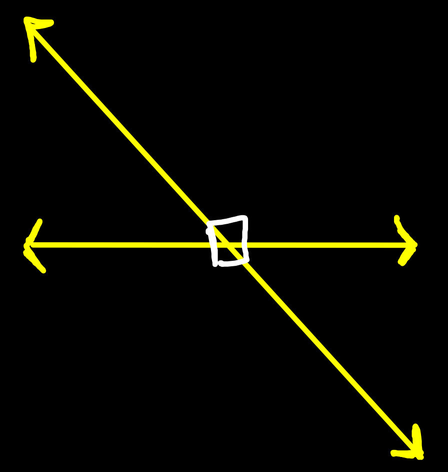
CONJECTURES ABOUT CENTRAL WEIGHTINGS

Julien COURTIEL (University of Caen)

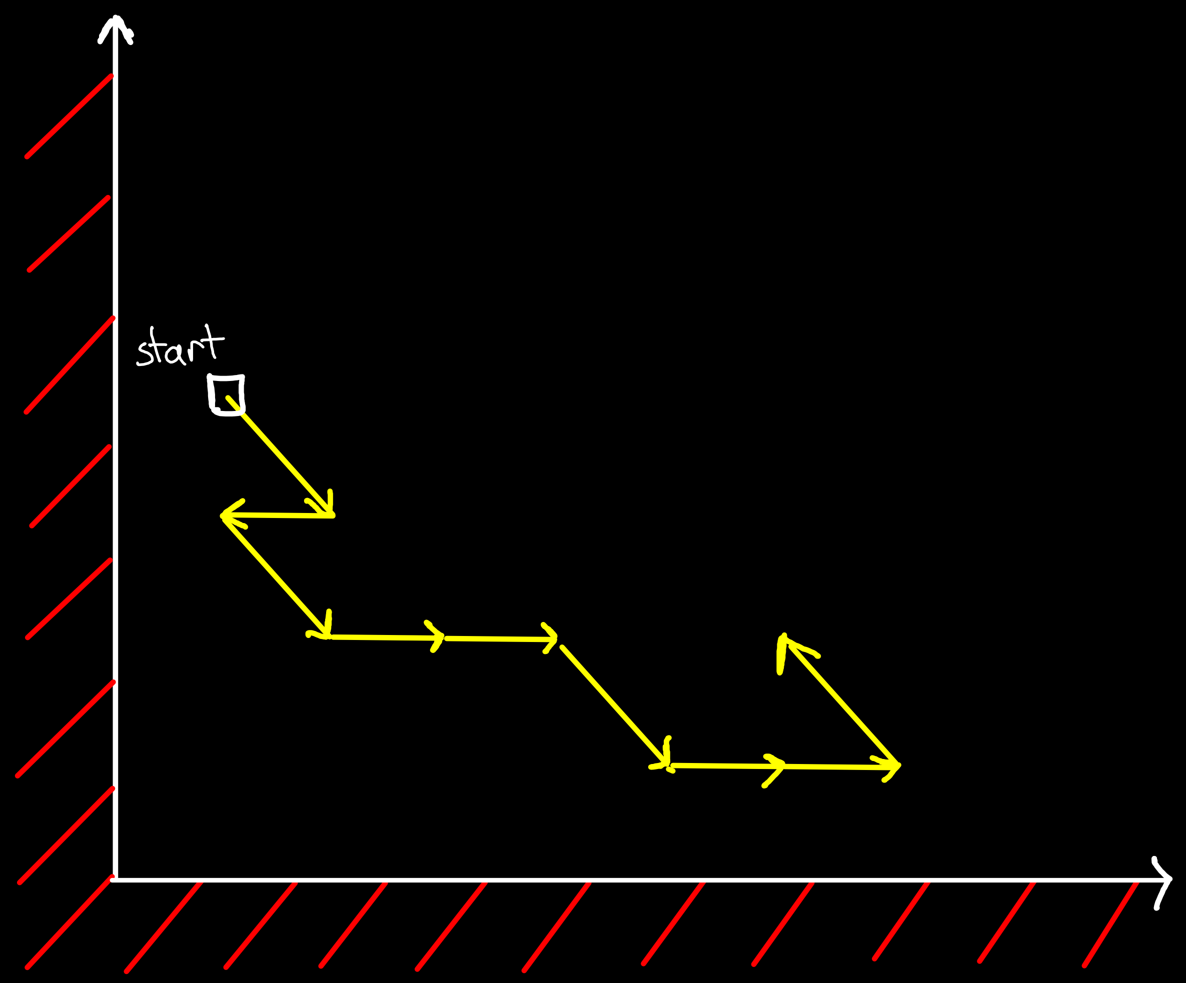
Lattice Walks at the Interface of Algebra, Analysis and Combinatorics (BIRS)

A joint work with S. Melczer, M. Mishna and K. Raschel.

