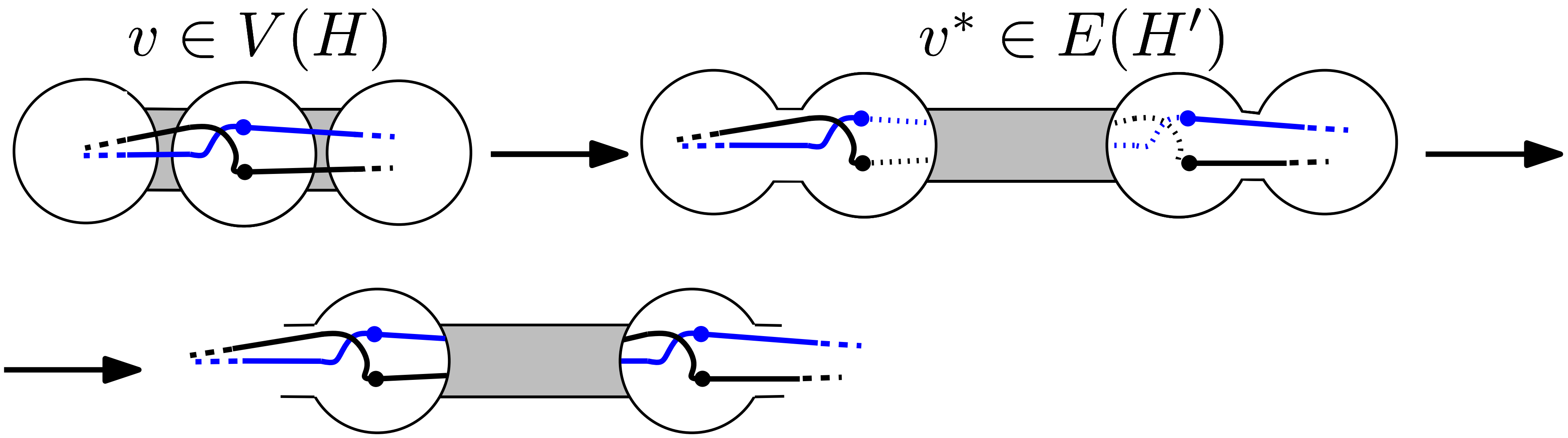

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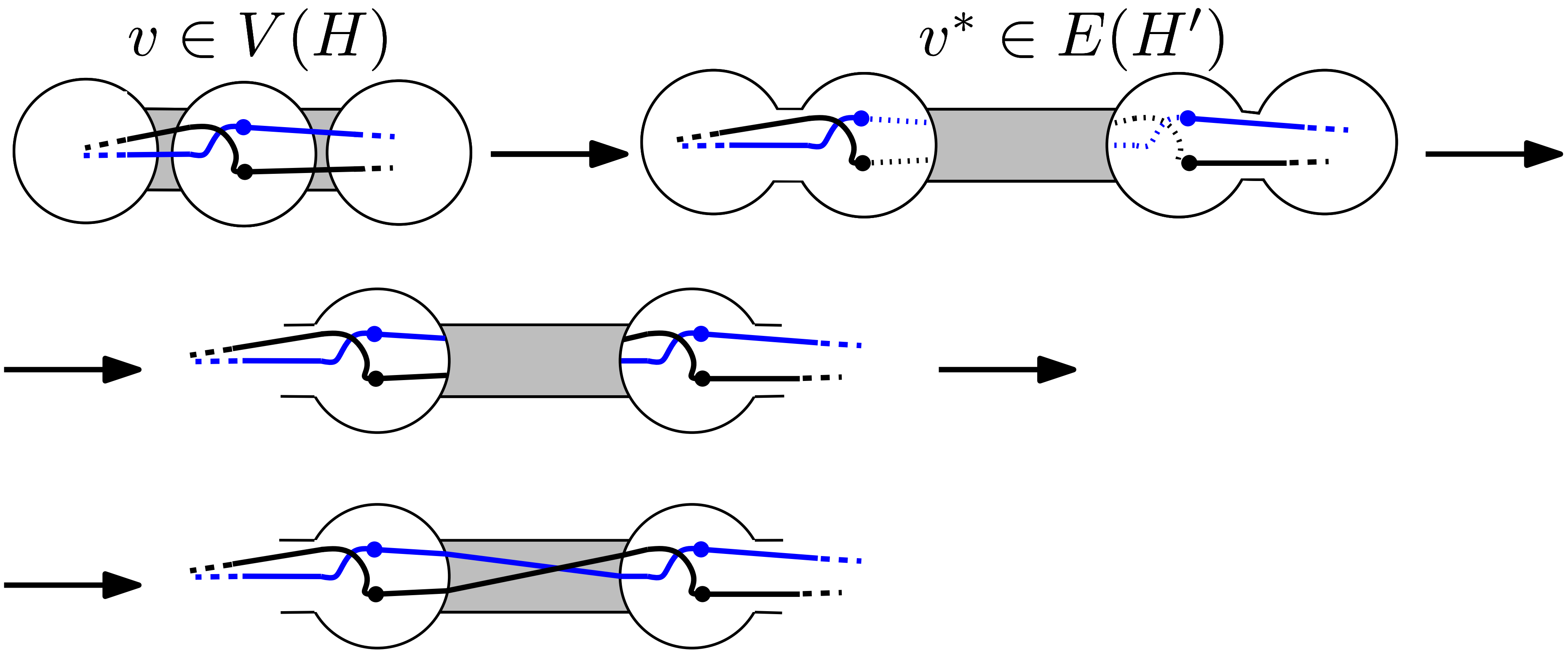

