Symmetry–Preserving Finite Element Methods: Preliminary Results

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Joint work with
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BIRS

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Statement of the Problem

Symmetry

Let G be a Lie group acting on $\mathbb{R}^{p+q} = \{(x, u)\}$:

$$x = \text{ independent variable(s)}$$
$$u = \text{ dependent variables(s)}$$

Example: $G = SE(2, \mathbb{R})$ acts on $\mathbb{R}^2$ via

$$X = x\cos\theta - u\sin\theta + a \qquad a,\, b,\, \theta \in \mathbb{R}$$
$$U = x\sin\theta + u\cos\theta + b$$

Definition: G is a symmetry group of the differential equation

$$\Delta(x, u^{(n)}) = 0$$

if it maps solutions to solutions:

$$\Delta(g \cdot (x, u^{(n)})) = 0 \qquad \text{whenever} \qquad \Delta(x, u^{(n)}) = 0$$

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Examples

- $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{u^3 + x^2 u - x - u}{x^3 + x u^2 - x + u}$ is invariant under the rotation group

$$X = x\cos\theta - u\sin\theta$$
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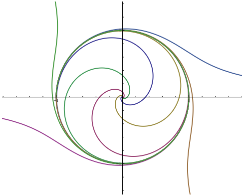

- Burgers' equation

$$u_t + u u_x = \nu u_{xx}, \qquad \nu > 0$$

admits the (non-maximal) symmetry group

$$X = \lambda(x + vt) + a, \qquad T = \lambda^2 t + b, \qquad U = \lambda^{-1}(u + v),$$

$a, b, v \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$

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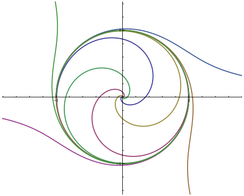

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  - sharp variations
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Overview

Most efforts have focused on finite difference equations

- Began with Dorodnitsyn in 1989
- Using Lie's infinitesimal approach, the focus was originally on the theoretical construction of symmetry-preserving schemes (Budd, Levi, Winternitz, ...)
  - Particularly fruitful for ODE
  - Mainly applied to time evolution PDE
- In 2001 Olver introduced the method of equivariant moving frames to construct finite difference symmetry-preserving schemes
- In recent years Bihlo, Nave et al. have focused on the numerical implementation:
  - evolution-projection techniques
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Other numerical methods

- Finite element
- Finite volume
- Spectral method
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We now consider

symmetries and finite elements

Disclaimer

- Preliminary investigation
- Comments/suggestions are welcome!

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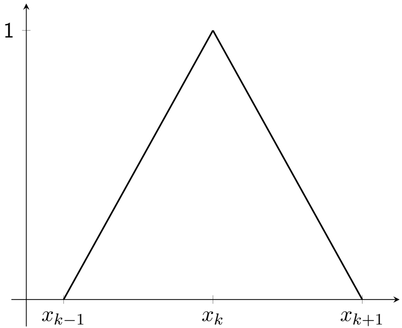
Finite Elements and Symmetries

Approximation

Subdivide $\mathbb{R}$:

We consider the hat functions

$$\phi_k(x) = \begin{cases} 0 & x \in (-\infty, x_{k-1}) \\ \dfrac{x - x_{k-1}}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k] \\ \dfrac{x_{k+1} - x}{x_{k+1} - x_k} & x \in (x_k, x_{k+1}] \\ 0 & x \in (x_{k+1}, \infty) \end{cases}$$



and approximate

$$u(x) \approx u_h(x) = \sum_{i=-\infty}^{\infty} u_i\, \phi_i(x) \qquad u_i = u(x_i)$$

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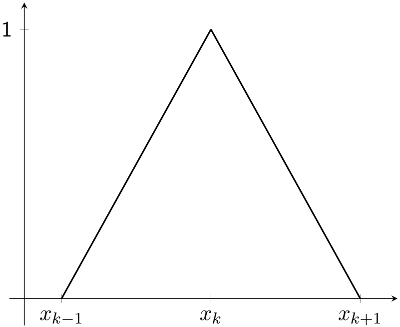

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Preserving the Decomposition

Let G be a Lie group acting on $\mathbb{R}^2 = \{(x, u)\}$

$$X = g \cdot x \qquad U = g \cdot u$$

Acting on $u_h(x) = \sum_{i=-\infty}^{\infty} u_i \, \phi_i(x)$:

$$g \cdot u_h := \sum_{i=-\infty}^{\infty} U_i \, (g \cdot \phi_i(x)) \qquad\qquad U_i = g \cdot u_i$$

We require

$$g \cdot \phi_i(x) = \Phi_i(x) \quad \Rightarrow \quad \text{projectable action} \quad \Rightarrow \quad g \cdot x = X(x)$$