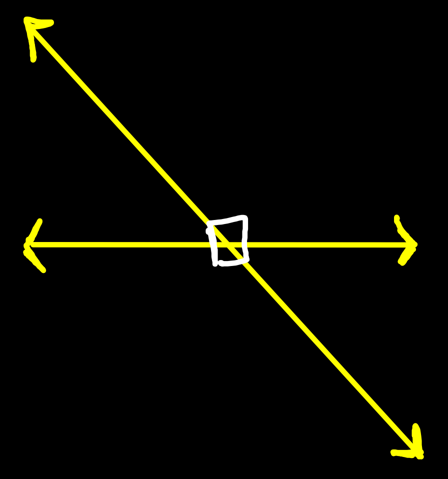
THE ORIGIN STORY



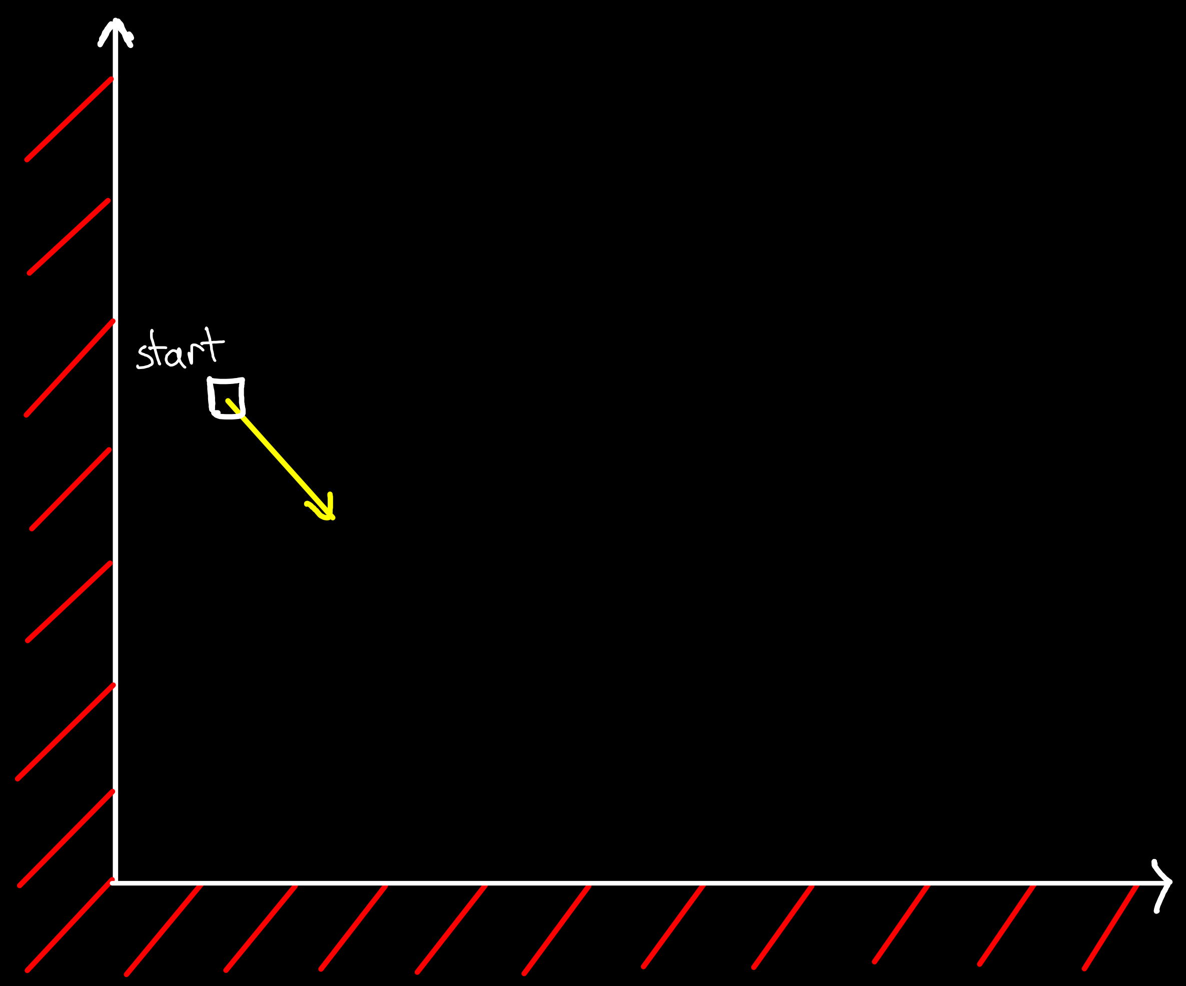
Gouyou-Beauchamps model



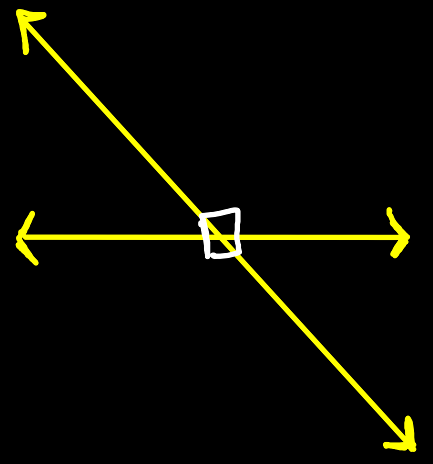
THE ORIGIN STORY



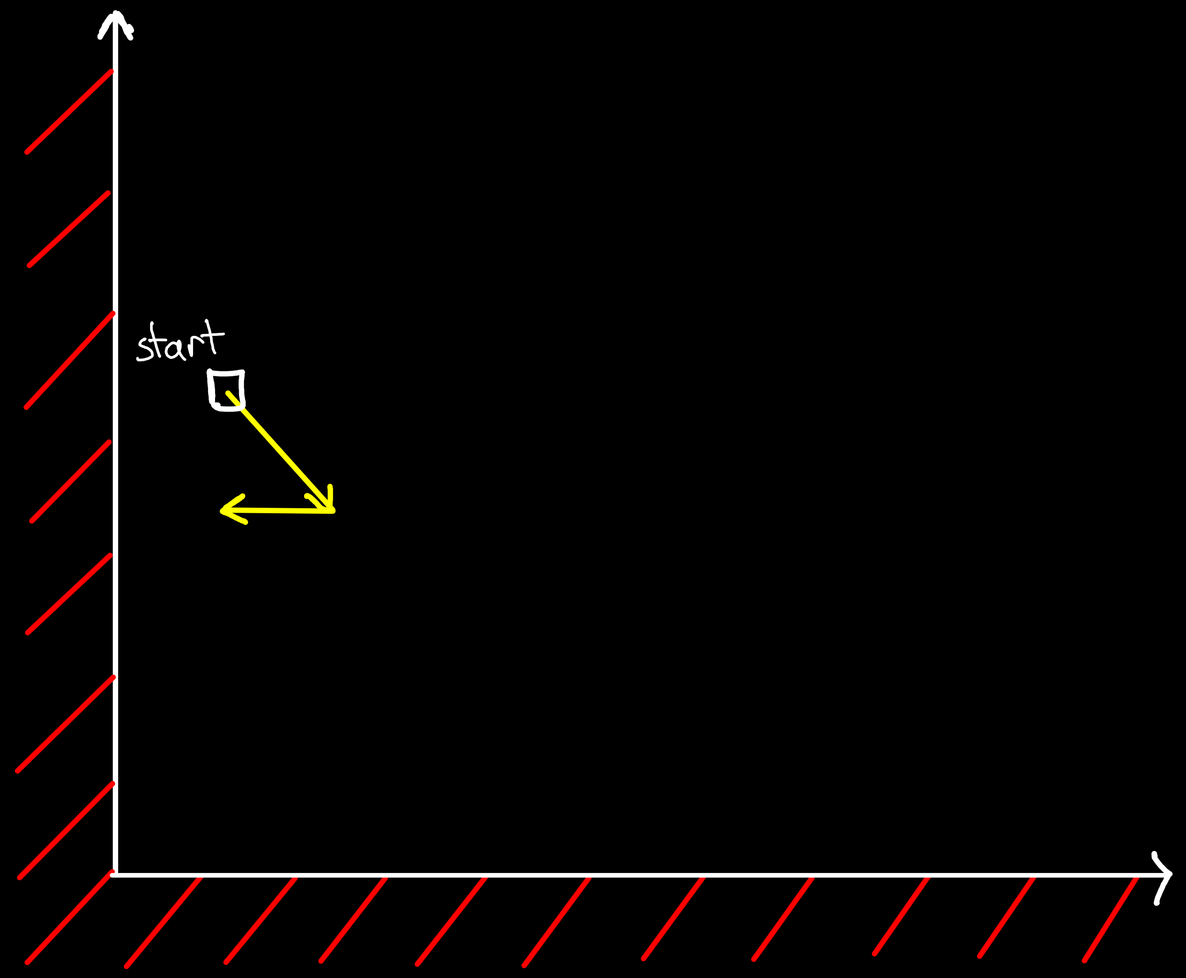
Gouyou - Beauchamps model



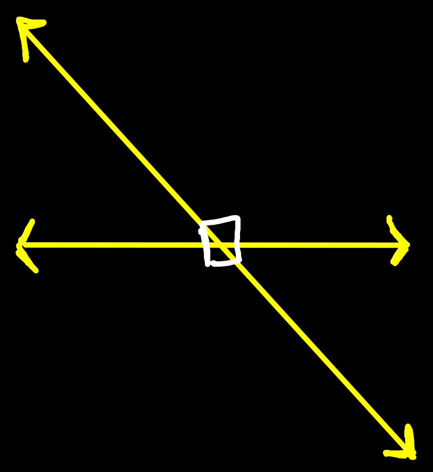
THE ORIGIN STORY



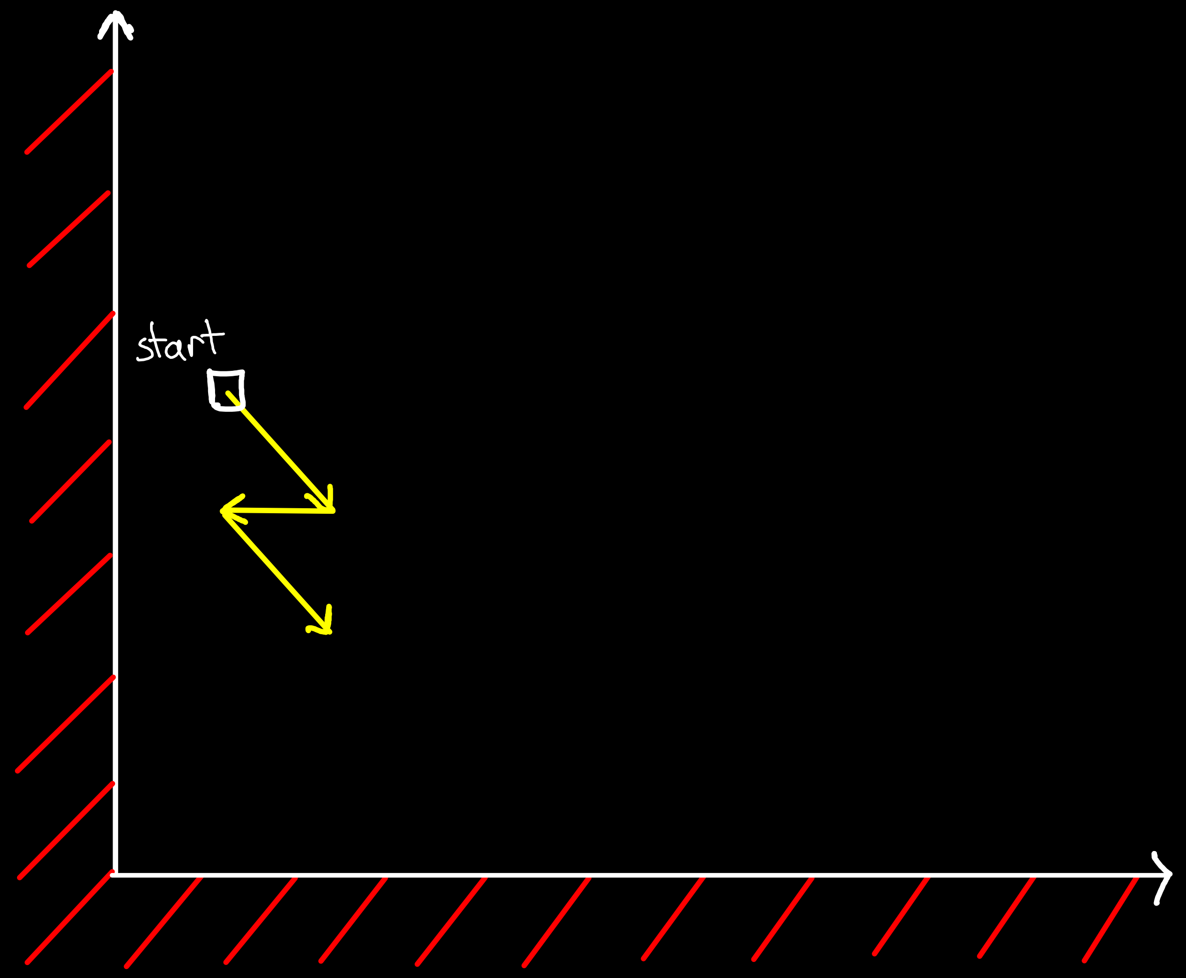
Gouyou - Beauchamps model



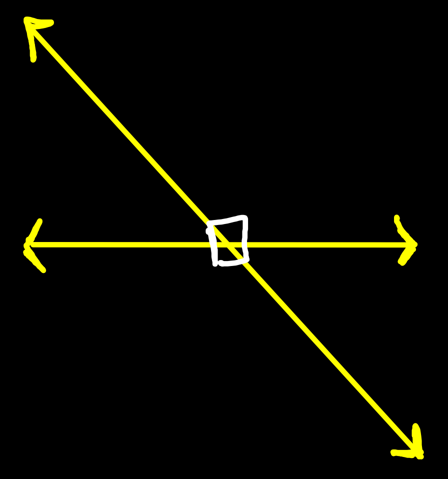
THE ORIGIN STORY



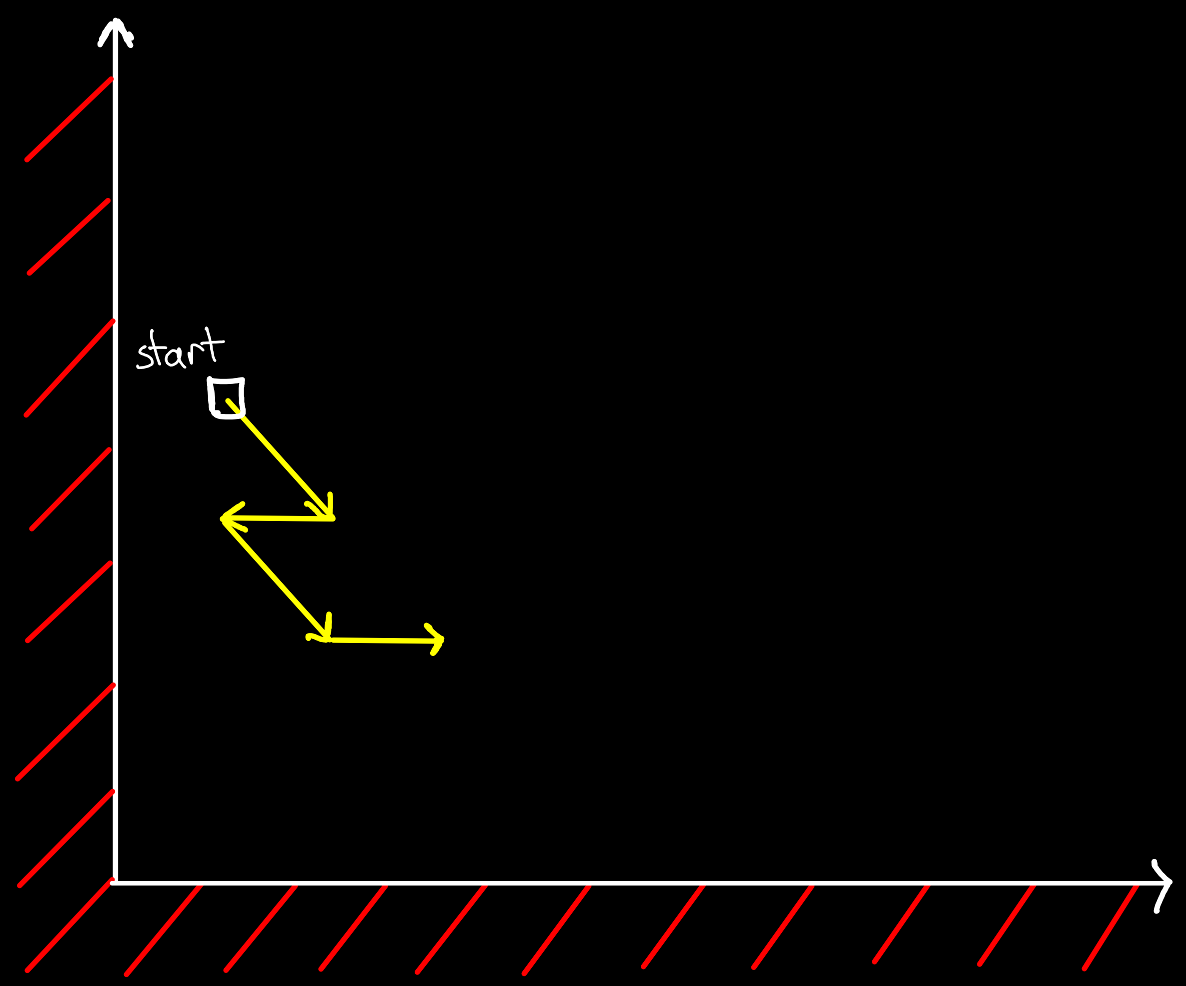
Gouyou - Beauchamps model



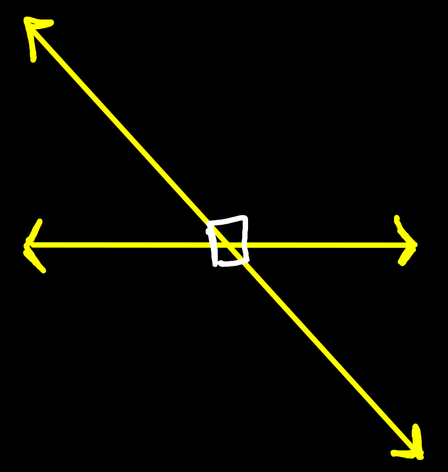
THE ORIGIN STORY



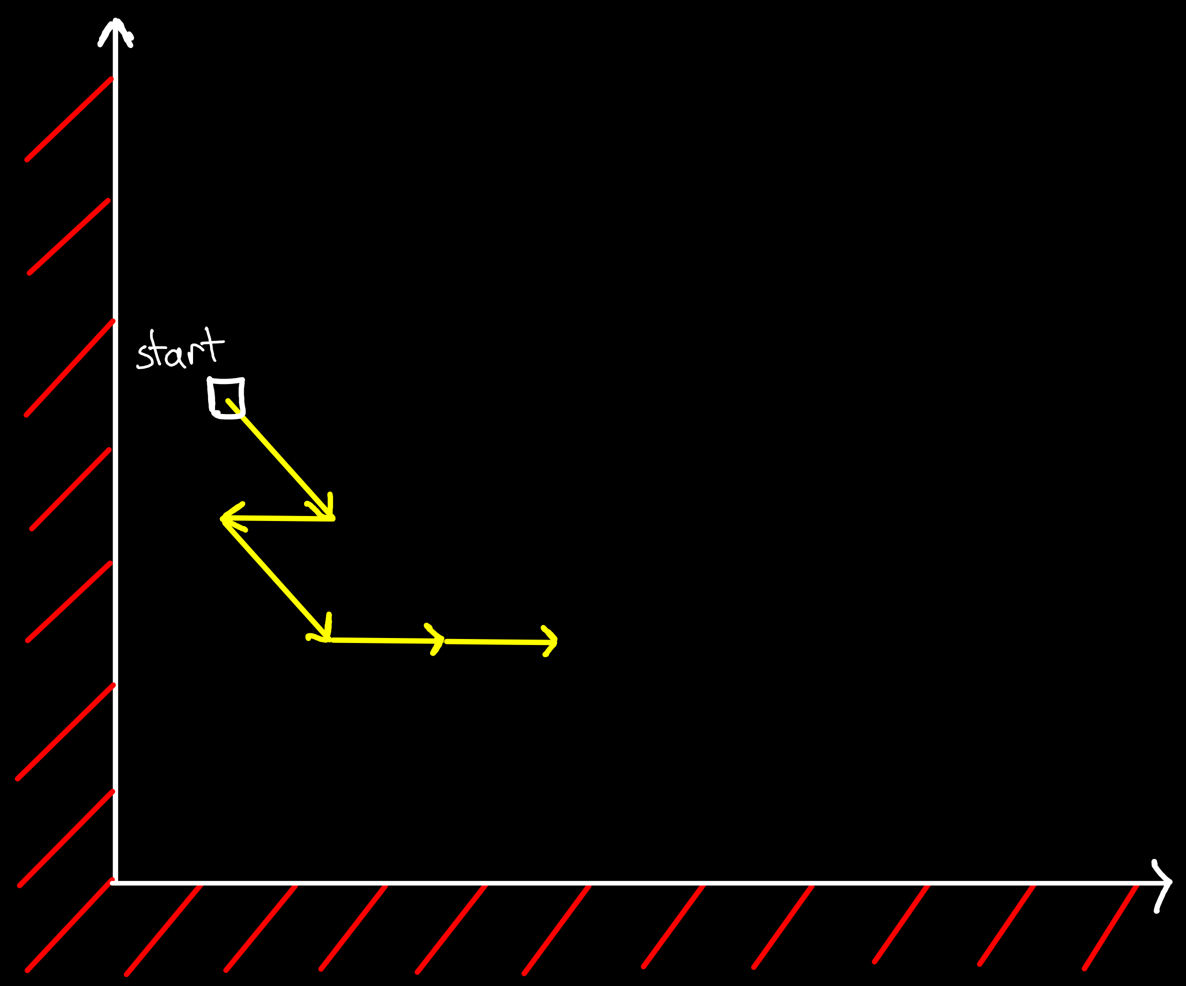
Gouyou-Beauchamps model



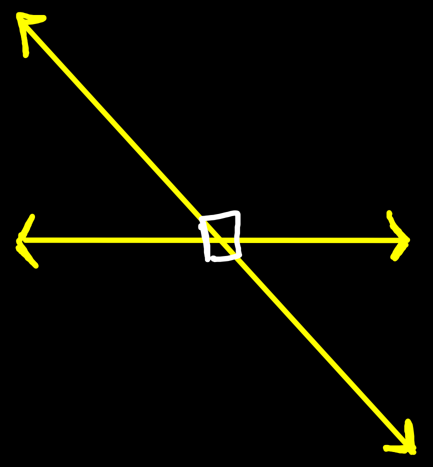
THE ORIGIN STORY



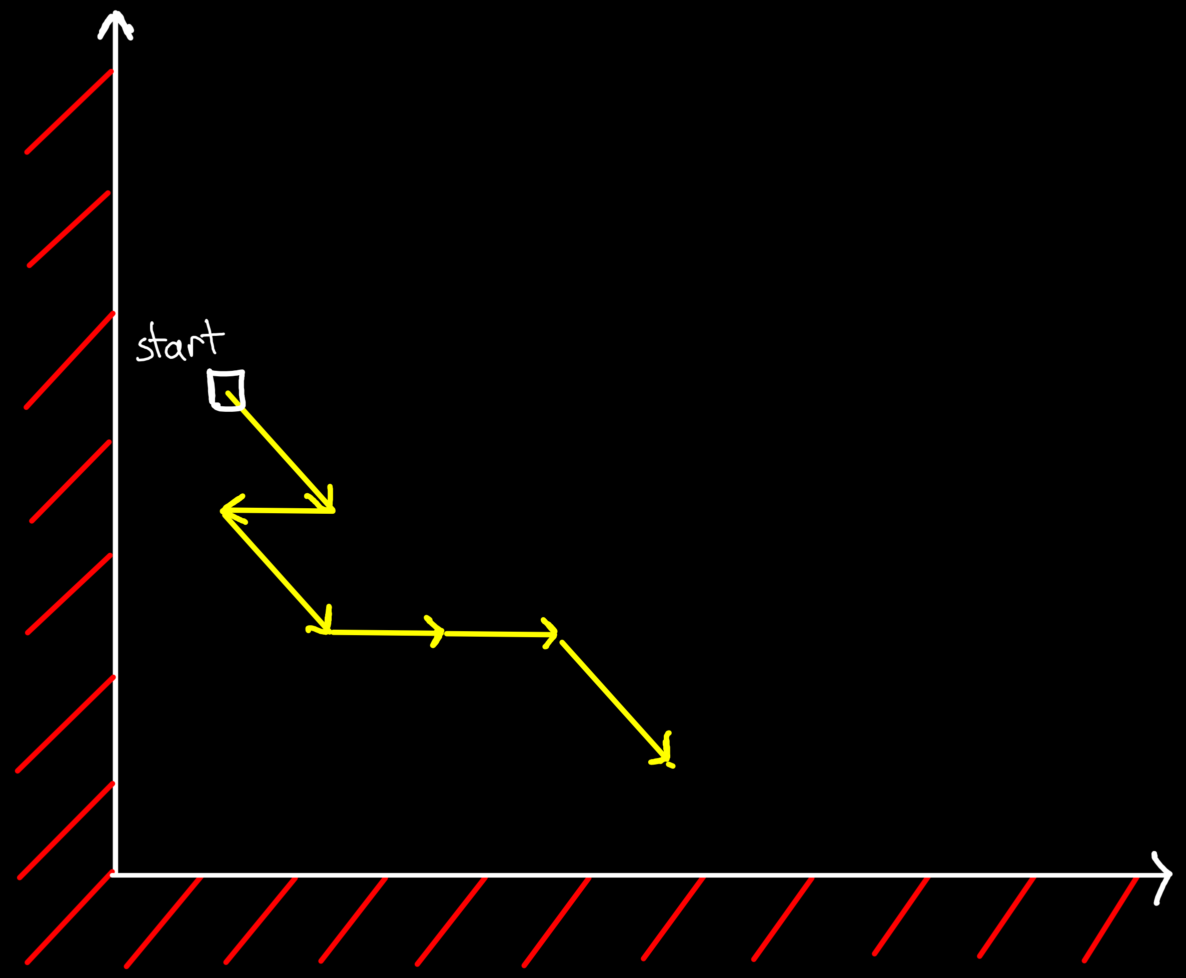
Gouyou - Beauchamps model



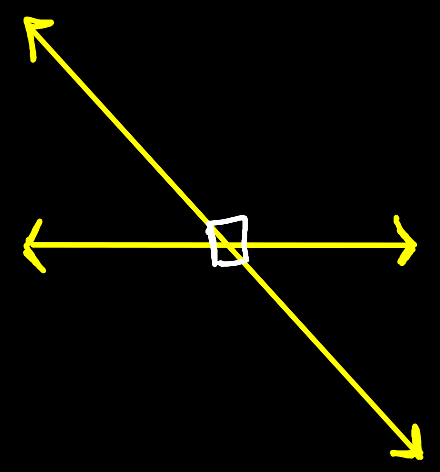
THE ORIGIN STORY



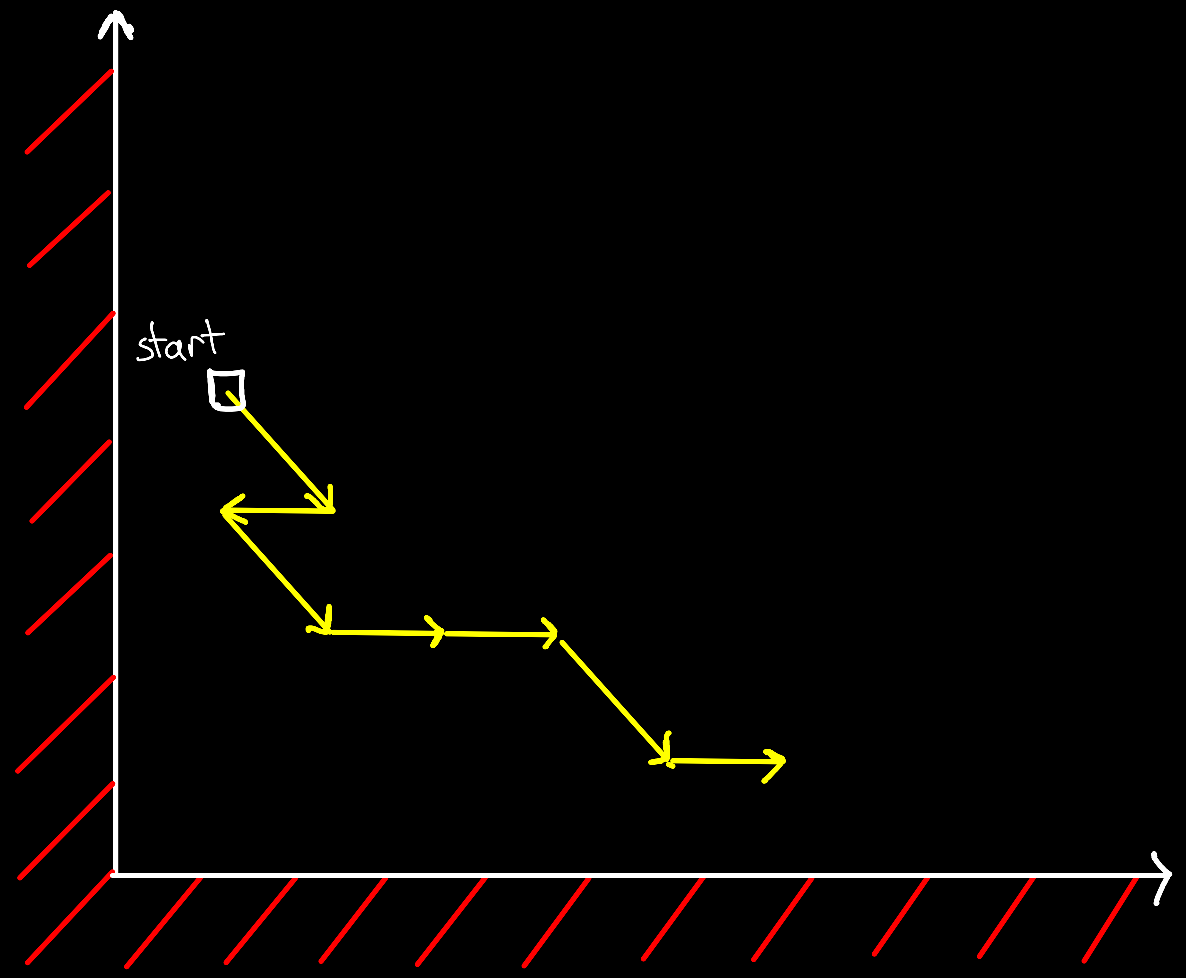
Gouyou - Beauchamps model



