Dimension reduction for the Landau-de Gennes theory of nematic liquid crystals.

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Phase Transitions Models, Banff

Figure: Logs in the Spirit Lake, Mt. St. Helens.

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Q-TENSOR THEORY

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Nontrivial information about LC configuration at x is given by the second moment

$$M(x) = \int_{\mathbb{S}^2} (\mathbf{n} \otimes \mathbf{n}) \rho(\mathbf{n}, x) d\mathbf{n}$$

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Q−tensor: $Q(x) = M(x) - M_{iso}$ so that Q vanishes in the isotropic state.

Nematic Q-tensor

$Q \in M^{3\times 3}_{sym}$ is a traceless tensor $\Rightarrow$ eigenvalues satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$ with a mutually orthonormal eigenframe $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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Uniaxial nematic: repeated nonzero eigenvalues $\lambda_1 = \lambda_2 \Rightarrow$ $Q = S\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}\right)$, where $S := \frac{3\lambda_3}{2}$ is the uniaxial nematic order parameter and $\mathbf{n} \in \mathbf{S}^2$ is the nematic director.

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Biaxial nematic: no repeated eigenvalues $\Rightarrow$ $Q = S_1\left(\mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{3}\mathbf{I}\right) + S_3\left(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}\mathbf{I}\right)$, where $S_1 := 2\lambda_1 + \lambda_3$ and $S_3 = \lambda_1 + 2\lambda_3$ are biaxial order parameters.

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Isotropic: all eigenvalues are equal zero $\Rightarrow$ $Q = 0$.

By construction, $\lambda_i \in \left[-\frac{1}{3}, \frac{2}{3}\right]$, where $i = 1, 2, 3$.

Landau-de Gennes Model

Bulk elastic energy density:

$$f_e(Q, \nabla Q) := \frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2}Q_{ik,j}Q_{ij,k} + \frac{L_3}{2}Q_{ij,j}Q_{ik,k} + \frac{L_4}{2}Q_{lk}Q_{ij,k}Q_{ij,l}$$

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Here $a(T)$ is temperature-dependent, $c > 0$, and $f_{LdG} \geq 0$ by adding an appropriate constant. Function of eigenvalues of Q only. Depending on T, minimum is either isotropic or nematic w/specific s.

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Surface energy density (Either strong or weak anchoring):

$$f_s(Q) := f(Q,\nu)$$

on the boundary of the container and $\nu \in \mathbb{S}^2$ is a normal to the surface of the liquid crystal.

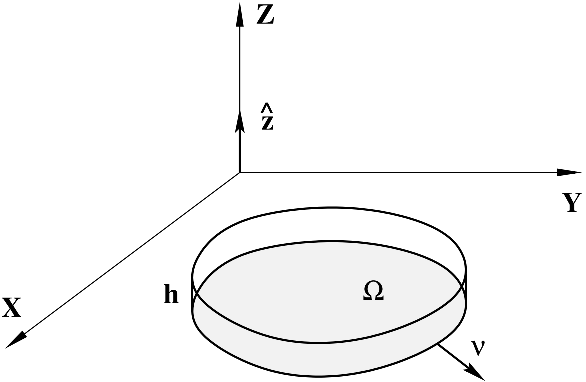

Figure: Geometry of the problem.

Here $\Omega \subset \mathbf{R}^2$ and $h > 0$ is small.

Nematic energy functional:

$$E[Q] := \int_{\Omega\times[0,h]} \{f_e(Q,\nabla Q) + f_{LdG}(Q)\}\ dV + \int_{\Omega\times\{0,h\}} f_s(Q,\hat{z})\, dA$$

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Uniaxial data on the lateral boundary of the film:

$$Q|_{\partial\Omega\times[0,h]} = g \in H^{1/2}(\partial\Omega;\mathcal{A}).$$

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Admissible class:

$$\mathcal{C}_h^g := \left\{ Q \in H^1\left(\Omega\times[0,h];\mathcal{A}\right) : Q|_{\partial\Omega\times[0,h]} = g \right\},$$

where $\mathcal{A}$ is the set of three-by-three symmetric traceless matrices.

