# Counterexample to a variant of the Hanani–Tutte theorem on the surface of genus 4

Radoslav Fulek and Jan Kynčl

Charles University, Prague

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

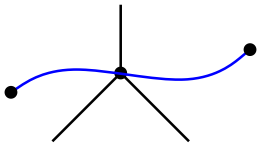
A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:

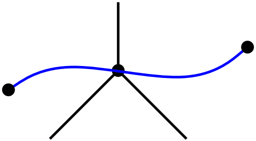


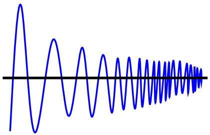
# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:



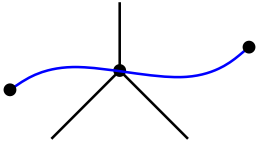


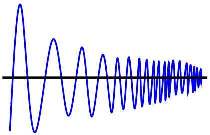
# Hanani–Tutte theorems

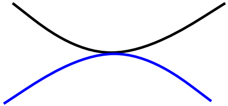
**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:





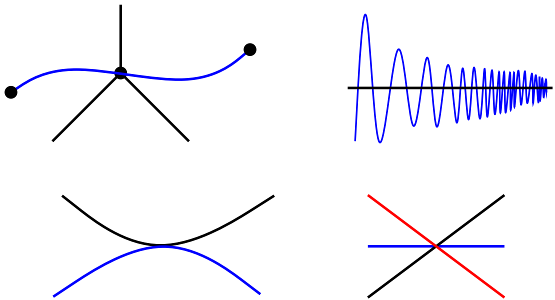


# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:

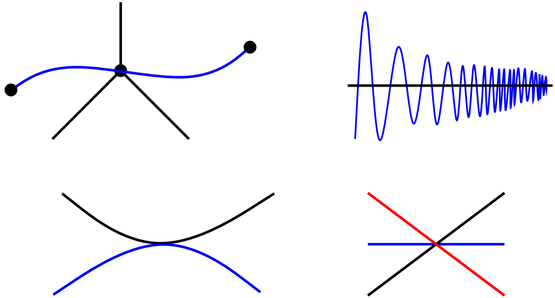


# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:

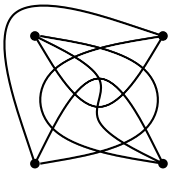


**embedding** = drawing with no crossings

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

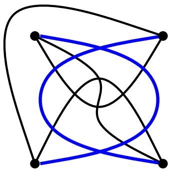
A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

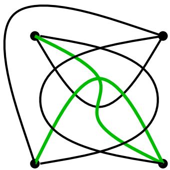
A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

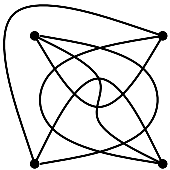
A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

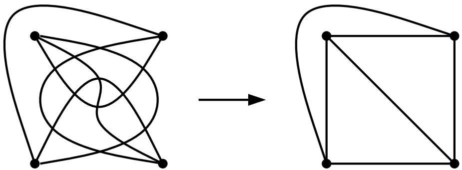
A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

**Weak Hanani–Tutte theorem:** (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph $G$ has an **even** drawing $D$ in the plane (every pair of edges crosses an even number of times), then $G$ is planar.
Moreover, $G$ has a plane embedding with the same rotation system as $D$.

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

**Weak Hanani–Tutte theorem:** (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph $G$ has an **even** drawing $D$ in the plane (every pair of edges crosses an even number of times), then $G$ is planar.
Moreover, $G$ has a plane embedding with the same rotation system as $D$.

- recommended reading:

- M. Schaefer, Hanani-Tutte and related results (2011)

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

**Weak Hanani–Tutte theorem:** (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph $G$ has an **even** drawing $D$ in the plane (every pair of edges crosses an even number of times), then $G$ is planar.
Moreover, $G$ has a plane embedding with the same rotation system as $D$.

- recommended reading:

- M. Schaefer, Hanani-Tutte and related results (2011)
- Fulek et al., Hanani-Tutte, Monotone Drawings, and Level-Planarity (2012)

# Hanani–Tutte theorems

**(Strong) Hanani–Tutte theorem:** (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

**Weak Hanani–Tutte theorem:** (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph $G$ has an **even** drawing $D$ in the plane (every pair of edges crosses an even number of times), then $G$ is planar.
Moreover, $G$ has a plane embedding with the same rotation system as $D$.