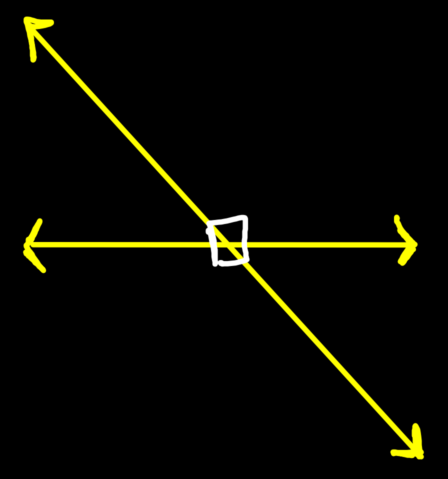
THE ORIGIN STORY



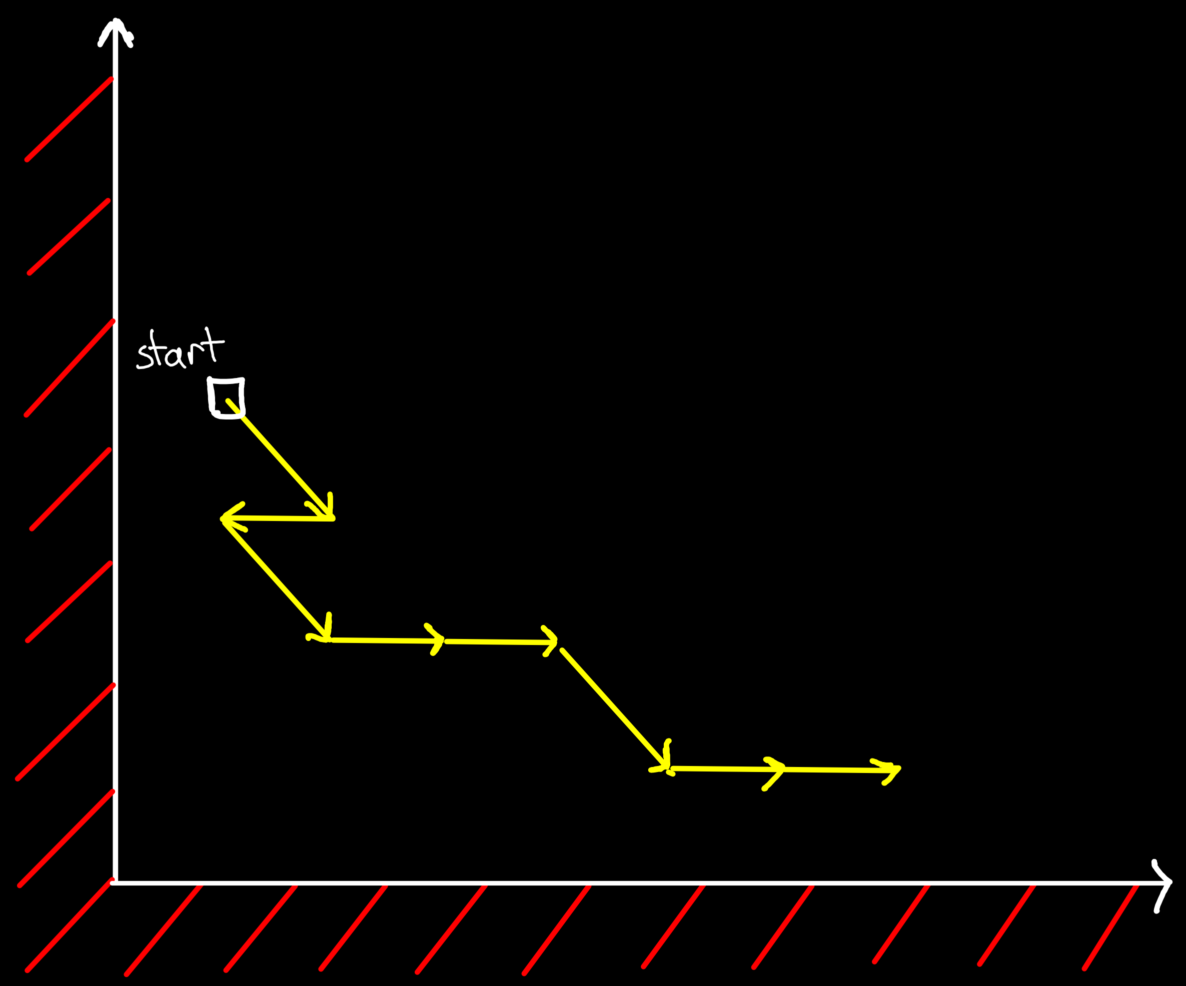
Gouyou - Beauchamps model



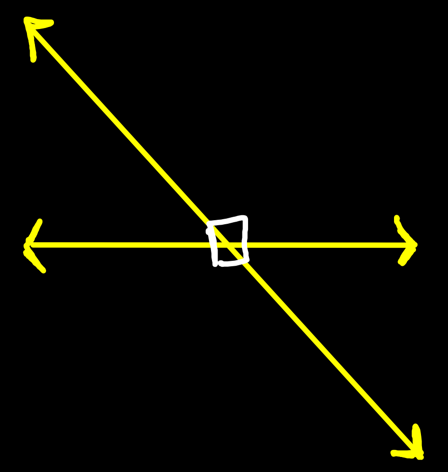
THE ORIGIN STORY



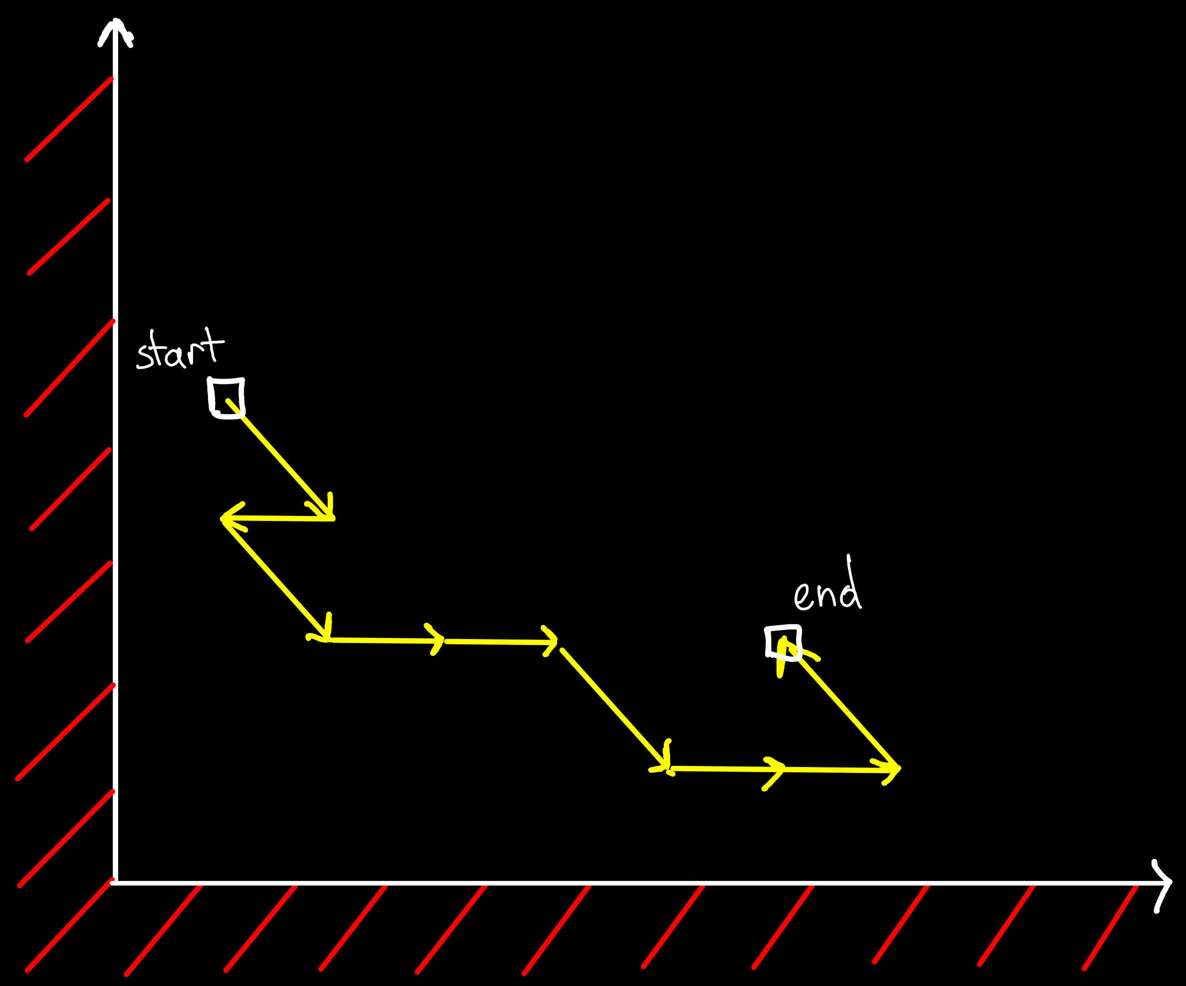
Gouyou - Beauchamps model



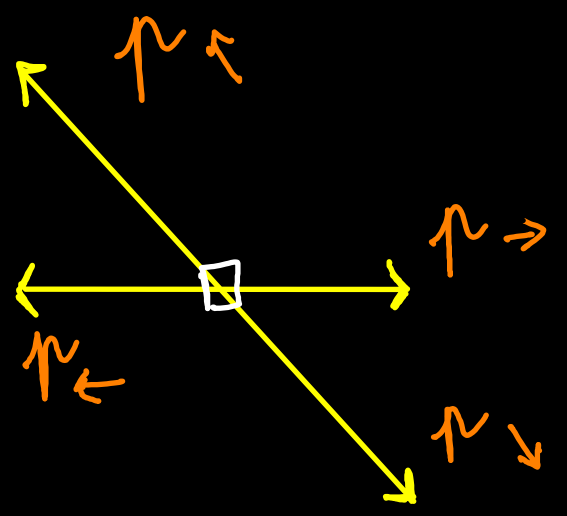
THE ORIGIN STORY



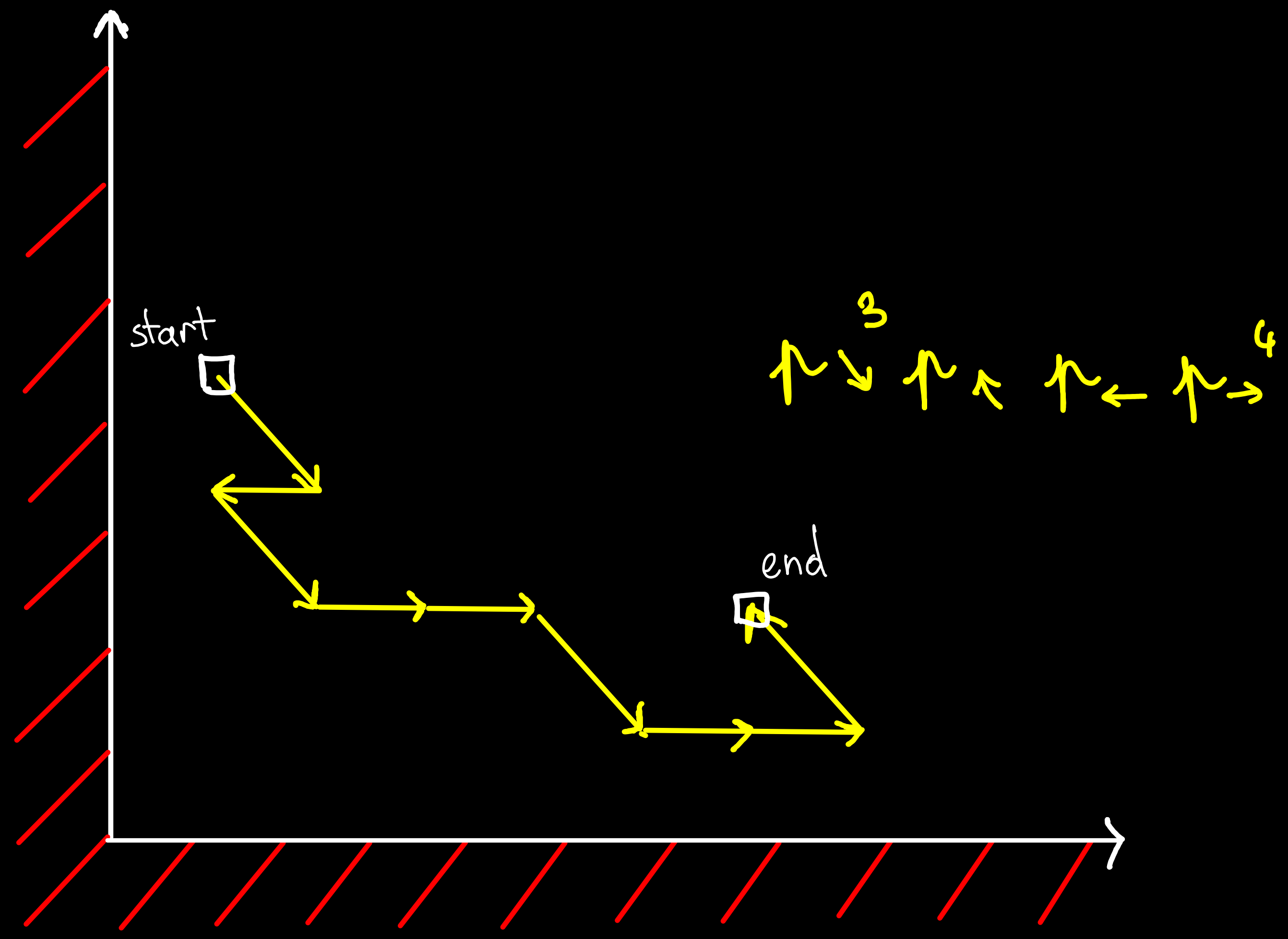
Gouyou - Beauchamps model



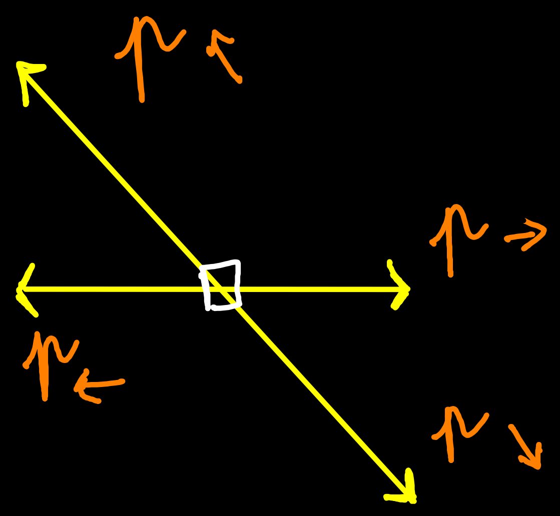
THE ORIGIN STORY



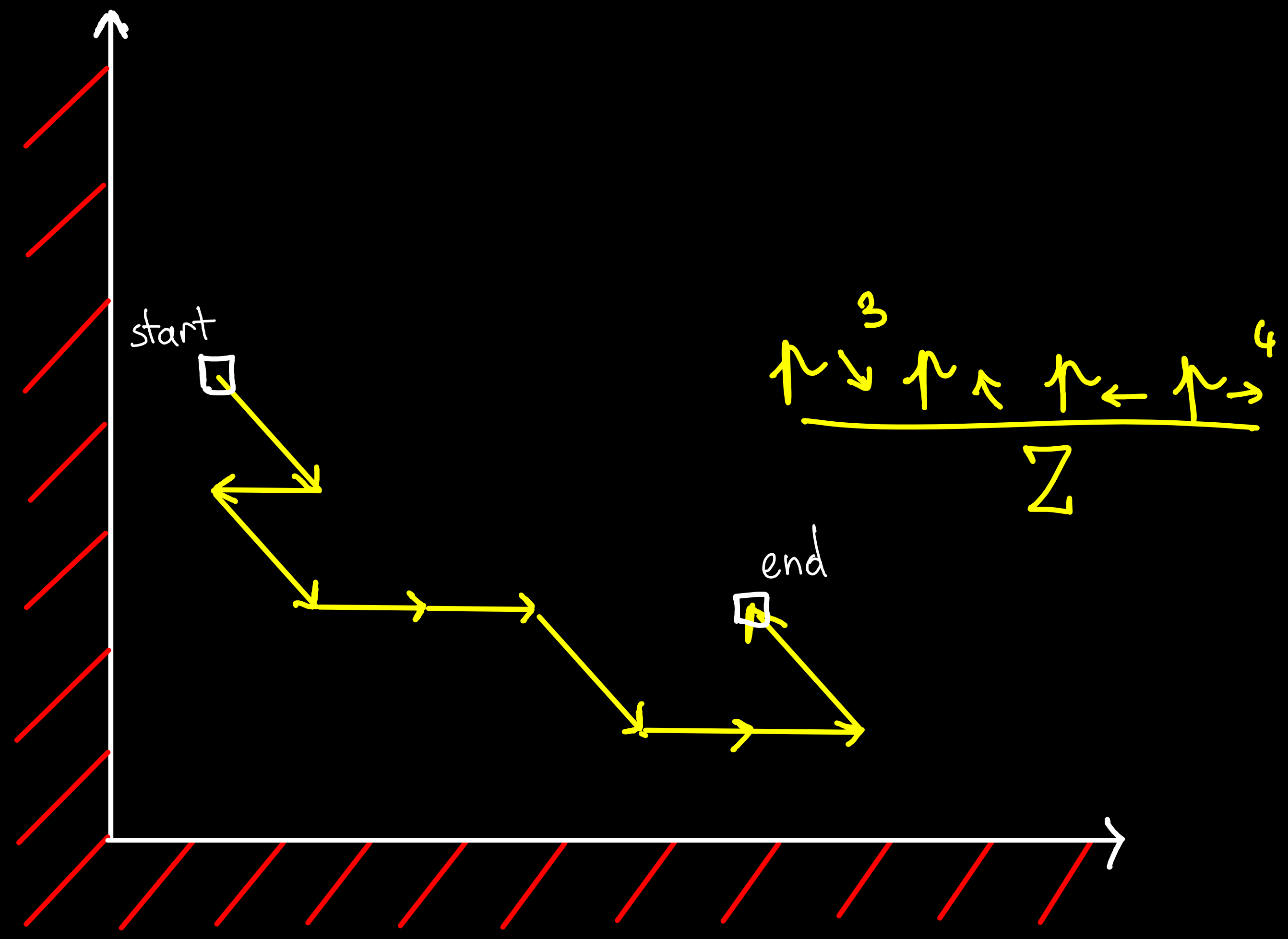
weighted Gouyou-Beauchamps model



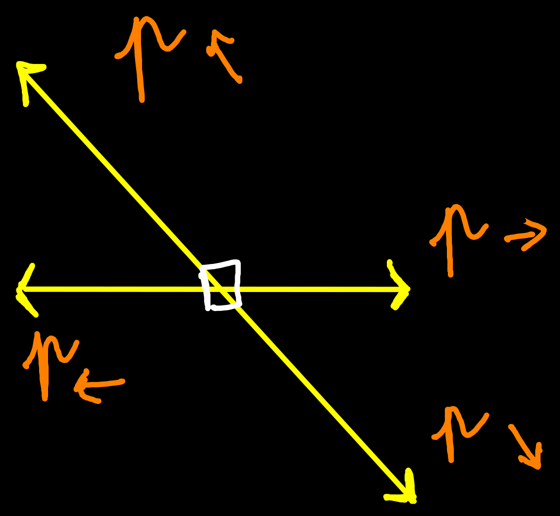
THE ORIGIN STORY



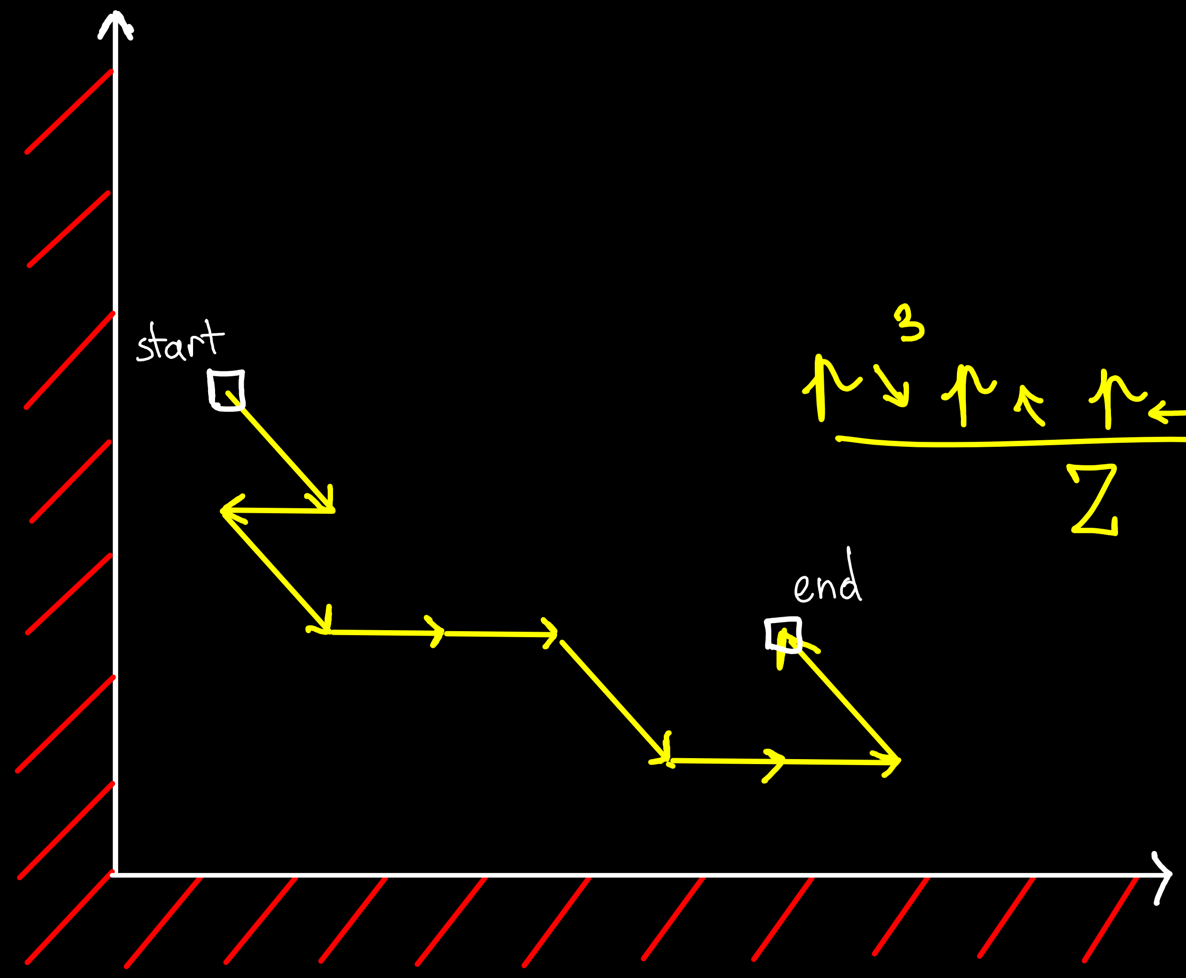
weighted Gouyou-Beauchamps model



THE ORIGIN STORY



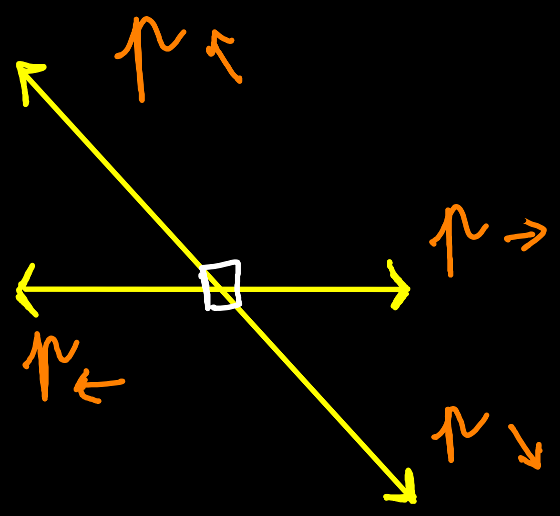
weighted Gouyou-Beauchamps model



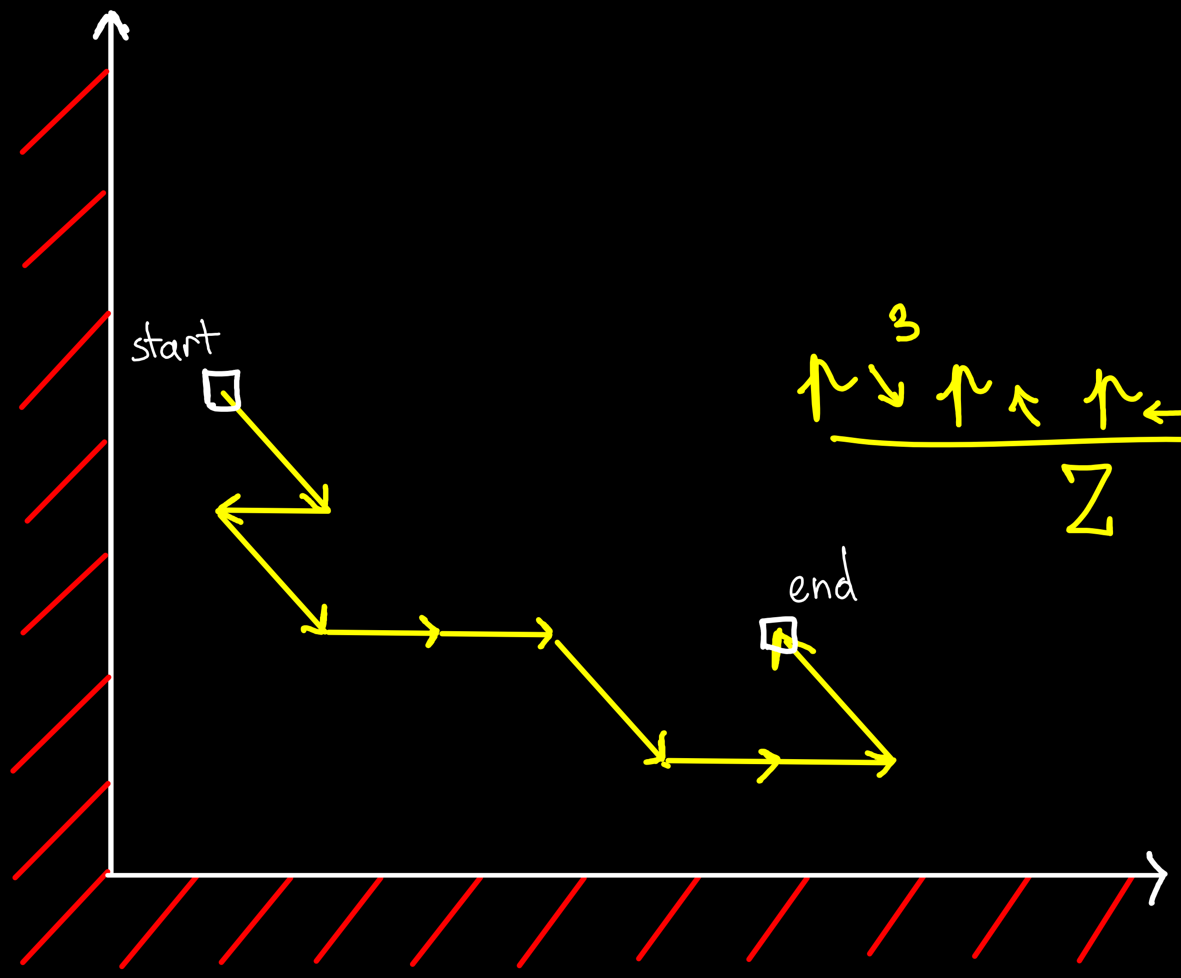
$$\frac{p_{\searrow}^{3}\, p_{\nwarrow}\, p_{\leftarrow}\, p_{\rightarrow}^{4}}{Z}$$

"good" set of weights : $p_{\nwarrow} \times p_{\searrow} = p_{\leftarrow} \times p_{\rightarrow}$