Osipov-Hess surface energy

"Bare" surface energy (Osipov-Hess):

$$f_s(Q, \hat{z}) := c_1(Q\hat{z} \cdot \hat{z}) + c_2 Q \cdot Q + c_3(Q\hat{z} \cdot \hat{z})^2 + c_4|Q\hat{z}|^2$$

where $c_i,\ i = 1, \ldots, 4$ are constants.

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where $c_i, \ i = 1, \dots, 4$ are constants.

Observe that:

$$Q \cdot Q = 2|Q\hat{z}|^2 - (Q\hat{z} \cdot \hat{z})^2 + Q_2 \cdot Q_2,$$

where

$$Q_2 := (\mathbf{I} - \hat{z} \otimes \hat{z})\, Q\, (\mathbf{I} - \hat{z} \otimes \hat{z})\,.$$

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Q is traceless $\Rightarrow$

$$\mathrm{tr}\,Q_2 + Q\hat{z}\cdot\hat{z} = 0.$$

In terms of x and Q_2:

$$f_s(Q,\hat{z}) = c_1(Q\hat{z}\cdot\hat{z}) + c_2 Q_2\cdot Q_2 + (c_3 - c_2)\,(Q\hat{z}\cdot\hat{z})^2 + (2c_2 + c_4)\,|Q\hat{z}|^2$$

This expression has a family of surface-energy-minimizing tensors that is

1. parameterized by at least one free eigenvalue
2. normal to the surface of the liquid crystal is an eigenvector

as long as $c_2 = 0$, $\alpha = c_3 + c_4 > 0$, and $\gamma = c_4 > 0$.

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as long as $c_2 = 0$, $\alpha = c_3 + c_4 > 0$, and $\gamma = c_4 > 0$. Then the surface energy has the form

$$f_s(Q,\hat{z}) = \alpha\left[(Q\hat{z}\cdot\hat{z}) - \beta\right]^2 + \gamma|(\mathbf{I} - \hat{z}\otimes\hat{z})\,Q\hat{z}|^2$$

where $\beta = -\frac{c_1}{2(c_3+c_4)}$.

Nondimensionalization

Let $L_4 = 0$ and

$$\tilde{x} = \frac{x}{D},\ \tilde{y} = \frac{y}{D},\ \tilde{z} = \frac{z}{h},\ F_\epsilon = \frac{2}{L_1 h} E,$$

where $D := \mathrm{diam}(\Omega)$. Set

$$\xi = \frac{L_1}{2D^2},\ \epsilon = \frac{h}{D},\ \delta = \sqrt{\frac{2\xi}{c}}$$

$$K_2 = \frac{L_2}{L_1},\ K_3 = \frac{L_3}{L_1}$$

$$A = \frac{a}{c},\ B = \frac{b}{c}$$

$$\tilde{\alpha} = \frac{\alpha}{\xi},\ \tilde{\gamma} = \frac{\gamma}{\xi}$$

$$F_\epsilon[Q] = \int_{\Omega\times[0,1]} \left(f_e(\nabla Q) + \frac{1}{\delta^2} f_{LdG}(Q) \right) dV + \frac{1}{\epsilon} \int_{\Omega\times\{0,1\}} f_s(Q,\hat{z})\, dA,$$

where

$$f_e(\nabla Q) := \left[|\nabla_{xy} Q|^2 + K_2 Q_{ik,j} Q_{ij,k} + K_3 Q_{ij,j} Q_{ik,k} \right] + \frac{2}{\epsilon} \left[K_2 Q_{i3,j} Q_{ij,3} + K_3 Q_{ij,j} Q_{i3,3} \right] + \frac{1}{\epsilon^2} \left[|Q_z|^2 + (K_2 + K_3) Q_{i3,3}^2 \right],$$

$$f_{LdG}(Q) = 2A \,\mathrm{tr}\left(Q^2\right) + \frac{4}{3} B \,\mathrm{tr}\left(Q^3\right) + \left(\mathrm{tr}\left(Q^2\right)\right)^2,$$

$$f_s(Q,\hat{z}) = \alpha\left[(Q\hat{z}\cdot\hat{z}) - \beta \right]^2 + \gamma \left| (\mathbf{I} - \hat{z}\otimes\hat{z})\, Q\hat{z} \right|^2.$$

