On Large Cuspidal Automorphic Forms

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Automorphic Forms, Mock Modular Forms and String Theory
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- $\mathcal{A}_2(G)$ is the set of equivalence classes of irreducible unitary representations π of $G(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\text{disc}}(X_G)$.
- $\mathcal{A}_{\text{cusp}}(G)$ is the subset of $\mathcal{A}_2(G)$ consisting of those π of $G(\mathbb{A})$ occurring in the cuspidal spectrum $L^2_{\text{cusp}}(X_G)$.

Theory of Endoscopic Classification

Theorem (Arthur, Mok, Kaletha-Minguez-Shin-White)

Let G^* be an F-quasisplit classical group and G be a pure inner form of G^* over F. For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$, there is a global Arthur parameter $\psi \in \Psi_2(G^*)$, which is G-relevant, such that

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$$\Pi_\psi[G^*] := \cup_G \Pi_\psi(G).$$
- ψ is G-relevant if the global packet $\Pi_\psi(G)$ is not empty.

Global Arthur Parameters $\Psi_2(G)$: Examples

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$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r$$

where $\psi_i = (\tau_i, b_i)$, with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$; $a_i, b_i \geq 1$; and $\sum_{i=1}^r a_i b_i = 2n$.

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- If $i \neq j$, either $\tau_i \ncong \tau_j$ or $b_i \neq b_j$, with the parity condition that $a_i \cdot b_i$ is even and $\psi_i \in \Psi_2(\mathrm{SO}^*_{a_i b_i + 1})$.

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- **Endoscopy Structure:** $2n = \sum_{i=1}^r a_i \cdot b_i$,

$$\begin{array}{ccccccc} \mathrm{SO}^*_{a_1 \cdot b_1 + 1} & \times & \cdots & \times & \mathrm{SO}^*_{a_r \cdot b_r + 1} & \Longrightarrow & \mathrm{SO}^*_{2n+1} \\ \Pi_{\psi_1}(\cdot) & \otimes & \cdots & \otimes & \Pi_{\psi_r}(\cdot) & \Longrightarrow & \Pi_{\psi}(\cdot) \end{array}$$

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- If $i \neq j$, either $\tau_i \not\cong \tau_j$ or $b_i \neq b_j$, with the parity:
  1. If $a_i \cdot b_i$ is even, then $\psi_i \in \Psi_2(\mathrm{SO}^*_{a_i b_i})$;
  2. If $a_i \cdot b_i$ is is odd, then $\psi_i \in \Psi_2(\mathrm{Sp}^*_{a_i b_i - 1})$.
- **Endoscopy Structure:** $2n+1 = \sum_{i=1}^r a_i \cdot b_i$,

$$\begin{array}{ccccc} \prod_{a_i b_i = 2l_i} \mathrm{SO}^*_{2l_i} & \times & \prod_{a_j b_j = 2l_j+1} \mathrm{Sp}^*_{2l_j} & \Longrightarrow & \mathrm{Sp}^*_{2n} \\ \otimes_{a_i b_i = 2l_i} \Pi_{\psi_i}(\cdot) & \otimes & \otimes_{a_j b_j = 2l_j+1} \Pi_{\psi_j}(\cdot) & \Longrightarrow & \Pi_\psi(\cdot) \end{array}$$

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$$\phi = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1)$$

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- If G is a pure inner form of G^*, then ${}^LG = {}^LG^*$.
- For $\phi \in \Phi_2(G^*)$, the endoscopic classification may define the global Arthur packet $\Pi_\phi(G^*)$ and also define the global Arthur packet $\Pi_\phi(G)$, which is non-empty if ϕ is G-relevant.

Endoscopic Classification and Langlands Functoriality

$$\mathcal{A}(\mathrm{GL}_{N_G})$$

$$\pi_\psi$$

$$\uparrow$$

$$\Psi_2(G^*)_G$$

$$\psi$$

$$\swarrow \qquad\qquad \searrow$$

$$\mathcal{A}_2(G) \cap \Pi_\psi(G) \qquad \Longleftrightarrow \qquad \Pi_\psi(G^*) \cap \mathcal{A}_2(G^*)$$

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- If ψ is cuspidal, how to **construct explicit modules** for the members in $\Pi_\psi(G) \cap \mathcal{A}_{\mathrm{cusp}}(G)$?
- This leads to the theory of twisted automorphic descents and endoscopy correspondences via integral transforms.

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- Over F, each $\overline{F}$-orbit reduces to an F-stable adjoint $G^*(F)$-orbits $\mathcal{O}^{\text{st}}$, and hence the F-stable adjoint orbits in the nilcone $\mathcal{N}(\mathfrak{g}^*)$ are also parameterized by the corresponding partitions of an integer $N = N_{G^*}$ of type G^*.