THE ORIGIN STORY



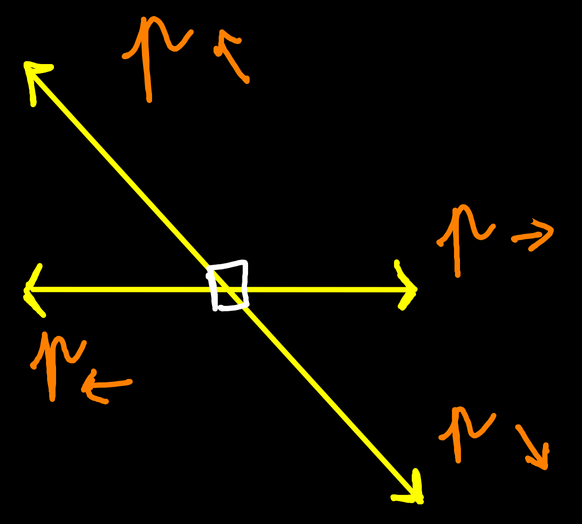
weighted Gouyou-Beauchamps model



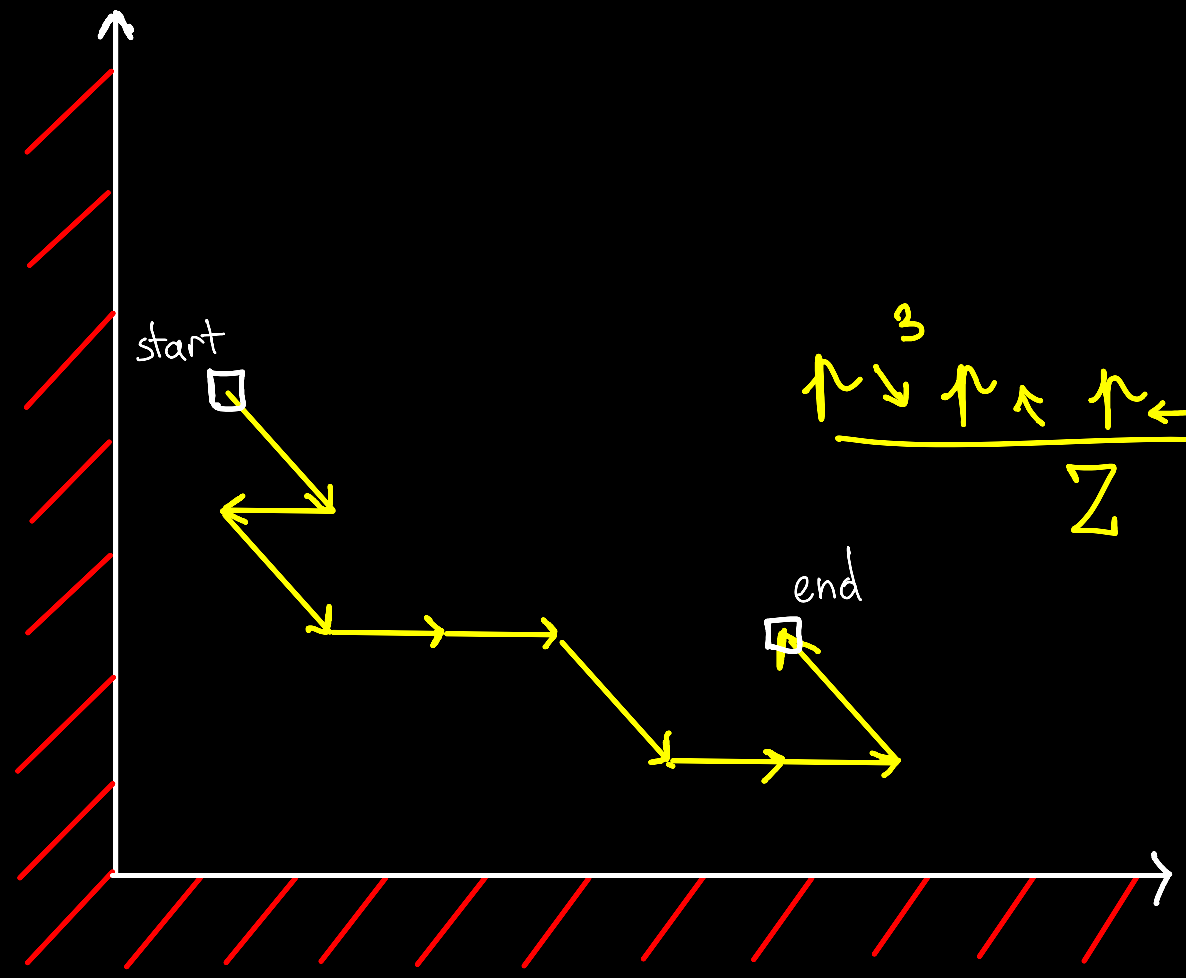
$$\frac{p_{\searrow}^{3}\, p_{\nwarrow}\, p_{\leftarrow}\, p_{\rightarrow}^{4}}{Z} = \frac{p_{\searrow}^{2}\, p_{\leftarrow}^{2}\, p_{\rightarrow}^{5}}{Z}$$

"good" set of weights : $p_{\nwarrow} \times p_{\searrow} = p_{\leftarrow} \times p_{\rightarrow}$

THE ORIGIN STORY



weighted Gouyou-Beauchamps model



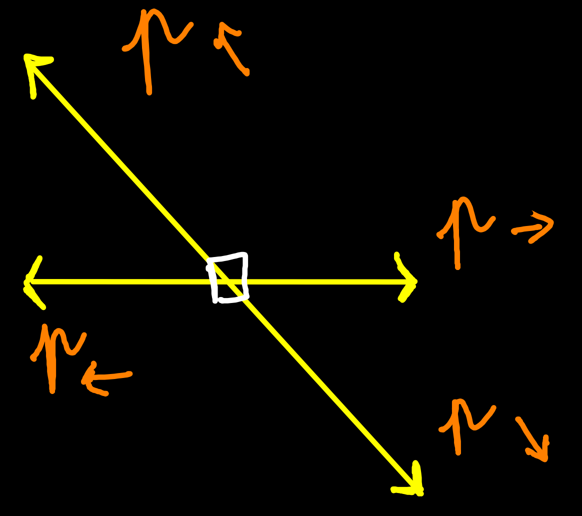
$$\frac{p_{\searrow}^{3}\, p_{\nwarrow}\, p_{\leftarrow}\, p_{\rightarrow}^{4}}{Z} = \frac{p_{\searrow}^{2}\, p_{\leftarrow}^{2}\, p_{\rightarrow}^{5}}{Z}$$

"good" set of weights : $p_{\nwarrow} \times p_{\searrow} = p_{\leftarrow} \times p_{\rightarrow}$

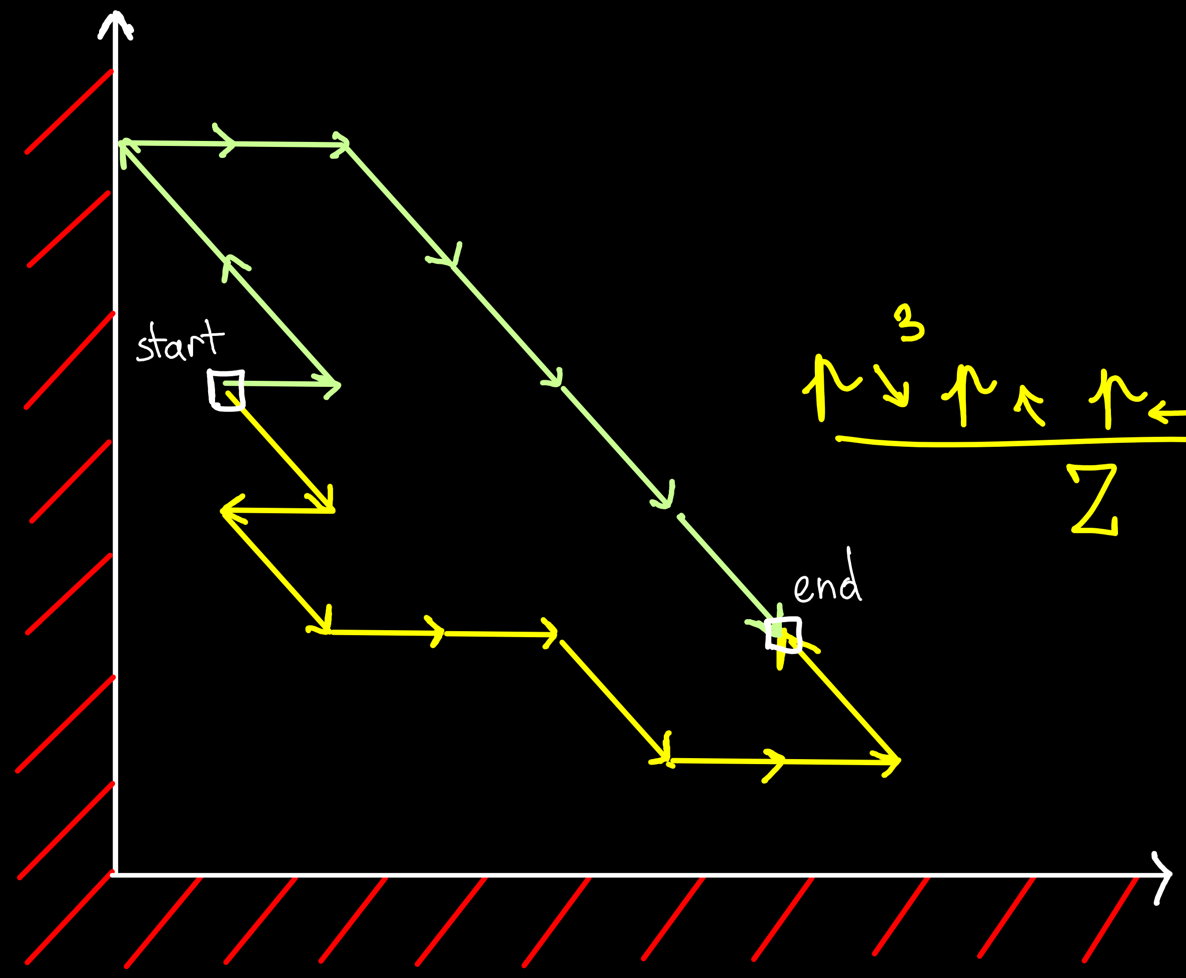
Interest? The probability of a walk only depends on the length, the start and end point.