Suppose for simplicity that $K_2 = K_3 = 0$ then for every $Q \in \mathcal{C}_1^g$

$$F_\epsilon[Q] = \int_{\Omega\times[0,1]} \left\{ |Q_x|^2 + |Q_y|^2 + \frac{1}{\epsilon^2}|Q_z|^2 + \frac{1}{\delta^2}\left(2A\,\mathrm{tr}\left(Q^2\right) + \frac{4}{3}B\,\mathrm{tr}\left(Q^3\right) + \left(\mathrm{tr}\left(Q^2\right)\right)^2\right)\right\} dV + \frac{1}{\epsilon}\int_{\Omega\times\{0,1\}} \left(\alpha\left[\left(Q\hat{z}\cdot\hat{z}\right) - \beta\right]^2 + \gamma\left|\left(\mathbf{I} - \hat{z}\otimes\hat{z}\right)Q\hat{z}\right|^2\right) dA,$$

and set

$$f_s(Q,\hat{z}) =: f_s^{(0)}(Q,\hat{z}) + \epsilon f_s^{(1)}(Q,\hat{z})$$

—this allows for different asymptotic regimes for α and γ.

Limiting problem

Let

$$F_0[Q] := \begin{cases} 2\int_\Omega \left\{ |\nabla_{xy} Q|^2 + \frac{1}{\delta^2} f_{LdG}(Q) + f_s^{(1)}(Q, \hat{z}) \right\} dA & \text{if } Q \in H^1_g, \\ +\infty & \text{otherwise.} \end{cases}$$

Here

$$H^1_g := \left\{ Q \in H^1(\Omega; \mathcal{D}) : Q|_{\partial\Omega} = g \right\}$$

and

$$\mathcal{D} := \left\{ Q \in \mathcal{A} : Q \in \operatorname{argmin}_{Q \in \mathcal{A}} f_s^{(0)}(Q) \right\},$$

for some boundary data $g : \partial\Omega \to \mathcal{D}$.

Theorem (G, Montero, Sternberg (2015))

Fix $g : \partial\Omega \to \mathcal{D}$ such that H^1_g is nonempty. Then $\Gamma\text{-}\lim_\epsilon F_\epsilon = F_0$ weakly in $\mathcal{C}^g_1$. Furthermore, if a sequence $\{Q_\epsilon\}_{\epsilon>0} \subset \mathcal{C}^g_1$ satisfies a uniform energy bound $F_\epsilon[Q_\epsilon] < C_0$ then there is a subsequence weakly convergent in $\mathcal{C}^g_1$ to a map in H^1_g.

Proof.

Idea: can use a trivial recovery sequence. Indeed, if $Q_\epsilon \equiv Q \in \mathcal{C}^g_1 \setminus H^1_g$ then $\lim_{\epsilon\to 0} F_\epsilon[Q_\epsilon] = +\infty = F_0[Q]$ and when $Q_\epsilon \equiv Q \in H^1_g$ then $F_\epsilon[Q_\epsilon] = F_0[Q_\epsilon] = F_0[Q]$ for all ϵ. □

Asymtotic regime

Let

$$f_s^{(0)} = \alpha\left[(Q\hat{z}\cdot\hat{z})-\beta\right]^2+\gamma|(\mathbf{I}-\hat{z}\otimes\hat{z})\,Q\hat{z}|^2 \text{ and } f_s^{(1)}\equiv 0 \quad\Rightarrow$$

(i). Admissible tensors satisfy $Q\hat{z}=\beta\hat{z}$

and

(ii). Thera are two types of $\mathcal{D}$-valued **uniaxial** Dirichlet data on $\partial\Omega$:

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(ii). Thera are two types of $\mathcal{D}$-valued **uniaxial** Dirichlet data on $\partial\Omega$:

- $Q=-3\beta\left(\mathbf{n}\otimes\mathbf{n}-\frac{1}{3}\mathbf{I}\right)$, where $\mathbf{n}\perp\hat{z}$ is **any** $\mathbb{S}^1$-valued field on $\partial\Omega$.

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- $Q=\frac{3\beta}{2}\left(\hat{z}\otimes\hat{z}-\frac{1}{3}\mathbf{I}\right)$.