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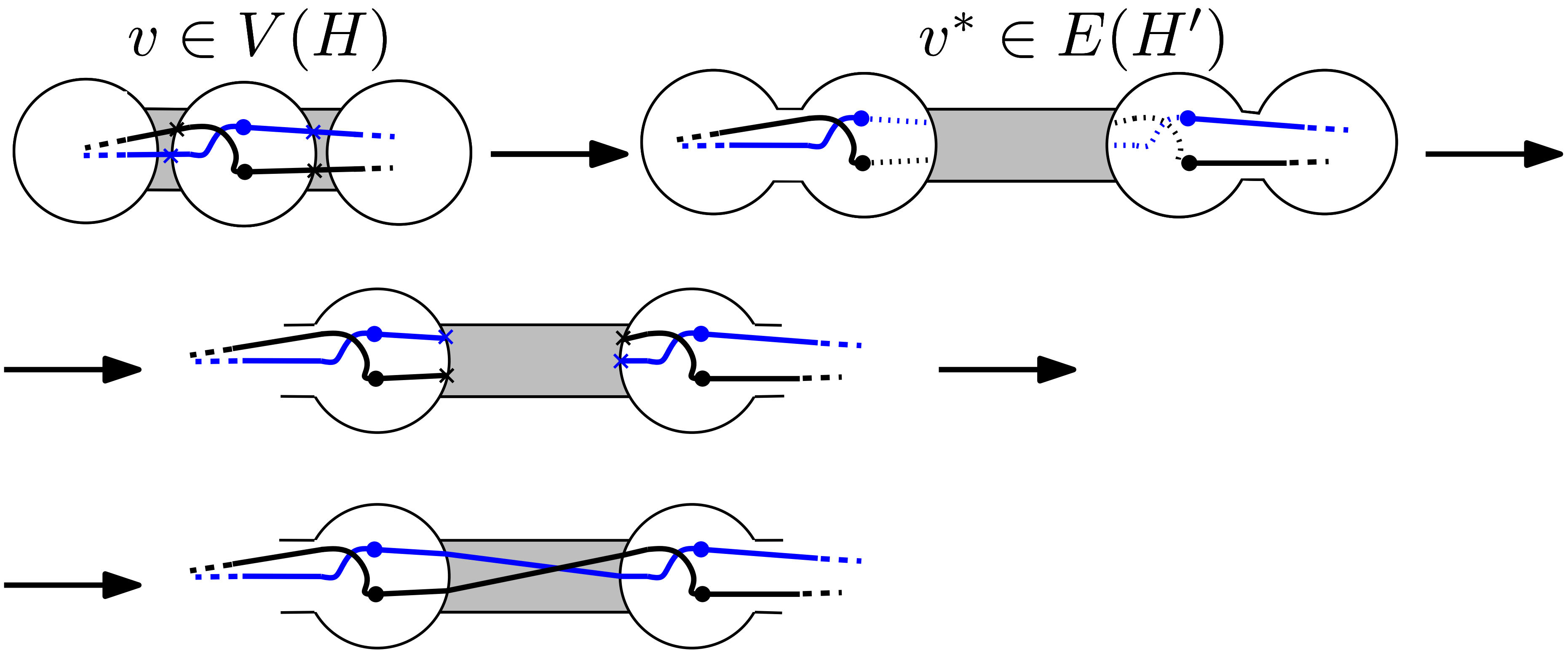

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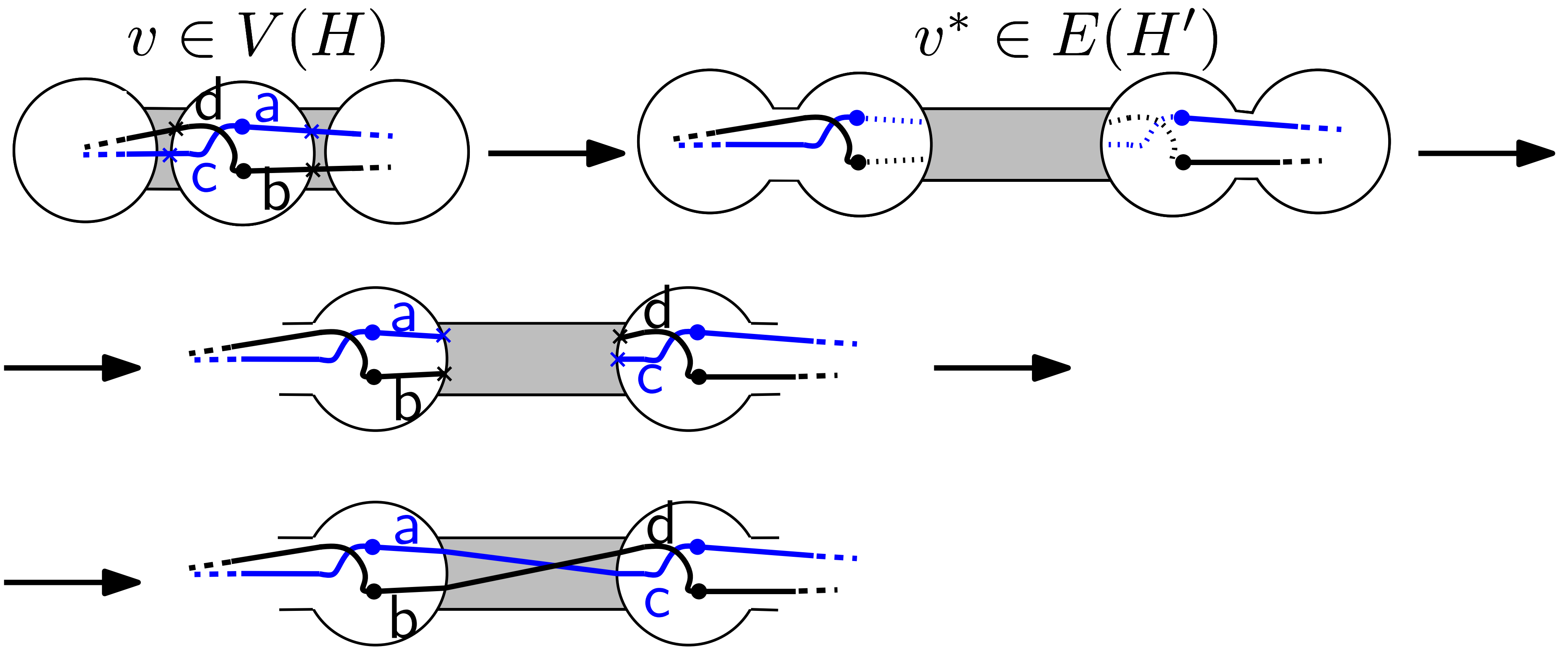