= "central probability"

THE ORIGIN STORY



weighted Gouyou-Beauchamps model



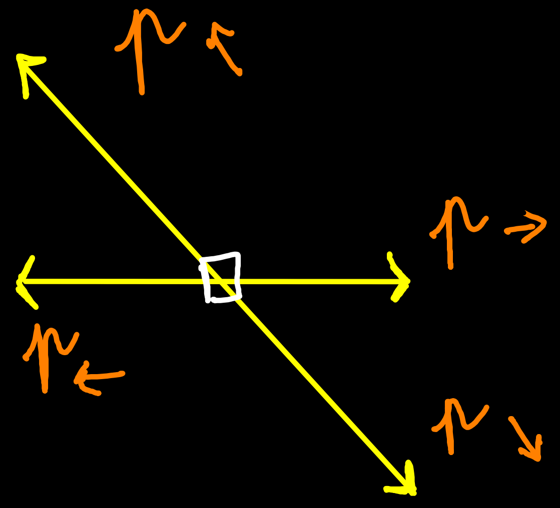
$$\frac{p_{\searrow}^{3}\, p_{\nwarrow}\, p_{\leftarrow}\, p_{\rightarrow}^{4}}{Z} = \frac{p_{\searrow}^{2}\, p_{\leftarrow}^{2}\, p_{\rightarrow}^{5}}{Z}$$

"good" set of weights : $p_{\nwarrow} \times p_{\searrow} = p_{\leftarrow} \times p_{\rightarrow}$

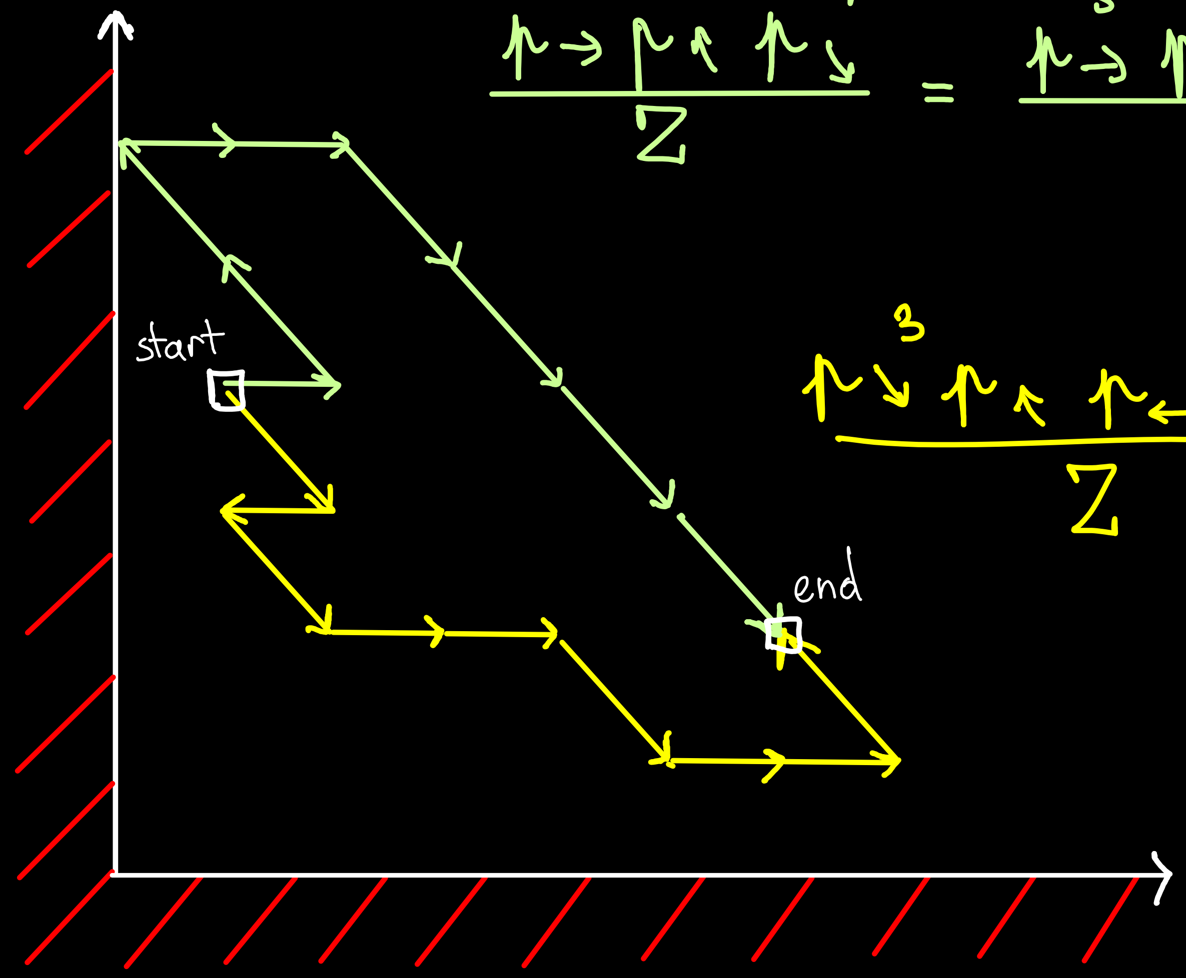
Interest? The probability of a walk only depends on the length, the start and end point.

= "central probability"

THE ORIGIN STORY



weighted Gouyou-Beauchamps model



$$\frac{p_{\rightarrow}^{3}\, p_{\nwarrow}^{2}\, p_{\searrow}^{4}}{Z} = \frac{p_{\rightarrow}^{5}\, p_{\leftarrow}^{2}\, p_{\searrow}^{2}}{Z}$$

$$\frac{p_{\searrow}^{3}\, p_{\nwarrow}\, p_{\leftarrow}\, p_{\rightarrow}^{4}}{Z} = \frac{p_{\searrow}^{2}\, p_{\leftarrow}^{2}\, p_{\rightarrow}^{5}}{Z}$$

"good" set of weights : $p_{\nwarrow} \times p_{\searrow} = p_{\leftarrow} \times p_{\rightarrow}$

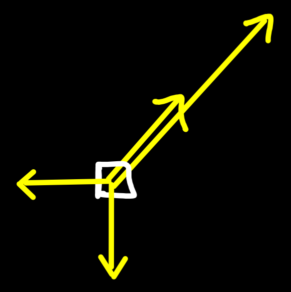
Interest? The probability of a walk only depends on the length, the start and end point.

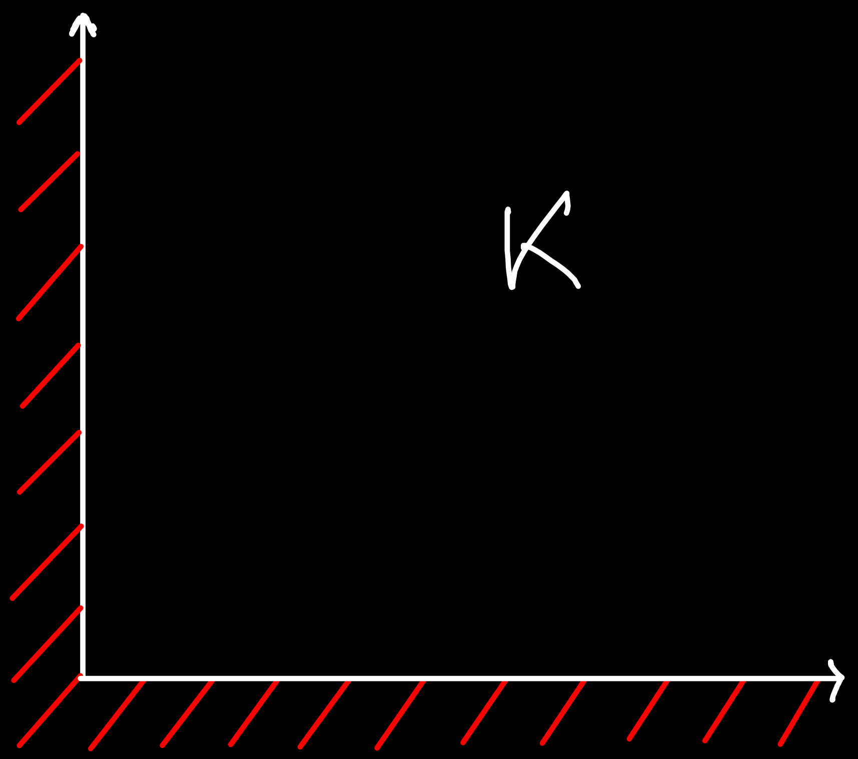
= "central probability"

CENTRAL WEIGHTING

Let $\mathcal{S}$ be a set of steps, K a cone of dimension $d =$ "$\dim(\mathcal{S})$"

Ex:

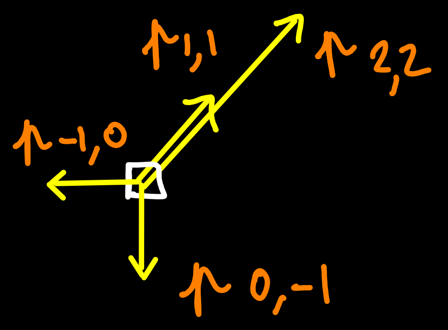
$\mathcal{S} =$ 

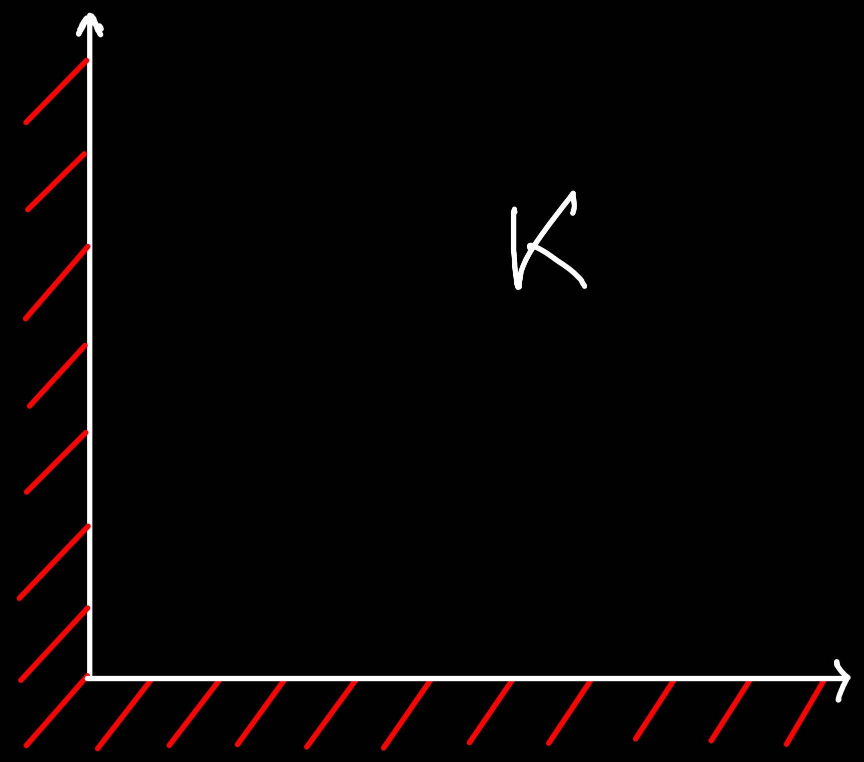


Central weighting

Let $\mathcal{S}$ be a set of steps, K a cone of dimension $d =$ "$\dim(\mathcal{S})$"

Ex:

$\mathcal{S} =$ 



An assignment of weights $\begin{array}{l} \mathcal{S} \longrightarrow \mathbb{Q}_{>0} \\ s \longmapsto p_s \end{array}$ is a central weighting if

the probability of a walk (conditioned to stay in K) is central,

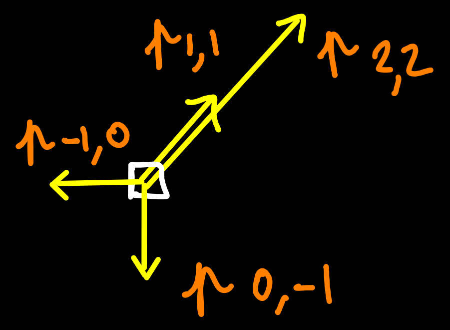
i.e two walks having the same length, start and end point, have the same probability.

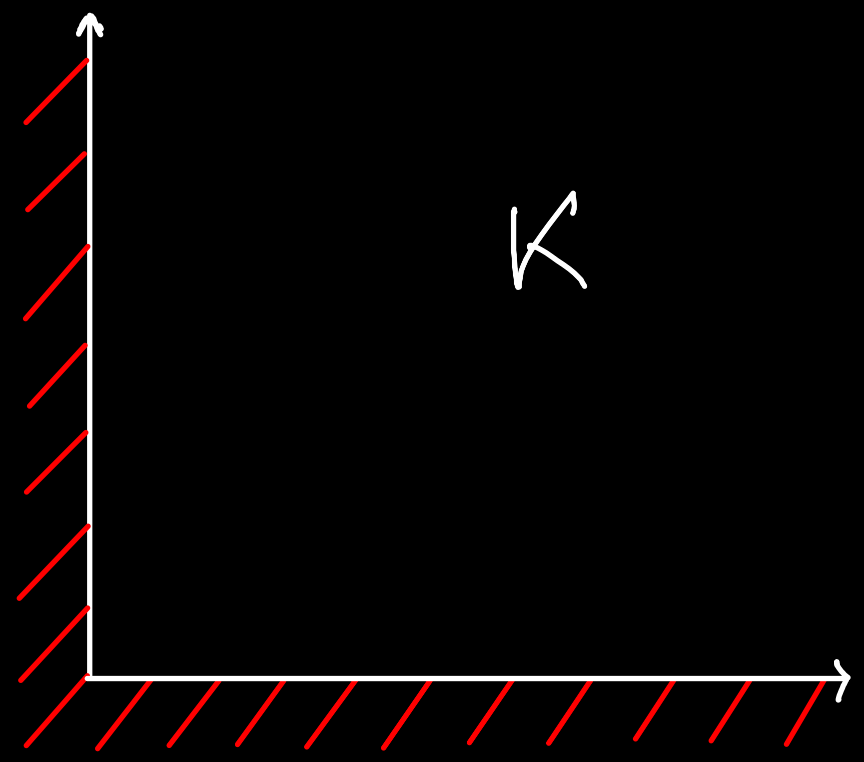
ALGEBRAIC IDENTITIES?

Can we find some necessary conditions for being central?

Ex:

$\mathcal{S}$ =





An assignment of weights $\begin{array}{l} \mathcal{S} \longrightarrow \mathbb{Q}_{>0} \\ s \longmapsto t_s \end{array}$ is a **central weighting** if

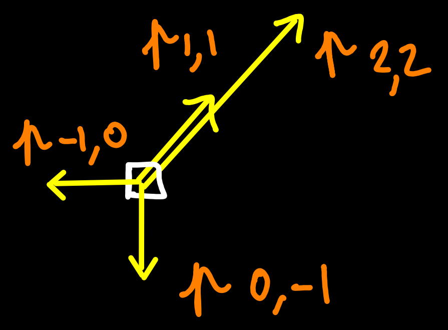
the probability of a walk (conditioned to stay in K) is central,

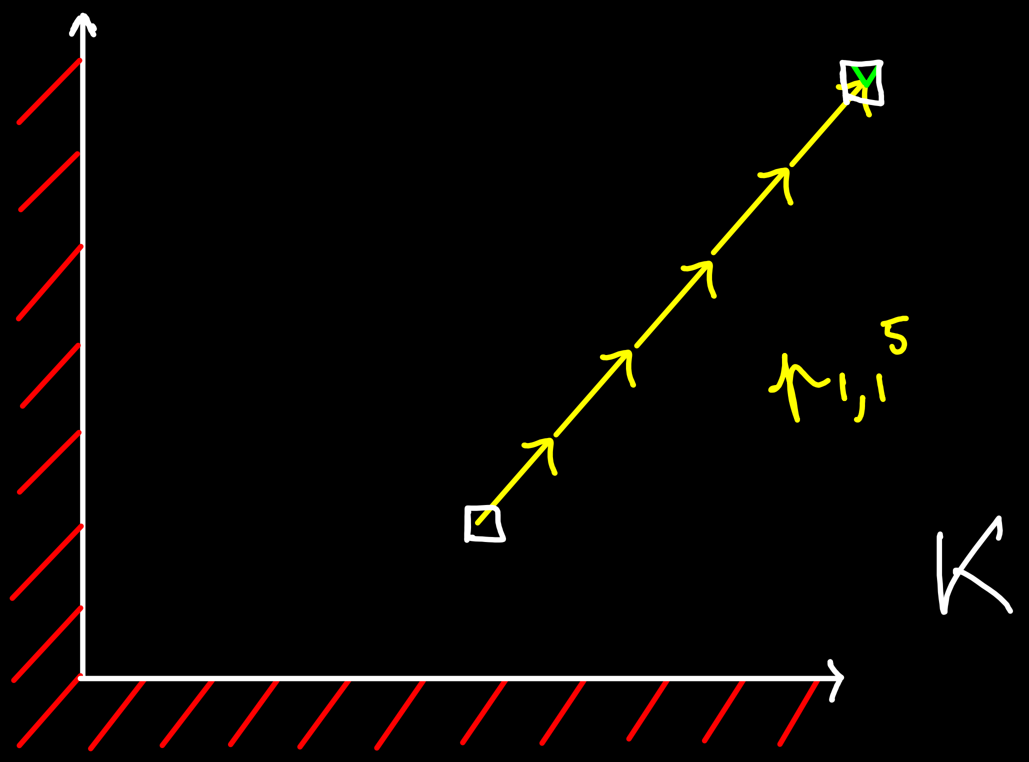
i.e two walks having the same length, start and end point, have the same probability.

ALGEBRAIC IDENTITIES?

Can we find some necessary conditions for being central?

Ex:

$\mathcal{S} =$ 



An assignment of weights $\begin{aligned} \mathcal{S} &\longrightarrow \mathbb{Q}_{>0} \\ s &\longmapsto \omega_s \end{aligned}$ is a central weighting if

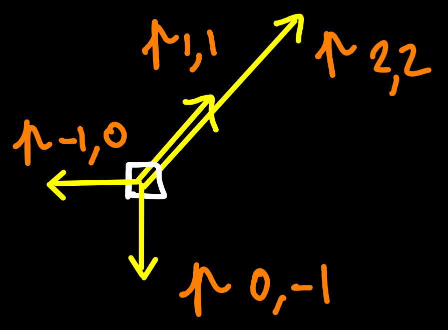
the probability of a walk (conditioned to stay in K) is central,

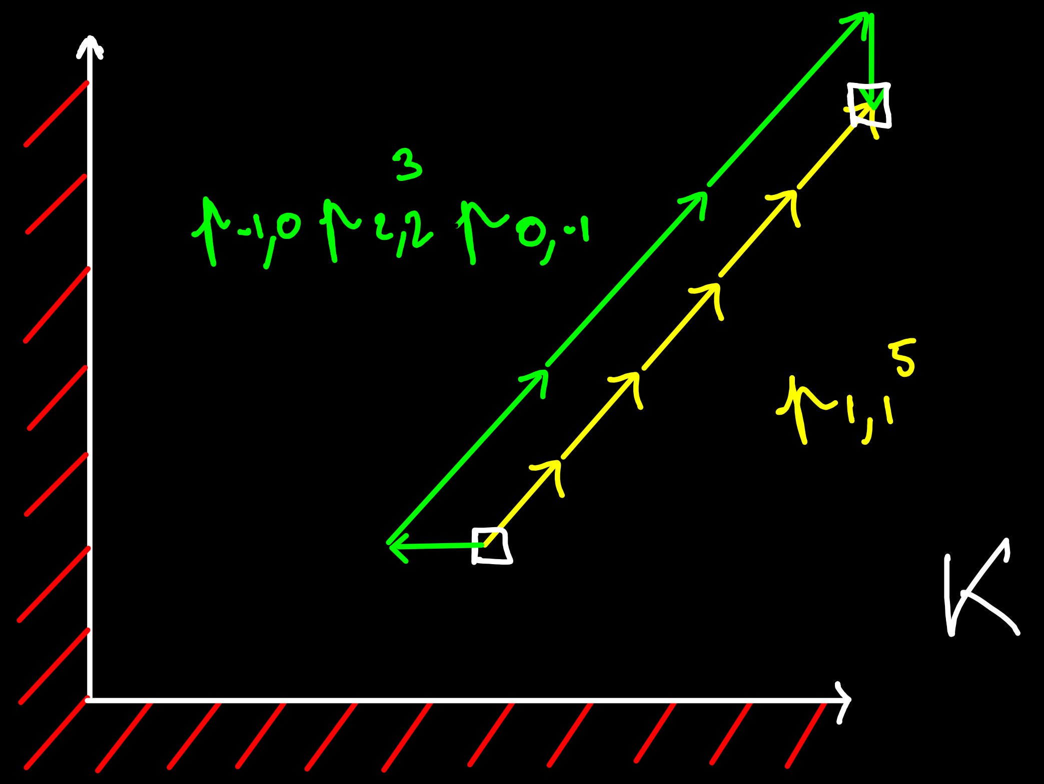
i.e two walks having the same length, start and end point, have the same probability.