Can represent $Q \in H^1_g$ as

$$Q = \begin{pmatrix} p_1 - \frac{\beta}{2} & p_2 & 0 \\ p_2 & -p_1 - \frac{\beta}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

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Then

$$F_0[Q] = \tilde{F}_0[\mathbf{p}] := \int_\Omega \left\{ 2|\nabla \mathbf{p}|^2 + \frac{1}{\delta^2} W(|\mathbf{p}|) \right\} dV,$$

where $\mathbf{p} = (p_1, p_2)$ and

$$W(t) = 4t^4 + \tilde{C}t^2 + \tilde{D},$$

with $\tilde{C} = 6\beta^2 - 4B\beta + 4A$ and $\tilde{D} \in \mathbb{R}$.

If

$$Q|_{\partial\Omega\times[0,1]} = \frac{3}{2}\beta\left(\hat{z}\otimes\hat{z} - \frac{1}{3}\mathbf{I}\right),$$

admissible functions satisfy the boundary condition

$$\mathbf{p}|_{\partial\Omega} = \mathbf{0}.$$

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The minimizer of

$$\tilde{F}_0[\mathbf{p}] = \int_\Omega \left\{2|\nabla\mathbf{p}|^2 + \frac{1}{\delta^2}W(|\mathbf{p}|)\right\} dV$$

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then has a constant angular component $\Rightarrow$ scalar minimization problem for $p := |\mathbf{p}|$ and

1. If $\tilde{C} \geq 0$ then the minimizer $p \equiv 0$.
2. If $\tilde{C} < 0$ then the minimizer p solves the problem

$$-\Delta p + \frac{1}{\delta^2}W'(p) = 0 \text{ in } \Omega, \quad p = 0 \text{ on } \partial\Omega.$$

Now suppose

$$Q|_{\partial\Omega\times[0,1]} = -3\beta\left(\mathbf{n}\otimes\mathbf{n}-\frac{1}{3}\mathbf{I}\right),$$

where $\mathbf{n}:\partial\Omega\to\mathbb{S}^1$.

We have

$$\mathbf{p} = -3\beta\left(n_1^2-\frac{1}{2}, n_1 n_2\right),$$

on $\partial\Omega$ where $|\mathbf{p}| = \frac{3\beta}{2}$. If $\mathbf{p}$ is smooth and nonvanishing, it has a well-defined winding number $d\in\mathbb{Z}$. We set the degree of g to be equal to $d/2$. Then $\mathbf{p}$ must vanish somewhere within a vortex core structure of a characteristic size of δ in Ω.

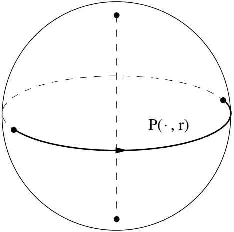
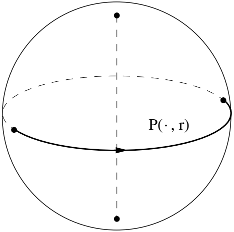


FIGURE: Geometry of the target space.

Topologically nontrivial boundary data will cause the director to "escape" from the xy-plane to the z-direction. The requirement that Q_0 takes values in $\mathcal{D}$ forces the escape to happen through biaxial states that are heavily penalized by the Landau-de Gennes energy.



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Degree of biaxiality:

$$\xi(Q)^2 := 1 - 6\frac{\left(\mathrm{tr}\,Q^3\right)^2}{\left(\mathrm{tr}\,Q^2\right)^3} = 1 - 27\,\frac{\beta^2\left(4p^2-\beta^2\right)^2}{(4p^2+3\beta^2)^3}$$

where $\xi(Q) = 0$ implies that Q is uniaxial.