- recommended reading:

- M. Schaefer, Hanani-Tutte and related results (2011)
- Fulek et al., Hanani-Tutte, Monotone Drawings, and Level-Planarity (2012)
- M. Schaefer, Toward a theory of planarity: Hanani-Tutte and planarity variants (2013)

# Hanani–Tutte theorems

**Unified Hanani–Tutte theorem:**

(Pelsmajer–Schaefer–Štefankovič, 2006; Fulek–K.–Pálvölgyi, 2016)

Let $G$ be a graph and let $W$ be a subset of vertices of $G$. Let $\mathcal{D}$ be an independently even drawing of $G$ in the plane where, in addition, every pair of edges with a common endpoint in $W$ crosses an even number of times.

Then $G$ has a plane drawing where the rotations of vertices from $W$ are the same as in $\mathcal{D}$.

# Hanani–Tutte theorems

**Unified Hanani–Tutte theorem:**
(Pelsmajer–Schaefer–Štefankovič, 2006; Fulek–K.–Pálvölgyi, 2016)

Let $G$ be a graph and let $W$ be a subset of vertices of $G$. Let $\mathcal{D}$ be an independently even drawing of $G$ in the plane where, in addition, every pair of edges with a common endpoint in $W$ crosses an even number of times.
Then $G$ has a plane drawing where the rotations of vertices from $W$ are the same as in $\mathcal{D}$.

- $W = \emptyset$: strong
- $W = V(G)$: weak

# Hanani–Tutte theorems on surfaces

**Weak Hanani–Tutte theorem on surfaces:**

(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph $G$ has an even drawing $\mathcal{D}$ on a surface $S$, then $G$ has an embedding on $S$ that preserves the embedding scheme of $\mathcal{D}$.

# Hanani–Tutte theorems on surfaces

**Weak Hanani–Tutte theorem on surfaces:**
(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph $G$ has an even drawing $\mathcal{D}$ on a surface $S$, then $G$ has an embedding on $S$ that preserves the embedding scheme of $\mathcal{D}$.

**(Strong) Hanani–Tutte theorem on the projective plane:**
(Pelsmajer–Schaefer–Stasi, 2009;
Colin de Verdière–Kaluža–Paták–Patáková–Tancer, 2016)

If a graph $G$ has an independently even drawing on the projective plane, then $G$ has an embedding on the projective plane.

# Hanani–Tutte theorems on surfaces

**Weak Hanani–Tutte theorem on surfaces:**
(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph $G$ has an even drawing $\mathcal{D}$ on a surface $S$, then $G$ has an embedding on $S$ that preserves the embedding scheme of $\mathcal{D}$.

**(Strong) Hanani–Tutte theorem on the projective plane:**
(Pelsmajer–Schaefer–Stasi, 2009;
Colin de Verdière–Kaluža–Paták–Patáková–Tancer, 2016)

If a graph $G$ has an independently even drawing on the projective plane, then $G$ has an embedding on the projective plane.

**Problem:** Can the strong Hanani–Tutte theorem be extended to other surfaces?

# Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

# Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

**Theorem 1:** There is a graph of genus 5 that has an independently even drawing on the orientable surface of genus 4.

# Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

**Theorem 1:** There is a graph of genus 5 that has an independently even drawing on the orientable surface of genus 4.

(disproves a conjecture of Schaefer and Štefankovič, 2013 )

# Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

**Theorem 1:** There is a graph of genus 5 that has an independently even drawing on the orientable surface of genus 4.

(disproves a conjecture of Schaefer and Štefankovič, 2013 )

- Unified Hanani–Tutte theorem does not generalize to the torus:

# Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

**Theorem 1:** There is a graph of genus 5 that has an independently even drawing on the orientable surface of genus 4.

(disproves a conjecture of Schaefer and Štefankovič, 2013 )

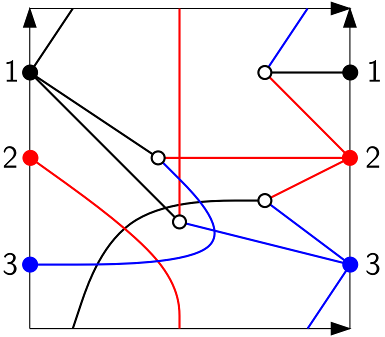
- Unified Hanani–Tutte theorem does not generalize to the torus:

**Theorem 2:** There is a graph $G$ with the following two properties.

1) The graph $G$ has an independently even drawing $\mathcal{D}$ on the torus, with a set $W$ of four vertices such that every pair of edges with a common endpoint in $W$ crosses an even number of times.
2) There is no embedding of $G$ on the torus with the same rotations of the vertices of $W$ as in $\mathcal{D}$.

# Proof of Theorem 2

1)




- $G = K_{3,4}$
- $W =$ the part with 4 vertices (empty circles)
- each vertex of $W$ has rotation $(1, 2, 3)$

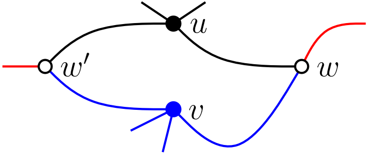
# Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of $G$ on an orientable surface $S$ of minimum genus such that the rotation of every vertex from $W$ is $(1, 2, 3)$.

# Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of $G$ on an orientable surface $S$ of minimum genus such that the rotation of every vertex from $W$ is $(1, 2, 3)$.

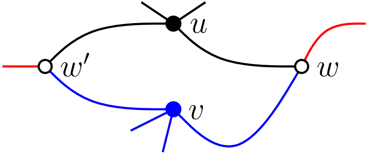
**forbidden:**



# Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of $G$ on an orientable surface $S$ of minimum genus such that the rotation of every vertex from $W$ is $(1, 2, 3)$.

**forbidden:**

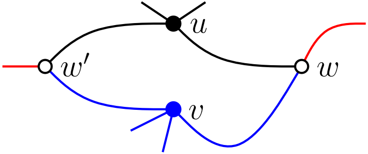



$\Rightarrow$ every facial walk has length at least 6, so $2e \geq 6f$

# Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of $G$ on an orientable surface $S$ of minimum genus such that the rotation of every vertex from $W$ is $(1, 2, 3)$.

**forbidden:**




$\Rightarrow$ every facial walk has length at least 6, so $2e \geq 6f$

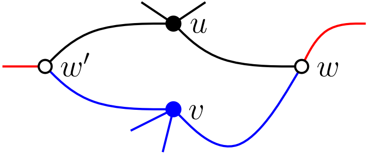
$\Rightarrow$ the Euler characteristic of $S$ satisfies

$$\chi(S) = v - e + f \leq \frac{1}{3}(3v - 2e) = \frac{1}{3}(21 - 24) = -1$$

# Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of $G$ on an orientable surface $S$ of minimum genus such that the rotation of every vertex from $W$ is $(1, 2, 3)$.

**forbidden:**




$\Rightarrow$ every facial walk has length at least 6, so $2e \geq 6f$

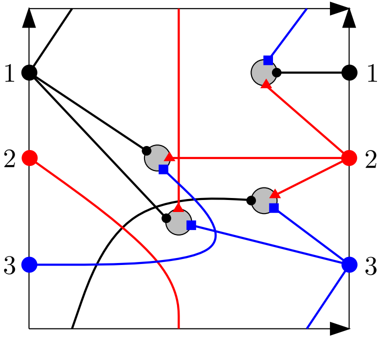
$\Rightarrow$ the Euler characteristic of $S$ satisfies

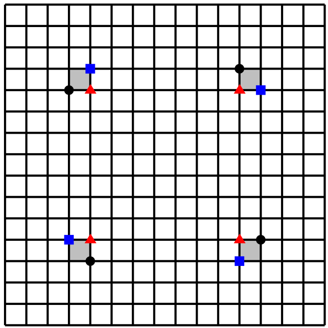
$$\chi(S) = v - e + f \leq \frac{1}{3}(3v - 2e) = \frac{1}{3}(21 - 24) = -1$$

$\Rightarrow$ the genus of $S$ is at least $\lceil (2+1)/2 \rceil = 2$.

# Proof of Theorem 1

independently even drawing of a graph $K$ on the orientable surface of genus 4:



- drill holes around the vertices of $W$ in the drawing from Theorem 2, split the vertices of $W$
- glue the resulting drawing (left) with a sufficiently large grid (right)

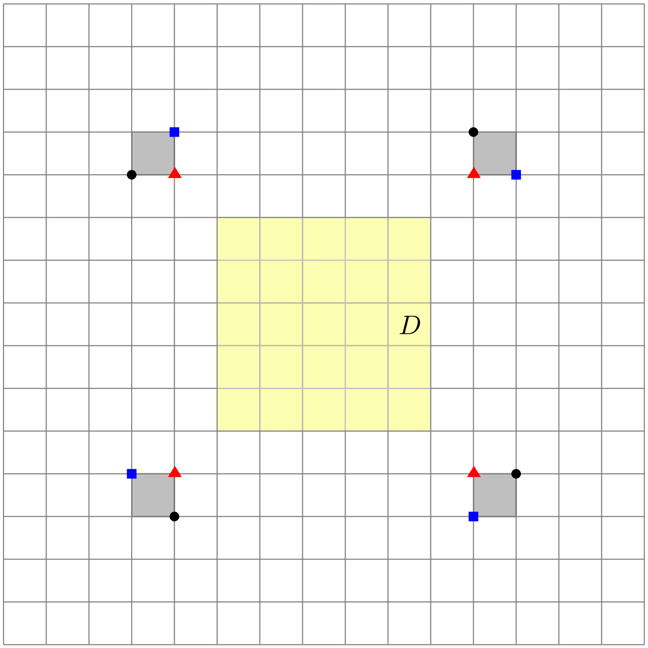
**idea:** the grid will fix the cyclic orders on the boundaries of the holes