ALGEBRAIC IDENTITIES?

Can we find some necessary conditions for being central?

Ex:

$\mathcal{S} =$ 



An assignment of weights $\mathcal{S} \longrightarrow \mathbb{Q}_{>0}$, $s \longmapsto \omega_s$ is a central weighting if

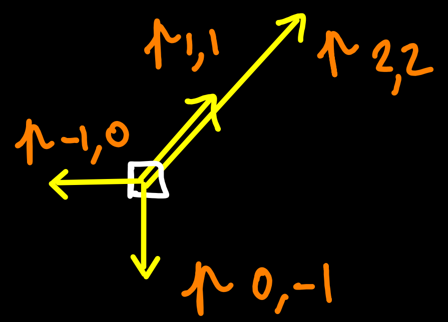
the probability of a walk (conditioned to stay in K) is central,

i.e two walks having the same length, start and end point, have the same probability.

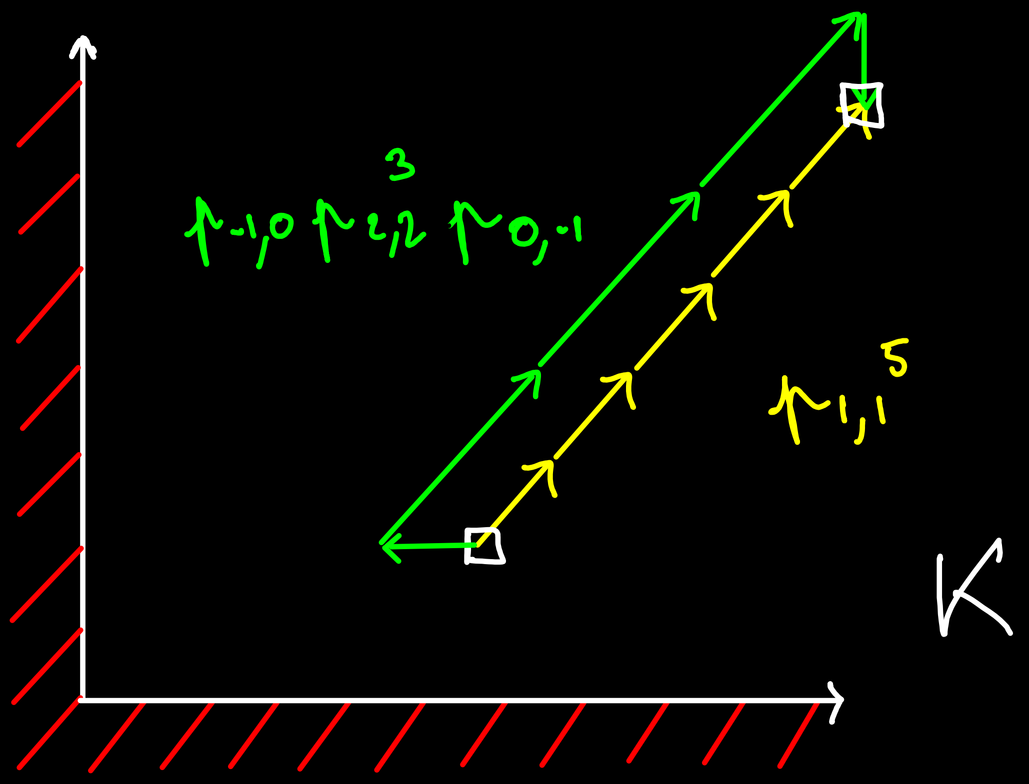
ALGEBRAIC IDENTITIES?

Can we find some necessary conditions for being central?

Ex:

$\mathcal{S} =$ 

central $\Rightarrow$ $p_{1,1}^{5} = p_{-1,0}\, p_{2,2}^{3}\, p_{0,-1}$



An assignment of weights $\begin{array}{l} \mathcal{S} \longrightarrow \mathbb{Q}_{>0} \\ s \longmapsto p_s \end{array}$ is a central weighting if

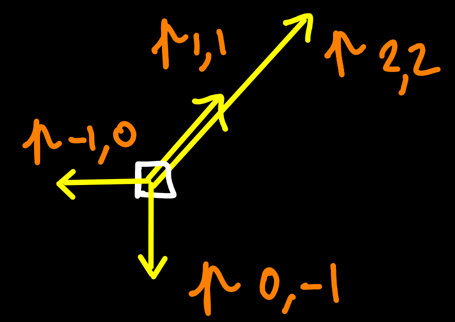
the probability of a walk (conditioned to stay in K) is central,

i.e two walks having the same length, start and end point, have the same probability.

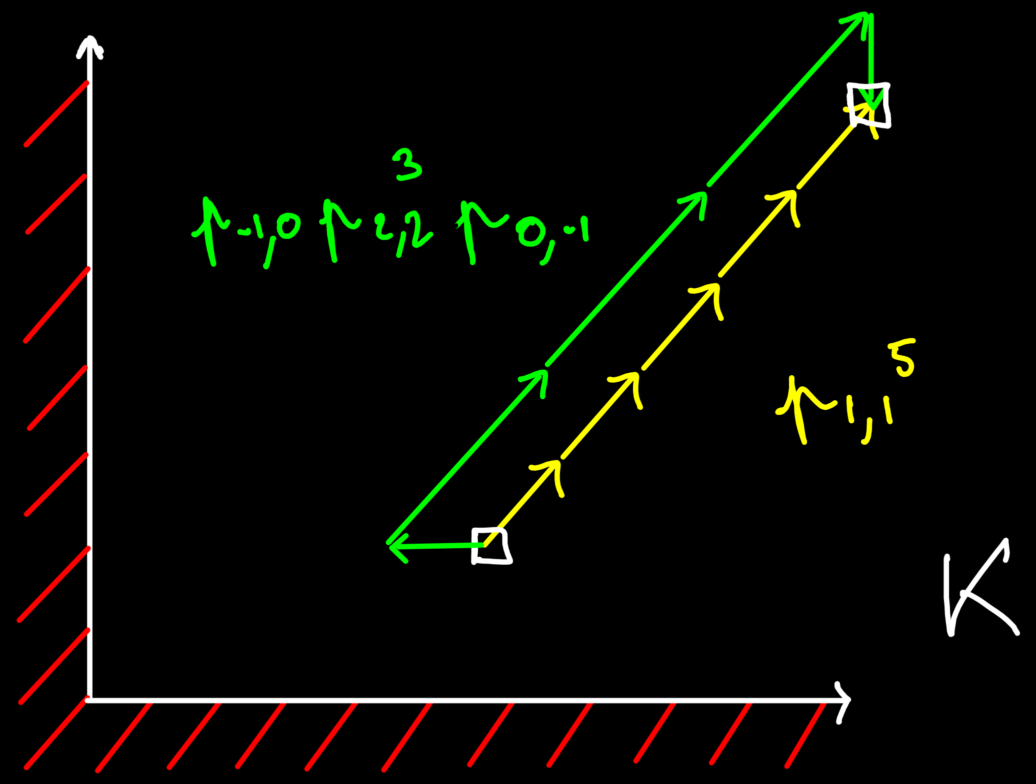
ALGEBRAIC IDENTITIES !

Can we find some necessary conditions for being central?

Ex:

$\mathcal{S} =$ 

central $\Rightarrow$ $r_{1,1}^{5} = r_{-1,0}\, r_{2,2}^{3}\, r_{0,-1}$



Prop. Let w_1, w_2 be two walks in $\mathbb{R}^d$ having the same length, start and end point.

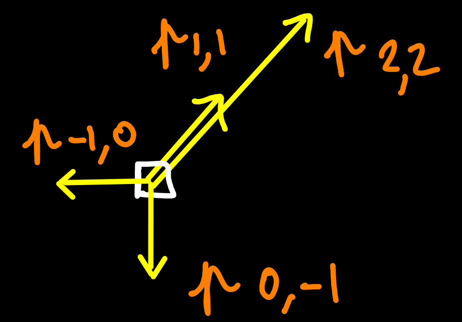
If the weighting is central,

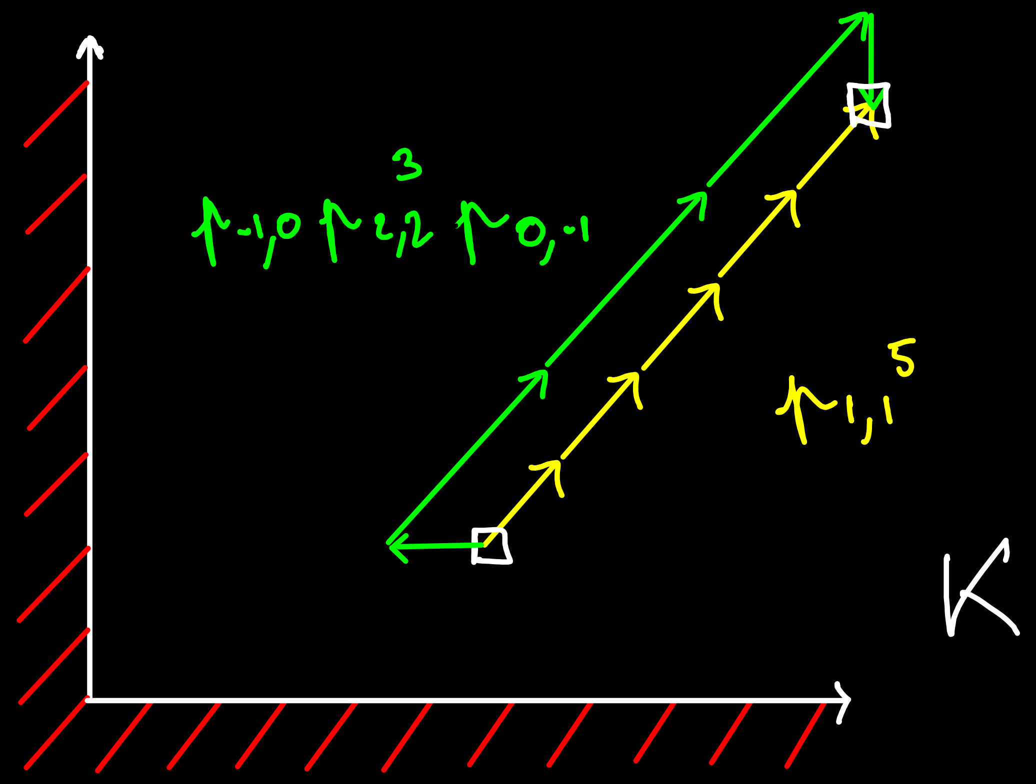
then $\displaystyle\prod_{\substack{s \text{ step in } w_1 \\ (\text{w. multiplicity})}} r_s = \prod_{\substack{s' \text{ step in } w_2 \\ (\text{w. multiplicity})}} r_{s'}$

ALGEBRAIC IDENTITIES!

Can we find some necessary conditions for being central?

Ex:

$\wp =$ 



central $\Rightarrow$ $\gamma_{1,1}^5 = \gamma_{-1,0}\gamma_{2,2}^3\gamma_{0,-1}$

converse?

Prop. Let w_1, w_2 be two walks in $\mathbb{R}^d$ having the same length, start and end point.

If the weighting is central,

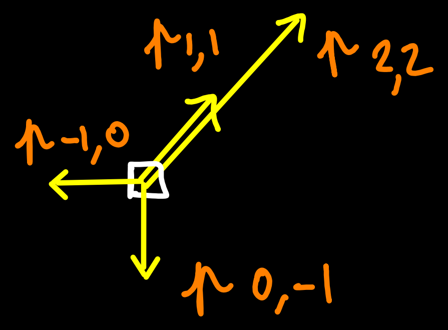
then $\prod_{s \text{ step in } w_1 \text{ (w. multiplicity)}} \gamma_s = \prod_{s' \text{ step in } w_2 \text{ (w. multiplicity)}} \gamma_{s'}$

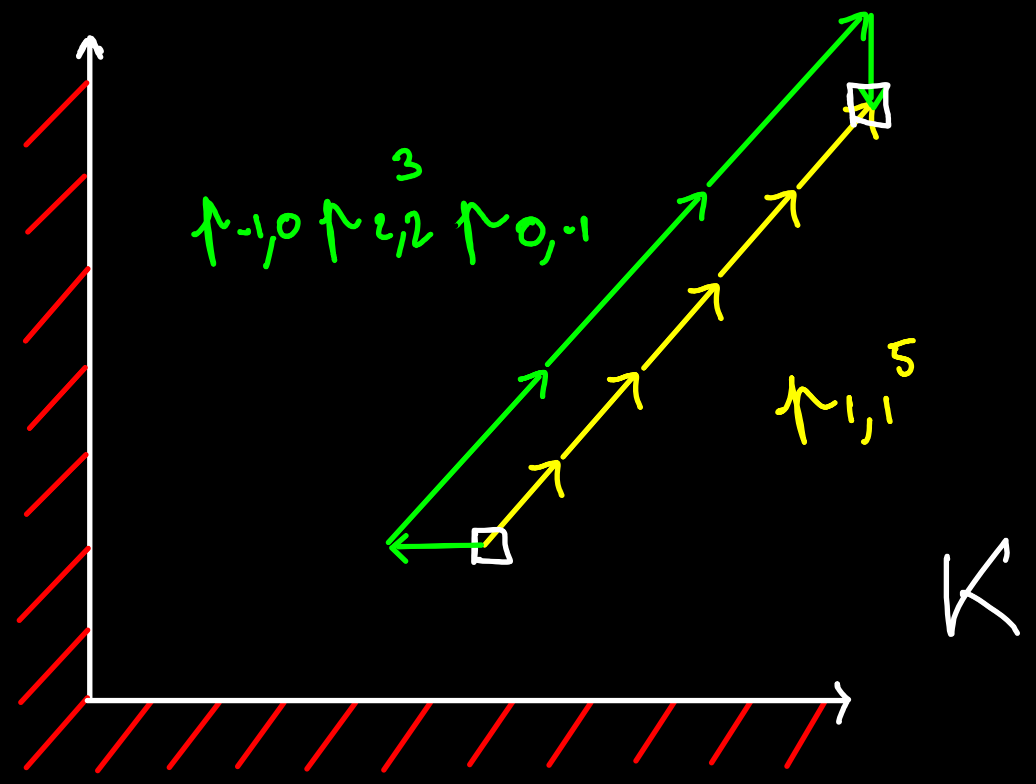
converse?

ALGEBRAIC IDENTITIES!

Can we find some necessary conditions for being central?

Ex:

$\rho =$ 



central $\Rightarrow$ $\rho_{1,1}^{5} = \rho_{-1,0}\,\rho_{2,2}^{3}\,\rho_{0,-1}$

converse? YES!

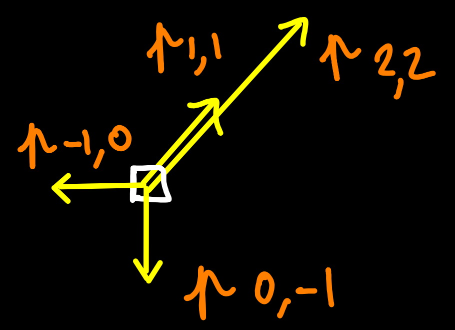
Prop. Let w_1, w_2 be two walks in $\mathbb{R}^d$ having the same length, start and end point.

If the weighting is central,

then $$\prod_{\substack{s \text{ step in } w_1 \\ \text{(w. multiplicity)}}} \rho_s = \prod_{\substack{s' \text{ step in } w_2 \\ \text{(w. multiplicity)}}} \rho_{s'}$$

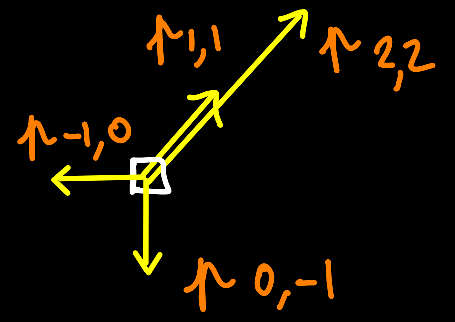
converse? Nye...

CHARACTERIZING CENTRAL WEIGHTINGS

$\varphi =$ 

$$r_{1,1}^{5} = r_{-1,0}\, r_{2,2}^{3}\, r_{0,-1}$$

CHARACTERIZING CENTRAL WEIGHTINGS

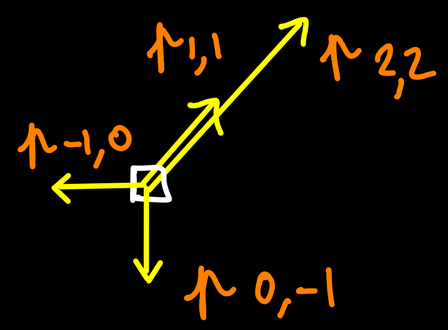
$\mathcal{S} =$ 

$$\rho_{1,1}^{5} = \rho_{-1,0}\,\rho_{2,2}^{3}\,\rho_{0,-1}$$

<u>Trick</u>: Write $\rho_{i,j}$ under the form $\rho_{i,j} = \alpha\beta^{i}\gamma^{j}$:

$$\begin{cases} \rho_{1,1} = \alpha\beta\gamma \\ \rho_{2,2} = \alpha\beta^{2}\gamma^{2} \\ \rho_{-1,0} = \alpha\beta^{-1} \\ \rho_{0,-1} = \alpha\gamma^{-1} \end{cases}$$

CHARACTERIZING CENTRAL WEIGHTINGS

$\mathcal{S} =$ 

$$r_{1,1}^5 = r_{-1,0}\, r_{2,2}^3\, r_{0,-1}$$

<u>Trick</u>: Write $r_{i,j}$ under the form $r_{i,j} = \alpha \beta^i \gamma^j$:

$$\begin{cases} r_{1,1} = \alpha \beta \gamma \\ r_{2,2} = \alpha \beta^2 \gamma^2 \\ r_{-1,0} = \alpha \beta^{-1} \\ r_{0,-1} = \alpha \gamma^{-1} \end{cases}$$

linear algebra

α, β and γ exist : $\alpha = r_{1,1}^2\, r_{2,2}^{-1}$ $\quad \beta = r_{1,1}^{-3}\, r_{2,2}^2\, r_{0,-1}$ $\quad \gamma = r_{1,1}^2\, r_{2,2}^{-1}\, r_{0,-1}^{-1}$

$\Downarrow$

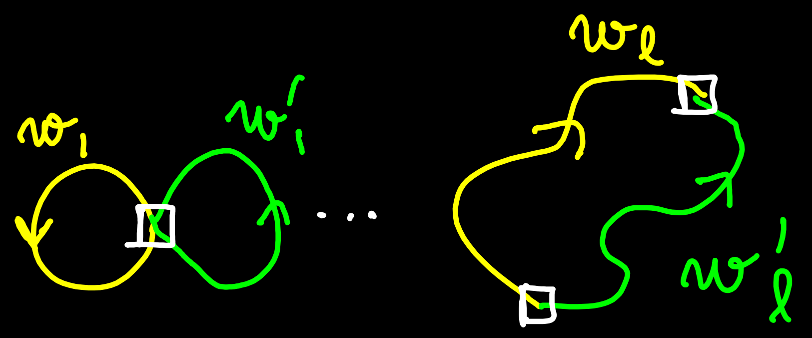
weight of a walk w $= \alpha^{\text{length}(w)}\, \beta^{x(\text{end}) - x(\text{start})}\, \gamma^{y(\text{end}) - y(\text{start})}$ $\Rightarrow$ The weighting is central!