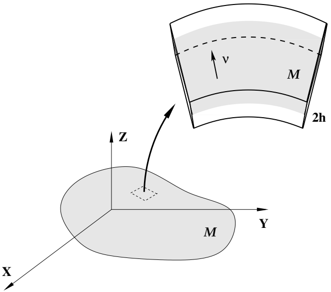

FIGURE: Geometry of the problem.

$$\Omega_h := \{X \in \mathbb{R}^3 : \; X = x + ht\nu(x) \text{ for } x \in \mathcal{M}, \; t \in (-1,1)\},$$

$$\mathcal{M}_{\pm h} := \{x \pm h\nu(x) : x \in \mathcal{M}\}$$

Variational Problem

Minimize

$$E[Q] := \int_{\Omega_h} \{f_e(\nabla Q) + f_{LdG}(Q)\}\, dV + \int_{\mathcal{M}_{-h}\cup\mathcal{M}_h} f_s(Q,\nu)\, dA\,.$$

in the class

$$\mathcal{C}_h^g := \left\{ Q \in H^1(\Omega_h; \mathcal{A}) : Q|_{\Omega_h^{\mathrm{lat}}} = g \right\},$$

of admissible functions. Here $\mathcal{A}$ is the set of three-by-three symmetric traceless matrices.

Coordinate System

Let $\tau : U \times [-1,1] \to \mathbb{R}^3$ given by

$$X(u,t) = x(u) + ht\nu(x(u)),$$

such that

$$X_t = h\nu, \qquad D_u X = D_u x\left(\mathbf{I} + htA\right),$$

where

$$A = -\mathbb{I}^{-1}\mathbb{II},$$

is the matrix of the shape operator $\nabla_{\mathcal{M}}\nu$ and $\mathbb{I}$ and $\mathbb{II}$ are the first and second fundamental forms for $\mathcal{M}$. Here

$$(\nabla_{\mathcal{M}}\nu)\,\nu = 0, \quad (\nabla_{\mathcal{M}}\nu)\,\mathbf{d_1} = \kappa_1\mathbf{d}_1, \quad (\nabla_{\mathcal{M}}\nu)\,\mathbf{d_2} = \kappa_2\mathbf{d}_2.$$

κ_i and $\mathbf{d}_i$, $i = 1,2$ are the principal curvatures and directions at $x(u)$, respectively.

Given $X \in \Omega_h$, let $x = \text{Proj}_{\mathcal{M}} X$ and $P_X = \mathbf{I} - \nu(x) \otimes \nu(x)$. Then

$$\nabla \mathbf{a} = \nabla \mathbf{a} \left(\mathbf{I} - P_X \right) + \nabla \mathbf{a} P_X .$$

so that

$$|\nabla \mathbf{a}|^2 = \nabla \mathbf{a} \cdot \nabla \mathbf{a} = |\nabla \mathbf{a} \left(\mathbf{I} - P_X \right)|^2 + |\nabla \mathbf{a} P_X|^2 .$$

Further

$$\nabla \mathbf{a} \left(I - P_X \right) = \frac{1}{h} \mathbf{a}_t \otimes \nu ,$$
$$\nabla \mathbf{a} \, P_X = D_u \mathbf{a} \left(I + htA \right)^{-1} (D_u x)^{-1} ,$$

Note: Setting $h = 0$ implies

$$\nabla \mathbf{a} \, P_X = D_u \mathbf{a} (D_u x)^{-1} = \nabla_{\mathcal{M}} \mathbf{a} .$$

Nondimensional Energy Functional

$$F_\epsilon[Q] = \int_{\Omega_1} \left(f_e(\nabla Q) + \frac{1}{\delta^2} f_{LdG}(Q) \right) dV + \frac{1}{\epsilon} \int_{\mathcal{M}_{-1} \cup \mathcal{M}_1} f_s(Q, \nu)\, dA,$$

Expanding in ε, we have

$$f_e(\nabla Q) = \frac{1}{2} \sum_{i=1}^{3} \left\{ \left| \nabla_{\mathcal{M}} Q_i + \frac{1}{\varepsilon} Q_{i,t} \otimes \nu \right|^2 + M_2 \left(\operatorname{div}_{\mathcal{M}} Q_i + \frac{1}{\varepsilon} Q_{i,t} \cdot \nu \right)^2 + M_3 \left(\nabla_{\mathcal{M}} Q_i + \frac{1}{\varepsilon} Q_{i,t} \otimes \nu \right) \cdot \left(\nabla_{\mathcal{M}} Q_i{}^T + \frac{1}{\varepsilon} \nu \otimes Q_{i,t} \right) \right\} + O(\varepsilon),$$