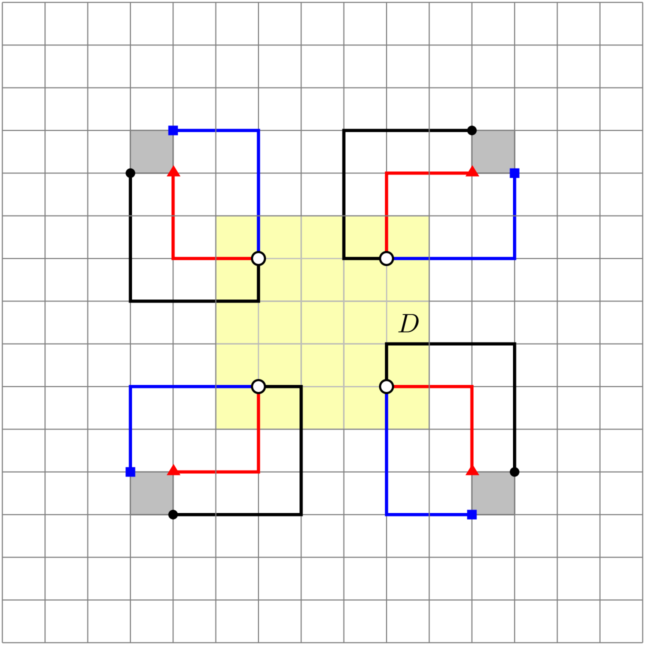
# Proof of Theorem 1

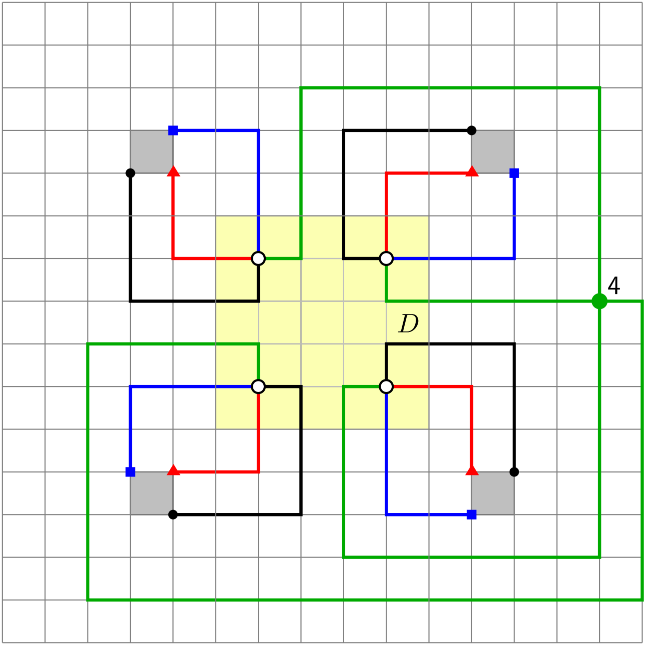
lower bound on the genus of $K$:

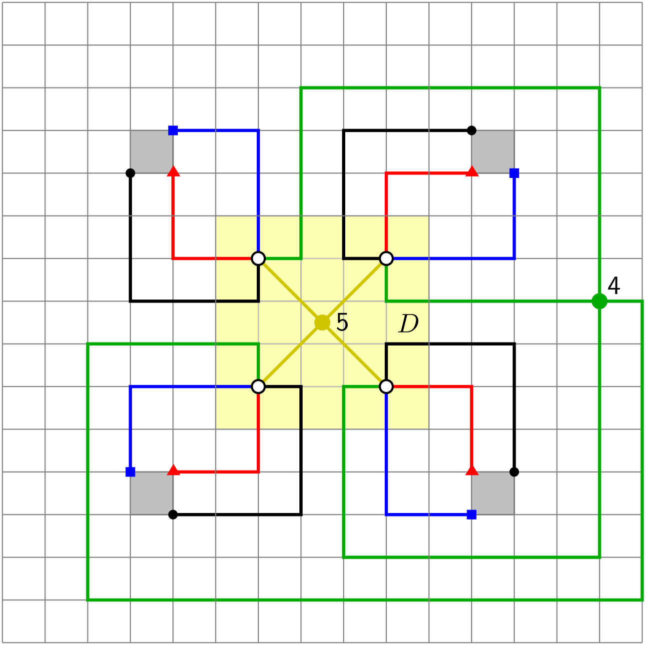
**Lemma:** (Geelen–Richter–Salazar, 2004; Thomassen, 1997; Mohar, 1992; Robertson–Seymour, 1990)

In every embedding of a large grid on a surface of fixed genus, a large portion of the grid is embedded in a planar way.


D


$D$


4
$D$


4
5
D