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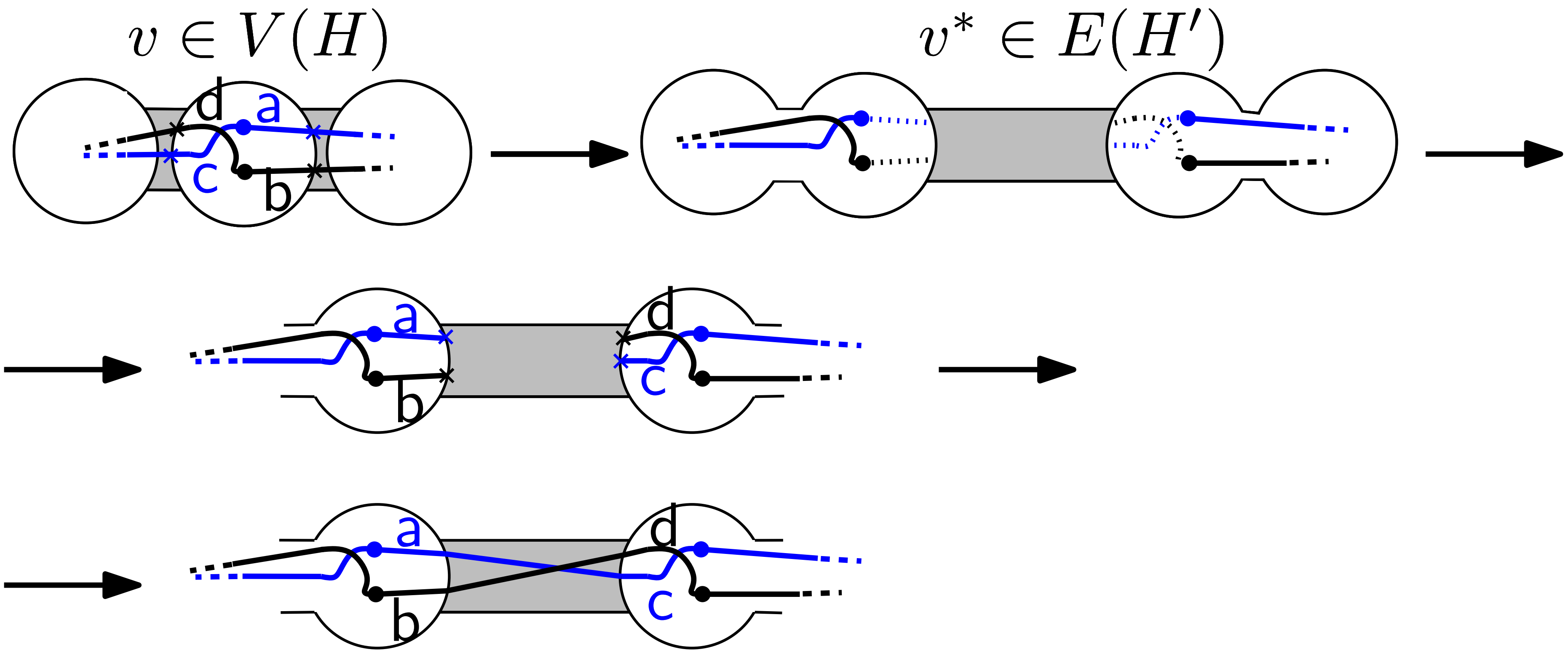

Since a crosses c evenly, and b crosses d evenly, the parity of the number of crossings between a and b is the same as between c and d if and only if the blue and black crosses do not alternate along the circle.