Limiting problem

Let

$$F_0[Q] := \begin{cases} \int_{\mathcal{M}} \left\{ f_e^0(\nabla_{\mathcal{M}} Q) + \frac{1}{\delta^2} f_{LdG}(Q) + 2 f_s^{(1)}(Q,\nu) \right\} dS & \text{if } Q \in H_g^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Here

$$f_e^0(\nabla_{\mathcal{M}} Q, \nu) := \min_{B \in \mathcal{A}} f_e(B \otimes \nu + \nabla_{\mathcal{M}} Q)$$

and the space

$$H_g^1 := \left\{ Q \in H^1(\mathcal{M};\mathcal{A}) : Q|_{\partial\mathcal{M}} = g, f_s^{(0)}(Q(x),\nu(x)) = 0 \text{ for a.e. } x \in \bar{\mathcal{M}} \right\}$$

for some uniaxial boundary data $g \in H^{1/2}(\partial\mathcal{M};\mathcal{A})$.

Note that, generally,

$$f_e^0(\nabla_{\mathcal{M}} Q) \neq |\nabla_{\mathcal{M}} Q|^2 + M_2 |\mathrm{div}_{\mathcal{M}} Q|^2 + M_3 \sum_{i=1}^{3} \nabla_{\mathcal{M}} Q_i \cdot (\nabla_{\mathcal{M}} Q_i)^T .$$

- True when $M_2 = M_3 = 0$.
- **Lemma:** Suppose that $M_3 = 0$ and $M_2 > -\frac{3}{5}$. Then

$$f_e^0(\nabla_{\mathcal{M}} Q, \nu) = \frac{1}{2}\left\{ |\nabla_{\mathcal{M}} Q|^2 + \frac{2M_2(M_2+1)}{M_2+2} |\mathrm{div}_{\mathcal{M}} Q|^2 - \frac{M_2^2}{(M_2+2)(2M_2+3)} (\nu \cdot \mathrm{div}_{\mathcal{M}} Q)^2 \right\}.$$

Theorem (G, Montero, Sternberg (2016))

Fix $g \in H^{1/2}(\partial\mathcal{M};\mathcal{A})$ *such that the set* H^1_g *is nonempty. Assume that* $-1 < M_3 < 2$, *and* $-\frac{3}{5} - \frac{1}{10}M_3 < M_2$*. Then* $\Gamma\text{-}\lim_\varepsilon F_\varepsilon = F_0$ *weakly in* $\mathcal{C}_1^g$*. Furthermore, if a sequence* $\{Q_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}_1^g$ *satisfies a uniform energy bound* $F_\varepsilon[Q_\varepsilon] < C_0$ *then there is a subsequence weakly convergent in* $\mathcal{C}_1^g$ *to a map in* H^1_g*.*

Example

$\mathcal{M}$ is a surface of revolution:

$$\Psi(s,\theta) = \begin{pmatrix} a(s)\cos\theta \\ a(s)\sin\theta \\ b(s) \end{pmatrix},$$

where $\theta \in [0, 2\pi]$ and $\mathbf{r}(s) := (a(s), b(s))$, $s \in [0, L]$ is a smooth curve in $\mathbb{R}^2$.

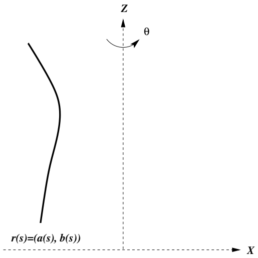

Figure: Radial Geometry.

Set $\mathbf{r}'(s) = (\cos\phi(s), \sin\phi(s))$ and introduce the eigenframe

$$\mathbf{T}(s,\theta) = \begin{pmatrix} \cos\phi(s)\cos\theta \\ \cos\phi(s)\sin\theta \\ \sin\phi(s) \end{pmatrix}, \qquad \mathbf{N}(s,\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix},$$

$$\nu(s,\theta) = \begin{pmatrix} -\sin\phi(s)\cos\theta \\ -\sin\phi(s)\sin\theta \\ \cos\phi(s) \end{pmatrix}.$$

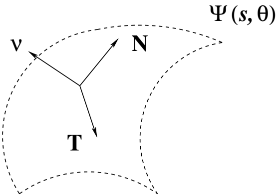

FIGURE: Eigenframe.