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$$\mathfrak{g}^* = \mathfrak{g}_{-r} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r.$$

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- Let ψ_F be a non-trivial additive character of $F \backslash \mathbb{A}$. The character ψ_X of $V_X(F)$ or $V_X(\mathbb{A})$ is defined by

$$\psi_X(v) = \psi_F(\mathrm{tr}(Y \log(v))).$$

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Fourier Coefficients and Nilpotent Adjoint Orbits

- The **Fourier coefficient** of $\varphi \in \pi \in \mathcal{A}_2(G^*)$ is defined by

$$\mathcal{F}^{\psi_X}(\varphi)(g) := \int_{V_X(F)\backslash V_X(\mathbb{A})} \varphi(vg)\psi_X(v)^{-1}dv.$$

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- $\mathfrak{p}^m(\varphi)$ is the set of all maximal partitions in $\mathfrak{p}(\varphi)$, according to the partial ordering of partitions.

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- This problem is closely related to the theory of *twisted automorphic descent*, and is an induction step towards the understanding of the *wave-front set* of π. The p-adic analogy was undertaken in my recent work with Lei Zhang.

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- In particular, the **Folklore Conjecture** is verified for all $\pi \in \mathcal{A}_2(\mathrm{GL}_n)$!

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- **Conjecture:** $\mathfrak{p}^m(\pi_\psi) = \{\eta_{\mathfrak{gl}_n^\vee, \mathfrak{gl}_n}(\underline{p}_\psi)\}$.

Maximal Fourier Coefficients and Arthur Parameters

- For $\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \in \Psi_2(G^*)$, where $\psi_i = (\tau_i, b_i)$ with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ and $b_i \geq 1$, $\underline{p}_{\psi} = [b_1^{a_1} \cdots b_r^{a_r}]$ is the partition of $N_{(G^*)^\vee}$ attached to $(\psi, (G^*)^\vee)$ and $\eta(\underline{p}_{\psi})$ is the Barbasch-Vogan duality of $\underline{p}_{\psi}$ from $(G^*)^\vee$ to G^*.

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- **Conjecture (J.-2014):**
  (1) For every $\pi \in \Pi_\psi(G^*) \cap \mathcal{A}_2(G^*)$, any partition $\underline{p} \in \mathfrak{p}^m(\pi)$ has the property that $\underline{p} \leq \eta(\underline{p}_\psi)$.
  (2) There exists at least one member $\pi \in \Pi_\psi(G) \cap \mathcal{A}_2(G)$ for some pure inner form G of G^* that have the property: $\eta(\underline{p}_\psi) \in \mathfrak{p}^m(\pi)$.

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- **Remark:** For a pure inner form G of G^*, assume that the global Arthur parameter ψ is G-relevant and the Barbasch-Vogan duality $\eta(\underline{p}_\psi)$ is a G-relevant partition of $N_G = N_{G^*}$ of type G^*. The definition of Fourier coefficients also work.

Examples of the Barbasch-Vogan duality

- $G = \mathrm{SO}_{2n+1}$ and $2n = ab$; Take $\psi = (\tau, b)$ for $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_a)$, and

$$b = \begin{cases} 2\ell, & \text{if } \tau \text{ is orthogonal,} \\ 2\ell + 1, & \text{if } \tau \text{ is symplectic.} \end{cases}$$

- $\underline{p}_\psi = [b^a]$ is the partition of $2n$ of type $(\psi, \mathrm{Sp}_{2n}(\mathbb{C}))$.
- The Barbasch-Vogan duality is given as follows:

$$\eta(\underline{p}_\psi) = \begin{cases} [(a+1)a^{b-2}(a-1)1] & \text{if } b = 2l \text{ and a is even;} \\ [a^b 1] & \text{if } b = 2l \text{ and a is odd;} \\ [(a+1)a^{b-1}] & \text{if } b = 2l + 1. \end{cases}$$

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- Take $G = \mathrm{Sp}_{2n}$ and $\psi = (\tau, 2b+1) \boxplus \boxplus_{i=2}^{r} (\tau_i, 1) \in \Psi_2(G)$.
- $\underline{p}_{\psi} = [(2b+1)^a (1)^{2m+1-a}]$ with $2m+1 = (2n+1) - 2ab$.
- When $a \leq 2m$ and a is even,

$$\begin{aligned}\eta(\underline{p}_{\psi}) =& \eta([(2b+1)^a(1)^{2m+1-a}]) = [(2b+1)^a(1)^{2m-a}]^t \\ =& [(a)^{2b+1}] + [(2m-a)] = [(2m)(a)^{2b}].\end{aligned}$$

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- This special case can be proved by the Arthur-Langlands transfer from G to GL_{N_G} and the Ginzburg-Rallis-Soudry descent (J.- Liu 2016).

On Large Cuspidal Spectrum

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- The **Question** remains when G is not quasisplit.

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- **Global Large Cuspidal Packet Conjecture (J.-Zhang):** Let G^* be the F-quasisplit pure inner form of G. For any generic global Arthur parameter ϕ of G^*, which is G-relevant, the global Arthur packet $\Pi_\phi(G)$ contains at least one **Large** cuspidal member.

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- One direction of the global Gan-Gross-Prasad conjecture holds for $\mathrm{U}_{n+2,n}$ (J.- Zhang 2015).