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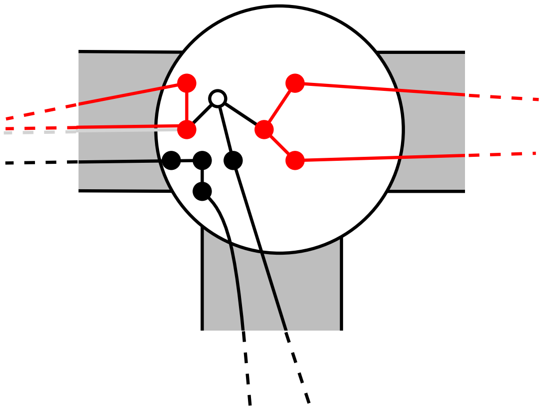

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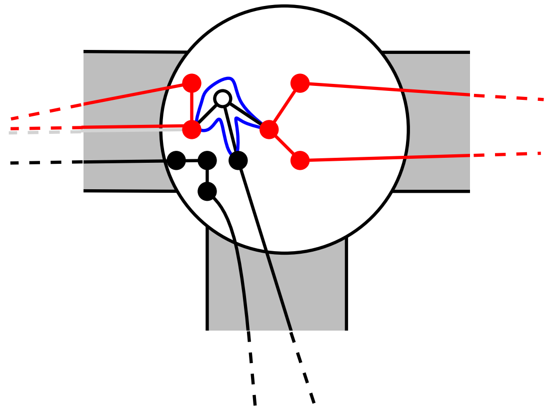