Q can be expressed in the form

$$Q = p_1(\mathbf{T}\otimes\mathbf{T} - \mathbf{N}\otimes\mathbf{N}) + p_2(\mathbf{T}\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{T}) + \frac{3\beta}{2}\left(\nu\otimes\nu - \frac{1}{3}I\right).$$

With $\beta = -1/3$, $f_s^{(1)} \equiv 0$, and $M_2 = M_3 = 0$:

$$\begin{aligned}|\nabla_{\mathcal{M}} Q|^2 &= |\mathbf{p}_{,s}|^2 + \frac{1}{a^2}|\mathbf{p}_{,\theta}|^2 + \frac{4\cos\phi}{a^2}\left(p_1 p_{2,\theta} - p_2 p_{1,\theta}\right) \\ &+ \left(\frac{4}{a^2} - 3\kappa_N^2 + \kappa_T^2\right)|\mathbf{p}|^2 - p_1\left(\kappa_N^2 - \kappa_T^2\right) := f_{el}(\nabla\mathbf{p},\mathbf{p}),\end{aligned}$$

$$f_{LdG}(Q) \to f_{LdG}(|\mathbf{p}|),$$

so that

$$E_0[Q] \to E_0[\mathbf{p}] = \int_{s_0}^{s_0+L}\int_0^{2\pi}\left(f_{el}(\nabla\mathbf{p},\mathbf{p}) + \frac{1}{\delta^2}f_{LdG}(|\mathbf{p}|)\right)a(s)\,d\theta\,ds.$$

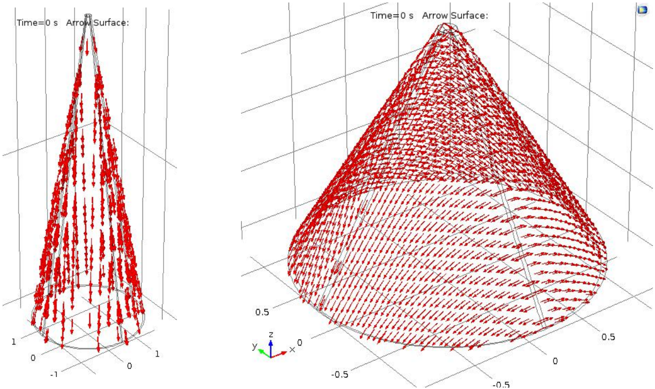

FIGURE: Minimizing configurations.

- Assume that $\mathcal{M}$ is a truncated cone: $\mathbf{r}(s) = (\cos\phi_0, \sin\phi_0)s$, where $s \in [s_0, s_0 + L]$.
- Impose natural boundary conditions on $\mathbf{p}$ on each orifice of the cone.
- Let $\delta \to 0$ so that $|\mathbf{p}| = const$; set $|\mathbf{p}| = 1$. Then

$$\mathbf{p} = (\cos\Psi(s,\theta), \sin\Psi(s,\theta)).$$

It follows that, up to a constant,

$$E_0[\Psi] = \int_{s_0}^{s_0+L}\int_0^{2\pi}\left(\Psi_{,s}^2 + \frac{1}{a^2(s)}\Psi_{,\theta}^2 + \frac{4\cos\phi_0}{a^2(s)}\Psi_{,\theta} - \frac{\sin^2\phi_0}{a^2(s)}\cos\Psi\right) a(s)d\theta\, ds$$

Can assume that $\Psi_{,s} \equiv 0$, then need to study

$$E_0[\Psi] = \int_0^{2\pi} \left(\Psi_{,\theta}^2 + 4\cos\phi_0 \Psi_{,\theta} - \sin^2\phi_0 \cos\Psi \right) \, d\theta,$$

subject to $\Psi(2\pi) = \Psi(0) + 2\pi k$ for some $k \in \mathbb{Z}$.

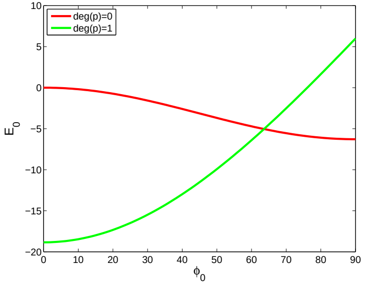

FIGURE: Energies of possible competitors.