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There's a restriction on

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Theorem (Lie): The largest Lie subgroup of $\mathcal{D}(\mathbb{R})$ is $SL(2, \mathbb{R})$:

$$X = g \cdot x = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad \alpha\delta - \beta\gamma = 1$$

The hat function ϕ_k transform according to

$$\Phi_k = g \cdot \phi_k = \left(\frac{\gamma\, x_k + \delta}{\gamma\, x + \delta} \right) \phi_k$$

and its derivative

$$\Phi_k' = g \cdot \phi_k' = (\gamma x_k + \delta)[(\gamma x + \delta)\phi_k'(x) - \gamma \phi_k(x)]$$

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New Paltz

Simple example

Consider

$$u'(x) = A'(x)\, u + B'(x) e^{A(x)}$$

The solution is

$$u(x) = (B(x) + C) e^{A(x)}$$

The ODE admits the symmetry

$$X = x \qquad U = u + \epsilon\, e^{A(x)} \qquad \epsilon \in \mathbb{R}$$

sending solutions to solutions:

$$C \to C + \epsilon$$

A weak form is given by

$$\int_{-\infty}^{\infty} [u(x)\, e^{-A(x)} + B(x)] \phi'(x)\, \mathrm{d}x = 0$$

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At the discrete level,

$$0 = \int_{-\infty}^{\infty} \Big[\sum_{i=-\infty}^{\infty} u_i \phi_i \, e^{-A(x)} - B(x) \Big] \phi_k' \, \mathrm{d}x$$

the weak form is not invariant under

$$X = x \qquad U_i = u_i + e^{A_i} \qquad A_i = A(x_i)$$

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Moving Frames

Given a non-invariant discrete weak form, we derive an invariant version using moving frames

Definition: Let G be a Lie group acting on a space M parametrized by z. A (right) moving frame is a map

$$\rho \colon M \to G$$

satisfying the G-equivariance

$$\rho(g \cdot z) = \rho(z)\, g^{-1}$$

- A moving frame is constructed by choosing cross-section $\mathcal{K} \subset M$ to the group orbits
- At $z \in M$, $\rho(z) \in G$ is the unique group element sending z onto $\mathcal{K}$:

$$\rho(z) \cdot z \in \mathcal{K}$$

Requires the action to be free and regular

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Invariantization

Definition: Let $\rho(z)$ be a moving frame. The invariantization of a function $F\colon M \to \mathbb{R}$ is the invariant

$$\iota(F)(z) = F(\rho(z) \cdot z)$$

Check:

$$\iota(F)(g \cdot z) = F(\rho(g \cdot z) \cdot g \cdot z)$$
$$= F(\rho(z) \cdot g^{-1} \cdot g \cdot z) = F(\rho(z) \cdot z) = \iota(F)(z)$$

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Example (Continuation)

Returning to the group action

$$X = x_i \qquad U_i = u_i + \epsilon\, e^{A_i} \qquad i \in \mathbb{Z}$$

we choose the cross-section

$$\mathcal{K} = \{u_k = 0\}$$

Solving the normalization equation

$$0 = U_k = u_k + \epsilon\, e^{A_k} \qquad \Rightarrow \qquad \epsilon = -u_k\, e^{-A_k}$$

Invariantizing $0 = \displaystyle\int_{-\infty}^{\infty} \Big[\sum_{i=-\infty}^{\infty} u_i \phi_i e^{-A(x)} - B(x) \Big] \phi_k' \,\mathrm{d}x$:

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$$X = x_i \qquad U_i = u_i + \epsilon\, e^{A_i} \qquad i \in \mathbb{Z}$$

we choose the cross-section

$$\mathcal{K} = \{u_k = 0\}$$

Solving the normalization equation

$$0 = U_k = u_k + \epsilon\, e^{A_k} \qquad \Rightarrow \qquad \epsilon = -u_k\, e^{-A_k}$$

Invariantizing $0 = \int_{-\infty}^{\infty} \Big[\sum_{i=-\infty}^{\infty} u_i \phi_i e^{-A(x)} - B(x) \Big] \phi_k' \,\mathrm{d}x$:

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Introducing

$$a_k = \left[\int e^{-A(x)}\, \mathrm{d}x \right]_{x=x_k} \qquad \langle f \rangle_{I_k} = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(x)\, \mathrm{d}x$$

the corresponding invariant scheme is

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Final Remarks

The previous considerations extend to

- higher order ODE (Done several examples involving 2$^{\text{nd}}$ order ODE)
- evolutive PDE in 1+1 variables using the method of lines (Considered Burgers' equation – Requires an adaptive mesh to preserve the symmetries)

Still needs to be done

- Consider other basis functions
  - higher order Lagrangian polynomials
  - splines
- consider boundary terms
- consider non-projectable group actions
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Comments and Suggestions are Welcome!!!

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