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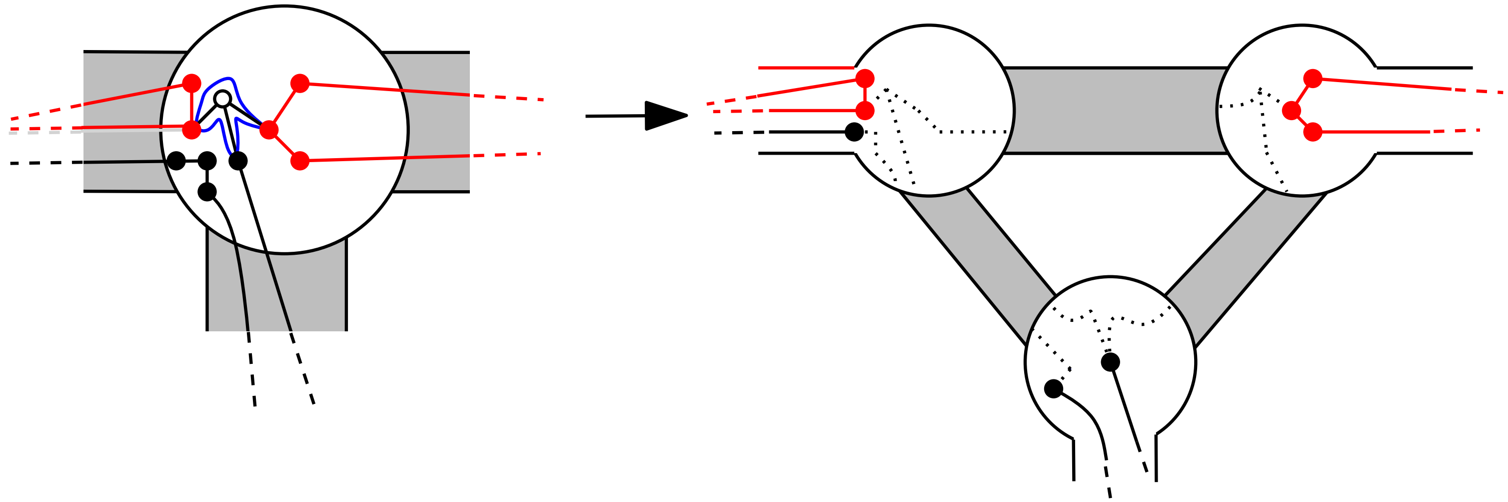

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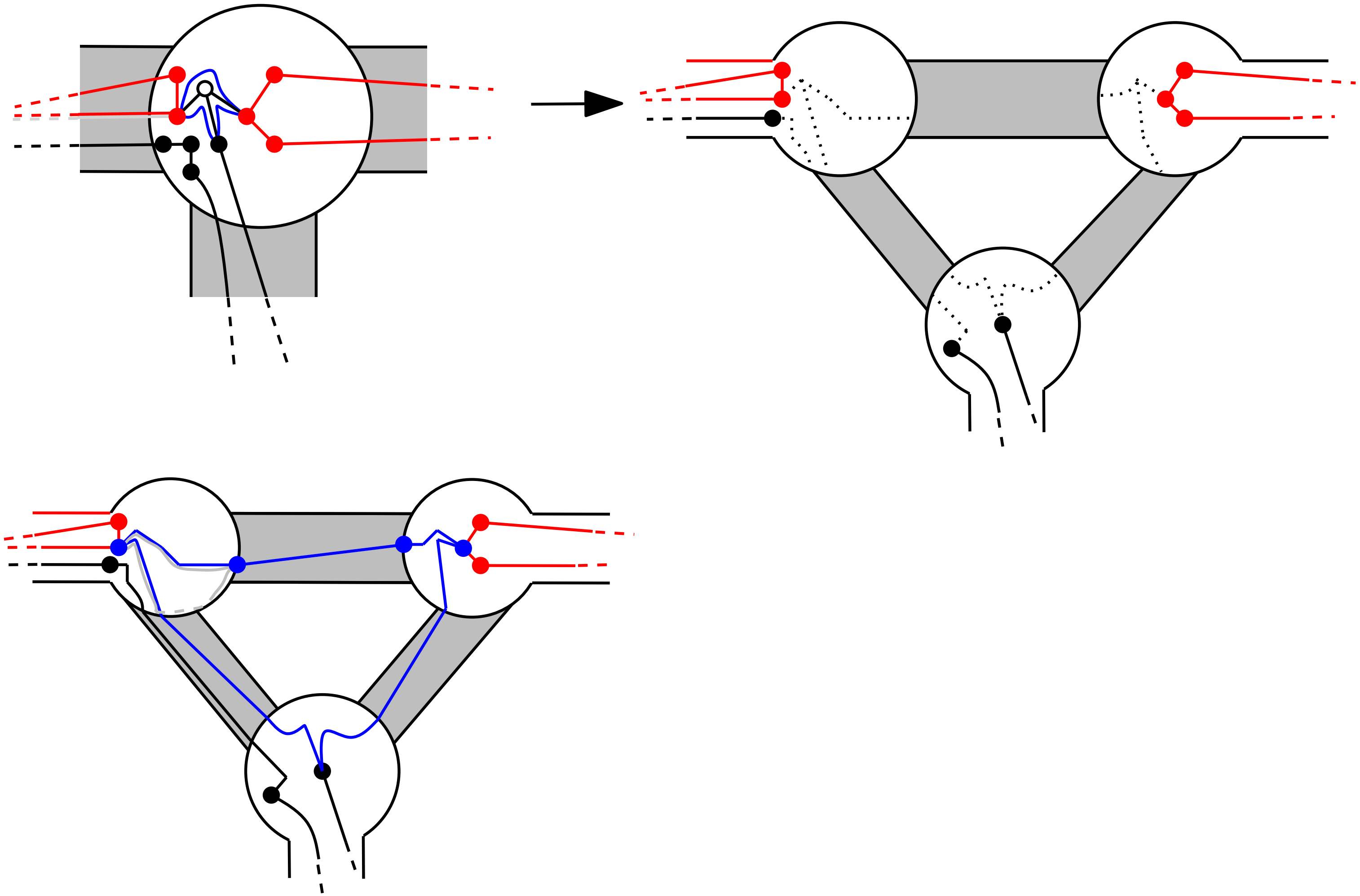

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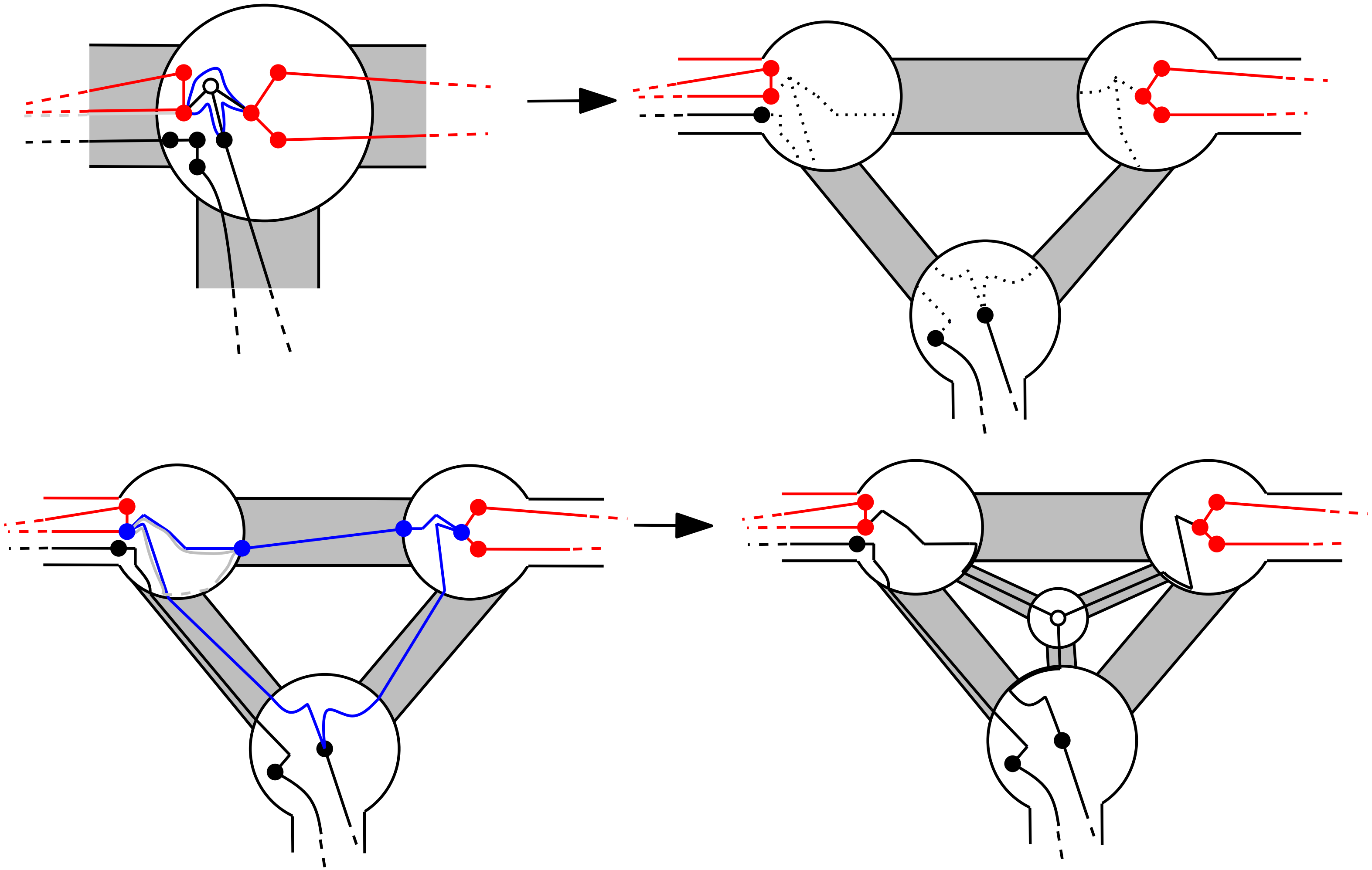

Open Problems

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A variant, where σ is unknown (challenging already for cycles).
A variant, where the handle-body is atomium-like.