CHARACTERIZING CENTRAL WEIGHTINGS

Theorem (i) Set $\ell = |\mathcal{S}| - d - 1$.

Let $(w_1, w_1'), \dots, (w_\ell, w_\ell')$ be ℓ "independent" pairs of walks in $\mathbb{R}^d$ such that for every $j \in \{1, \dots, \ell\}$, w_j and w_j' have the same length, start and end point

and $$\prod_{\substack{s \text{ step in } w_j \\ \text{(w. multiplicity)}}} \uparrow_s \;=\; \prod_{\substack{s' \text{ step in } w_j' \\ \text{(w. multiplicity)}}} \uparrow_{s'}.$$



$\Downarrow$

(ii) There exist $\alpha, \beta_1, \beta_2, \dots, \beta_d$ such that

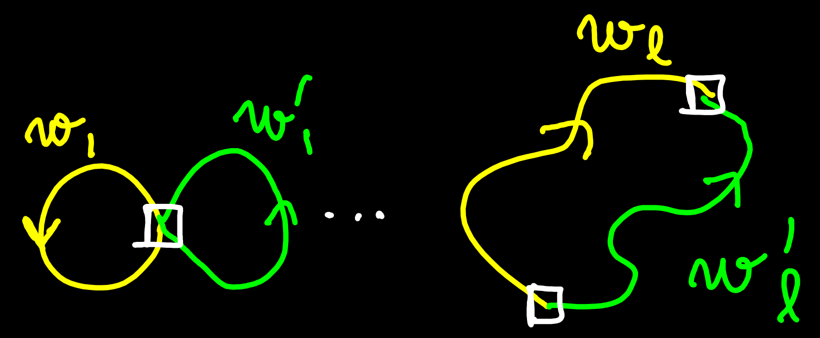
for every step $(s_1, s_2, \dots, s_d) \in \mathcal{S}$, $\uparrow_{s_1, s_2, \dots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$

CHARACTERIZING CENTRAL WEIGHTINGS

Theorem (i) Set $\ell = |\mathcal{S}| - d - 1$.

Let $(w_1, w_1'), \dots, (w_\ell, w_\ell')$ be ℓ "independent" pairs of walks in $\mathbb{R}^d$ such that for every $j \in \{1, \dots, \ell\}$, w_j and w_j' have the same length, start and end point

and
$$\prod_{\substack{s \text{ step in } w_j \\ \text{(w. multiplicity)}}} p_s = \prod_{\substack{s' \text{ step in } w_j' \\ \text{(w. multiplicity)}}} p_{s'}.$$



⇓

(ii) There exist $\alpha, \beta_1, \beta_2, \dots, \beta_d$ such that for every step $(s_1, s_2, \dots, s_d) \in \mathcal{S}$, $p_{s_1, s_2, \dots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$

⇓

(iii) The weighting is central.

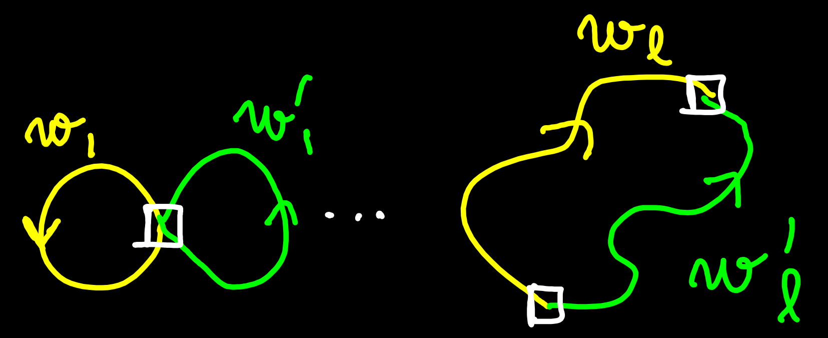
CHARACTERIZING CENTRAL WEIGHTINGS

Theorem

(i) Set $\ell = |\mathcal{S}| - d - 1$.

Let $(w_1, w_1'), \dots, (w_\ell, w_\ell')$ be ℓ "independent" pairs of walks in $\mathbb{R}^d$ such that for every $j \in \{1, \dots, \ell\}$, w_j and w_j' have the same length, start and end point

and $\displaystyle\prod_{\substack{s \text{ step in } w_j \\ (\text{w. multiplicity})}} \uparrow_s = \prod_{\substack{s' \text{ step in } w_j' \\ (\text{w. multiplicity})}} \uparrow_{s'}$.

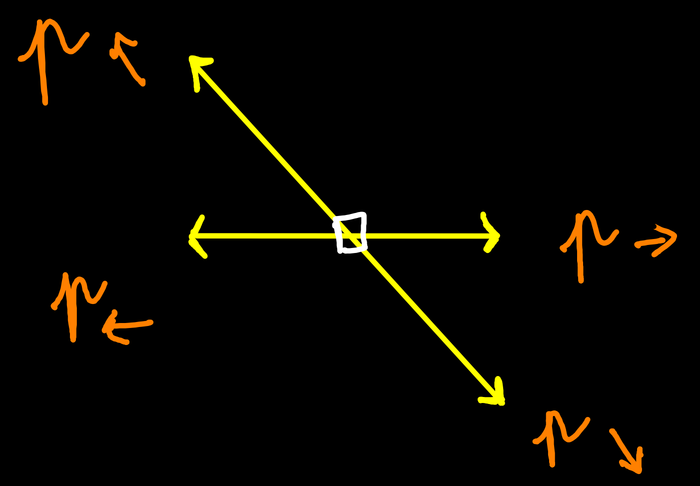


(ii) There exist $\alpha, \beta_1, \beta_2, \dots, \beta_d$ such that for every step $(s_1, s_2, \dots, s_d) \in \mathcal{S}$, $\uparrow_{s_1, s_2, \dots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$

(iii) The weighting is central.

(i) ⇒ (ii) ⇒ (iii) ⇒ (i)

AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



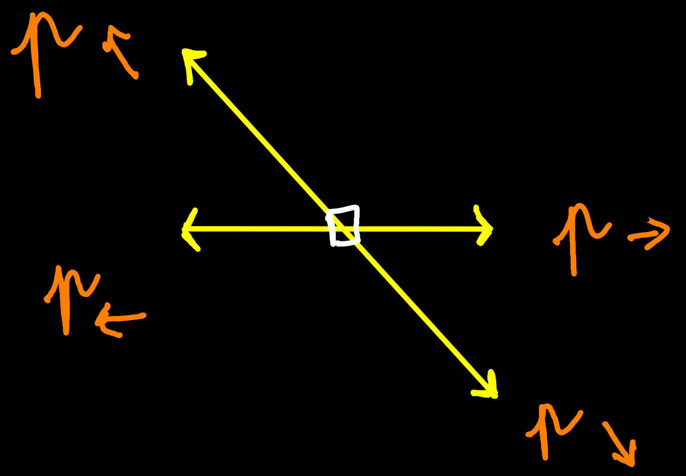
weighted Gouyou-Beauchamps model

Central weighting:

$\nu_{\leftarrow} = \alpha\beta^{-1}$ $\quad \nu_{\rightarrow} = \alpha\beta$

$\nu_{\nwarrow} = \alpha\beta^{-1}\gamma$ $\quad \nu_{\searrow} = \alpha\beta\gamma^{-1}$

AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



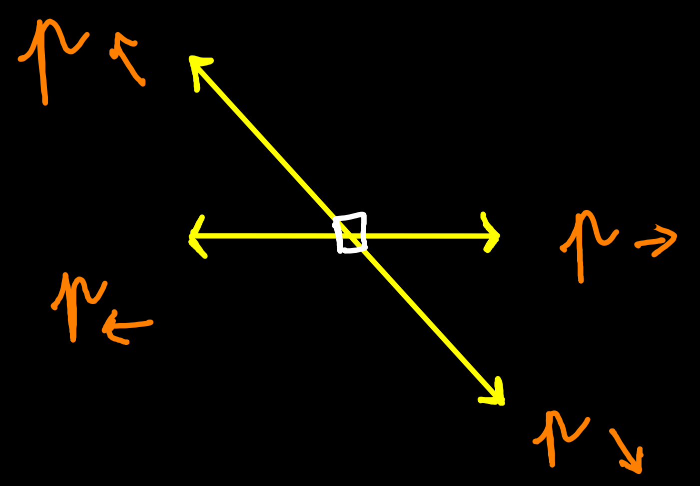
weighted Gouyou-Beauchamps model

Central weighting:

$$\rho_{\leftarrow} = \cancel{\alpha}\, \beta^{-1} \qquad \rho_{\rightarrow} = \cancel{\alpha}\, \beta$$

$$\rho_{\nwarrow} = \cancel{\alpha}\, \beta^{-1}\gamma \qquad \rho_{\searrow} = \cancel{\alpha}\, \beta\, \gamma^{-1}$$

AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



weighted Gouyou-Beauchamps model

Central weighting:

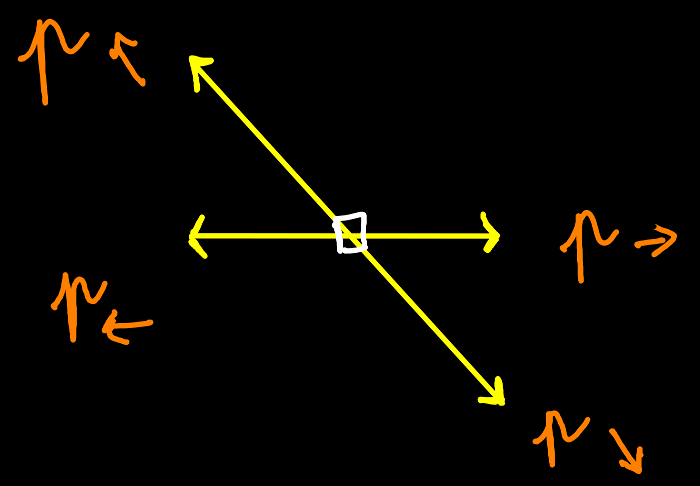
$$\nu_{\leftarrow} = \not{\alpha}\, \beta^{-1} \qquad \nu_{\rightarrow} = \not{\alpha}\, \beta$$

$$\nu_{\nwarrow} = \not{\alpha}\, \beta^{-1}\gamma \qquad \nu_{\searrow} = \not{\alpha}\, \beta\, \gamma^{-1}$$

Theorem weight of GB-walks of length n in the ¼-plane starting at the origin

$$\sim K_{[n]}\, \rho^{n}\, n^{\alpha}$$

AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



weighted Gouyou-Beauchamps model

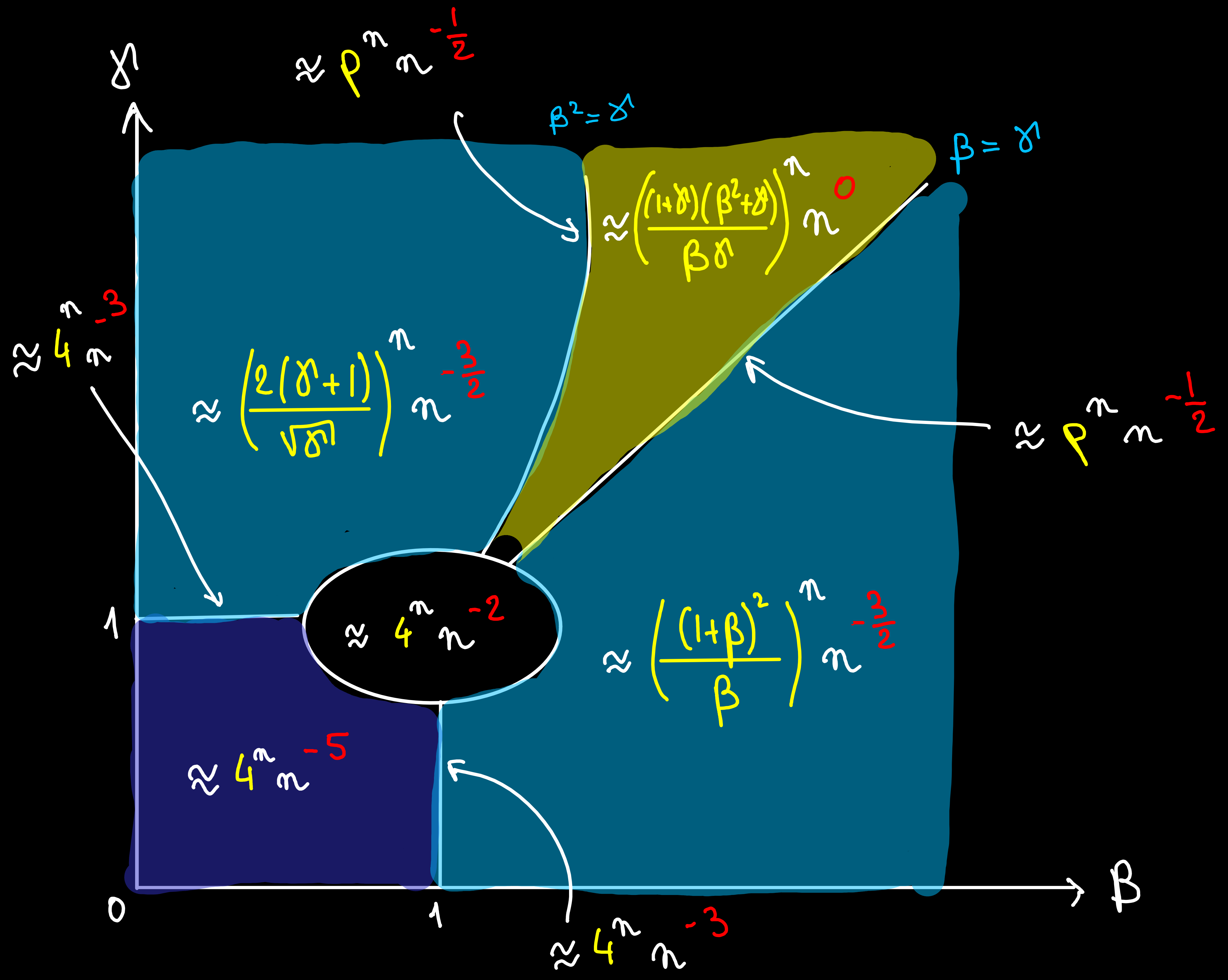
Central weighting:

$\omega_{\leftarrow} = \cancel{\alpha}\, \beta^{-1}$ $\qquad \omega_{\rightarrow} = \cancel{\alpha}\, \beta$

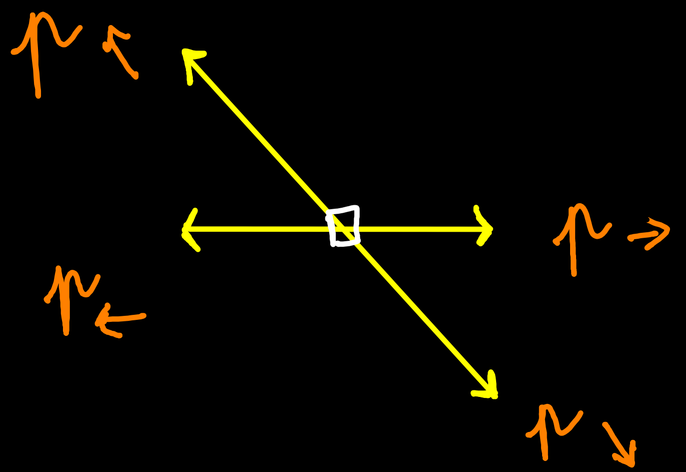
$\omega_{\nwarrow} = \cancel{\alpha}\, \beta^{-1}\gamma$ $\qquad \omega_{\searrow} = \cancel{\alpha}\, \beta\, \gamma^{-1}$

<u>Theorem</u> weight of GB-walks of length n in the $1/4$-plane starting at the origin

$$\sim K_{[n]}\, \rho^n\, n^{\alpha}$$



AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



weighted Gouyou-Beauchamps model

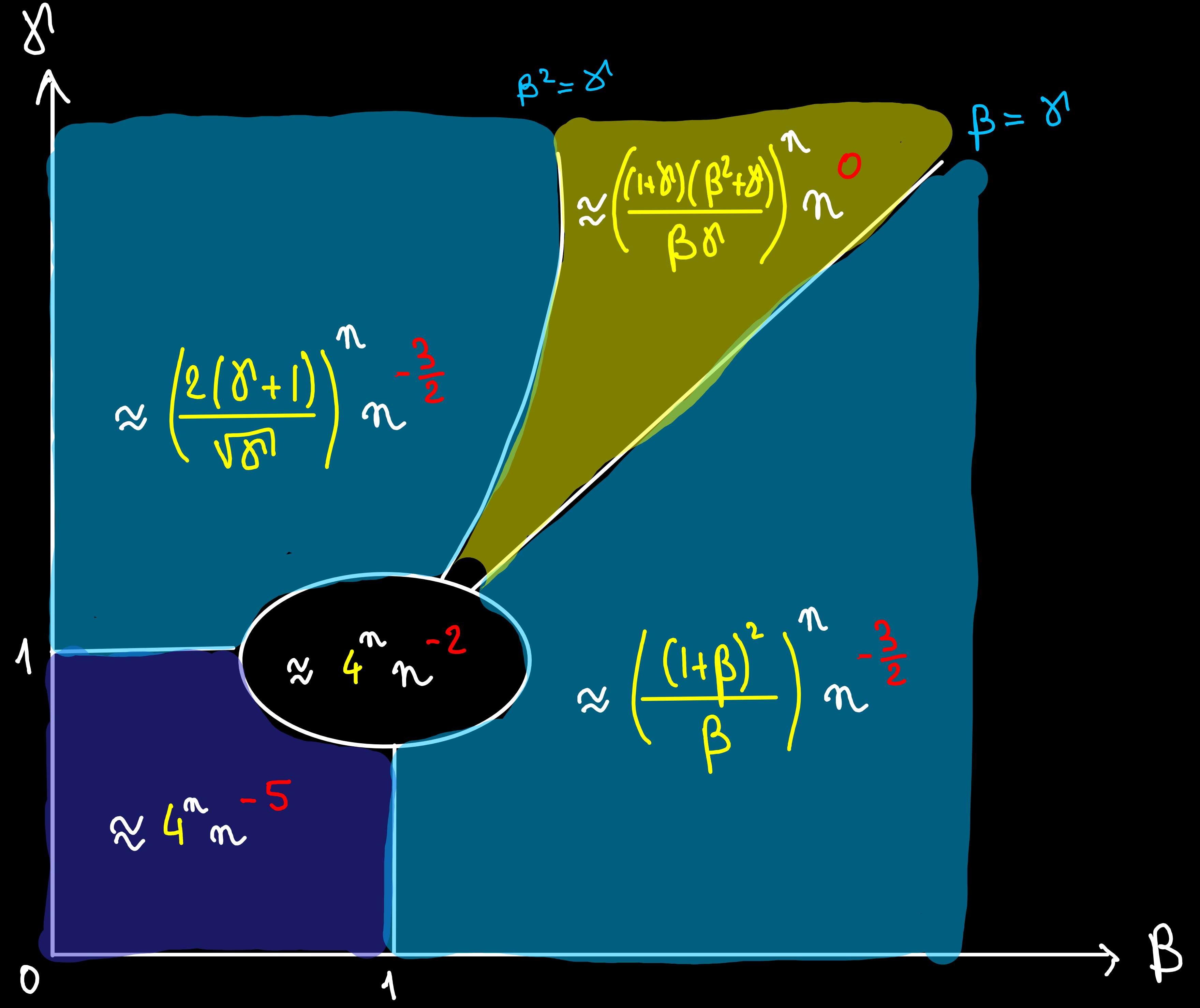
Central weighting:

$\rho_{\leftarrow} = \alpha\!\!\!/ \, \beta^{-1}$ $\quad \rho_{\rightarrow} = \alpha\!\!\!/ \, \beta$

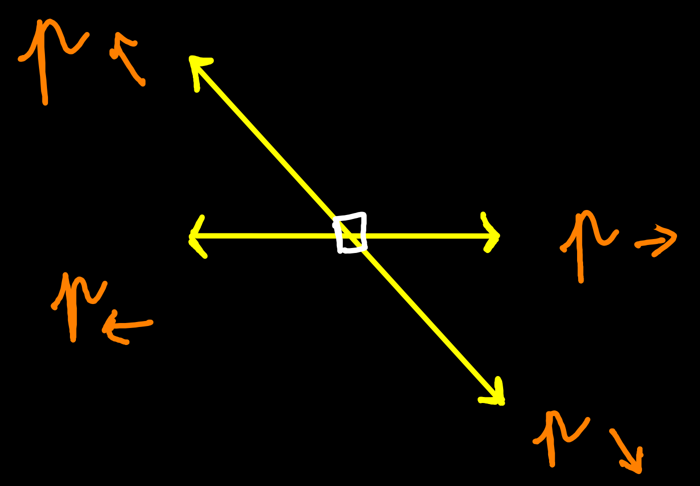
$\rho_{\nwarrow} = \alpha\!\!\!/ \, \beta^{-1}\gamma$ $\quad \rho_{\searrow} = \alpha\!\!\!/ \, \beta\gamma^{-1}$

<u>Theorem</u> weight of GB-walks of length n in the ¼-plane starting at the origin

$$\sim K_{[n]} \, \rho^n \, n^{\alpha}$$



AN ASYMPTOTIC RESULT [C. Melczer Mishna Raschel]



weighted Gouyou-Beauchamps model

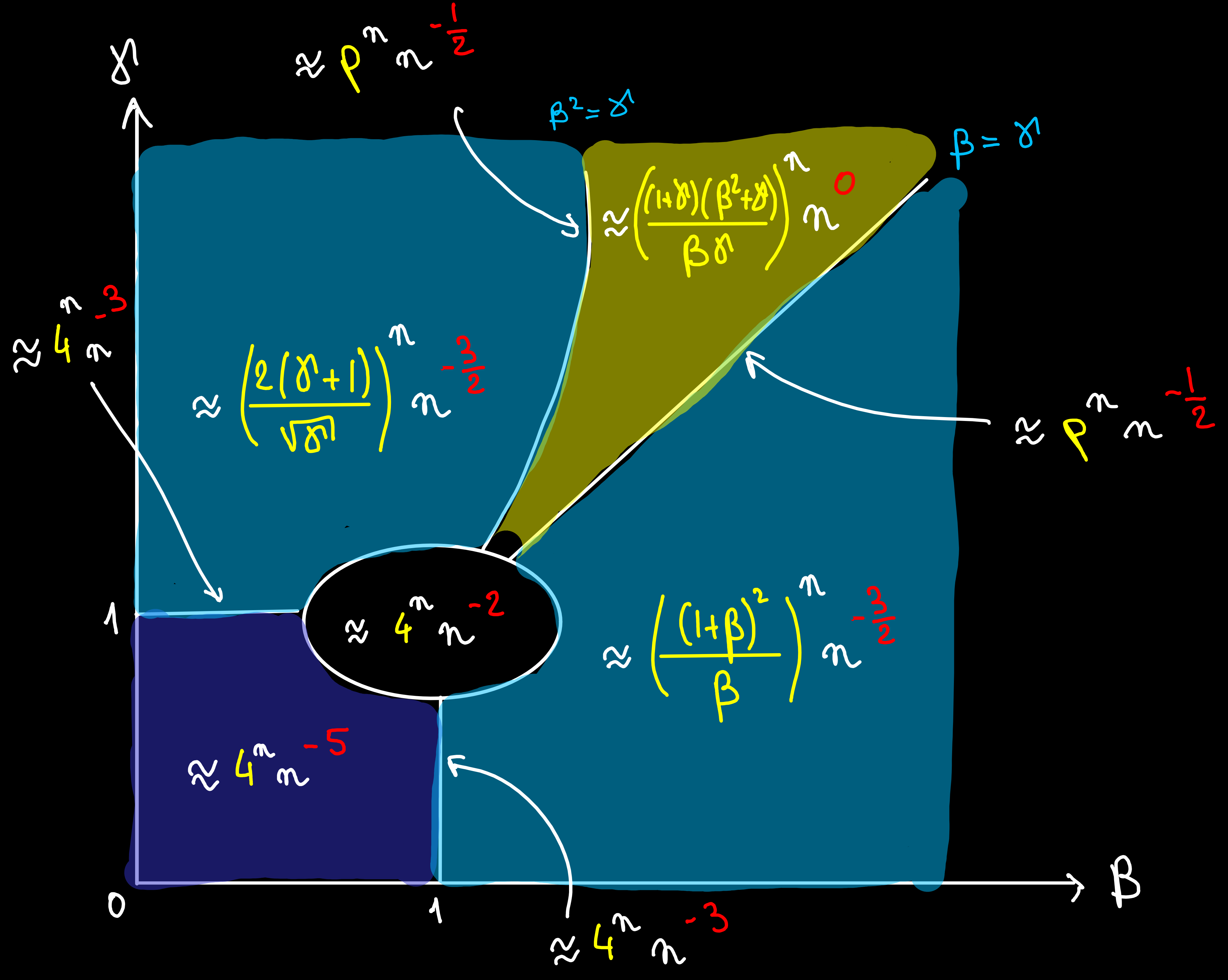
Central weighting:

$\rho_\leftarrow = \alpha\beta^{-1}$ $\rho_\rightarrow = \alpha\beta$

$\rho_\nwarrow = \alpha\beta^{-1}\gamma$ $\rho_\searrow = \alpha\beta\gamma^{-1}$

Theorem weight of GB-walks of length n in the 1/4-plane starting at the origin

$$\sim K_{[n]}\, \rho^n n^\alpha$$



THE CONJECTURE OF GARBIT, MUSTAPHA & RASCHEL

Theorem 1. *For a standard Brownian motion with drift δ, as $n \to \infty$,*

$$f_n := \mathbb{E}_\delta^y[e^{-\langle y^*, B_n\rangle}, \tau_C > n] = \kappa' \cdot n^{-\alpha} \cdot (1 + o(1)), \tag{9}$$

where κ' is some positive constant and α is given in Figure 1.

	$d^* = 0$	$d^* \in \partial K \setminus \{0\}$	$d^* \in K^o$
$x^* = 0$	zero drift $\alpha = p_1/2$	boundary drift $\alpha = 1/2$	interior drift $\alpha = 0$
$x^* \in \partial K^* \setminus \{0\}$	polar boundary drift $\alpha = p_1/2 + 1$ (if $d = 2$)	non-polar exterior drift $\alpha = 3/2$	impossible (orthogonality)
$x^* \in (K^*)^o$	polar interior drift $\alpha = \alpha_1 + 1$	impossible (x^* local minimum)	impossible (x^* local minimum)

FIGURE 1. Values of α in Theorem 1 according to the positions of x^* and d^*. Constants α_1 and p_1 are characteristics of the cone $C = MK$ and are defined in [19]. The indications on the drift refer to the drift δ of $(B_t)_{t\geqslant 0}$.

THE CONJECTURE OF GARBIT, MUSTAPHA & RASCHEL

Framework: K = quarter of plane

$\mathcal{S} \subset \{\text{2D small steps}\}$

central weighting $\mu_{i,j} = \beta^i \gamma^j$

THE CONJECTURE OF GARBIT, MUSTAPHA & RASCHEL

Framework: K = quarter of plane

$\mathcal{S} \subset \{$2D small steps$\}$

central weighting $\rho_{i,j} = \beta^i \gamma^j$

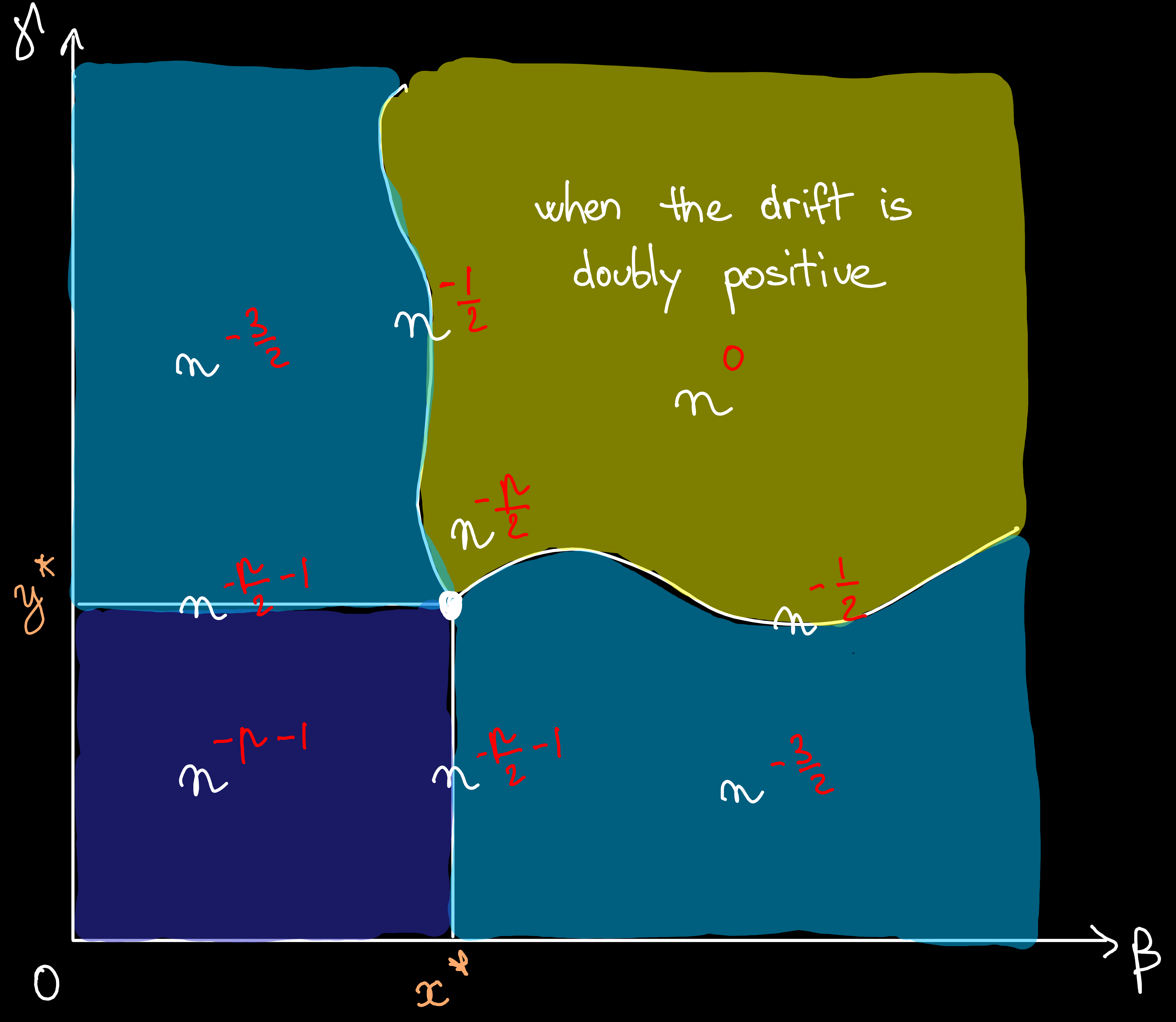
Caption

$$(x^*, y^*) = \operatorname{argmin} S(x,y)$$

$$S(x,y) = \sum_{(i,j)\in\mathcal{S}} \rho_{i,j}\, x^i y^j$$

$$p = \pi / \arccos(-c)$$

$$c = \frac{\partial^2 S}{\partial x \partial y}(x^*,y^*) \Big/ \sqrt{\frac{\partial^2 S}{\partial x^2}(x^*,y^*)\frac{\partial^2 S}{\partial y^2}(x^*,y^*)}$$



GENERATING FUNCTIONS

Let $Q(x_1, \dots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \dots, i_d)$ and $Q_p(x_1, \dots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$.

GENERATING FUNCTIONS

Let $Q(x_1, \ldots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \ldots, i_d)$

and $Q_p(x_1, \ldots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$.

Consequence 1: $Q_p(x_1, \ldots, x_d; z)$ D-finite $\Leftrightarrow$ $Q(x_1, \ldots, x_d; z)$ D-finite (if the weights are central & rational)

GENERATING FUNCTIONS

Let $Q(x_1, \ldots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \ldots, i_d)$

and $Q_p(x_1, \ldots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$.

Consequence 1: $Q_p(x_1, \ldots, x_d; z)$ D-finite $\Leftrightarrow$ $Q(x_1, \ldots, x_d; z)$ D-finite (if the weights are central & rational)

Consequence 2: weight of excursions of length n $=$ $\alpha^n \times$ number of excursions of length n

GENERATING FUNCTIONS

Let $Q(x_1, \ldots, x_d; z)$ be the <u>unweighted</u> generating function of walks in K starting at the origin and ending at $(i_1, \ldots, i_d)$

and $Q_p(x_1, \ldots, x_d; z)$ be the weighted analogue.

<u>Prop</u> If the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$.

? What about the converse? ? ?

GENERATING FUNCTIONS

Let $Q(x_1, \dots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \dots, i_d)$

and $Q_p(x_1, \dots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$.

?. What about the converse? ? ? ?

Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

GENERATING FUNCTIONS

Let $Q(x_1, \dots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \dots, i_d)$

and $Q_p(x_1, \dots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$.

? What about the converse? ? ? ?

Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

GENERATING FUNCTIONS

Let $Q(x_1, \ldots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \ldots, i_d)$

and $Q_p(x_1, \ldots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$.

?? What about the converse? ? ?

Conjecture If $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

GENERATING FUNCTIONS

Let $Q(x_1, \dots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \dots, i_d)$

and $Q_p(x_1, \dots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$.

? What about the converse? ? ?

Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

GENERATING FUNCTIONS

Let $Q(x_1, \ldots, x_d; z)$ be the <u>unweighted</u> generating function of walks in K starting at the origin and ending at $(i_1, \ldots, i_d)$

and $Q_p(x_1, \ldots, x_d; z)$ be the weighted analogue.

<u>Prop</u> If the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$.

?. What about the converse?

<u>Conjecture</u> If $Q_p(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \ldots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

GENERATING FUNCTIONS

Let $Q(x_1, \dots, x_d; z)$ be the unweighted generating function of walks in K starting at the origin and ending at $(i_1, \dots, i_d)$

and $Q_p(x_1, \dots, x_d; z)$ be the weighted analogue.

Prop If the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$;

then $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$.

? What about the converse? ? ?

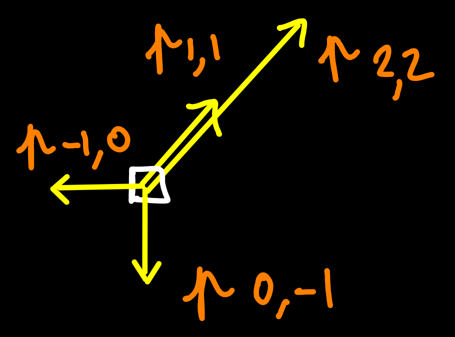
Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

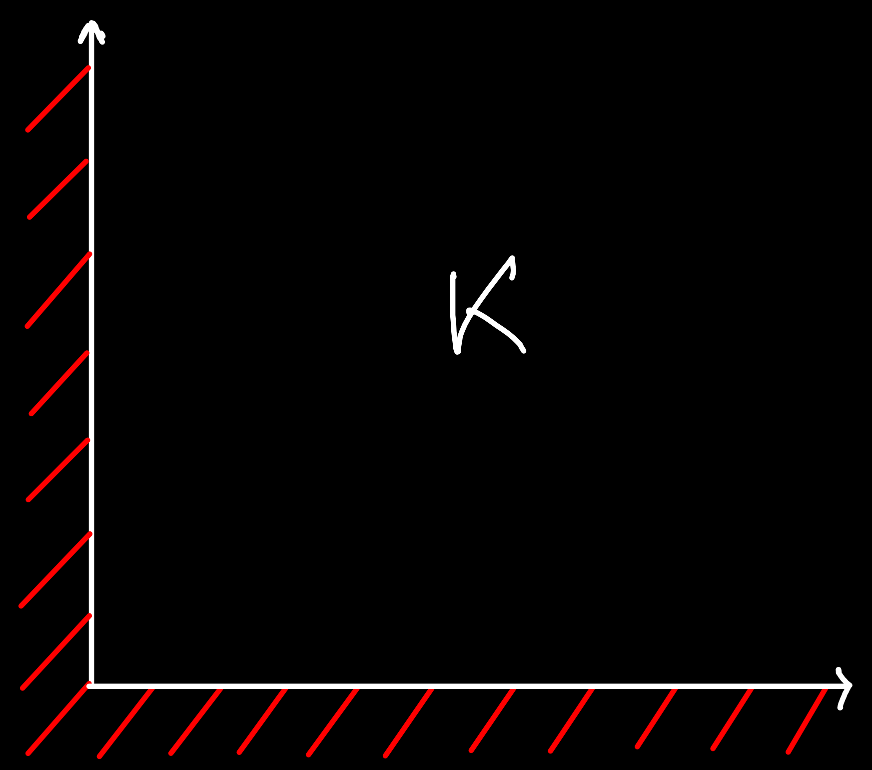
THE CONJECTURE

<u>Conjecture</u> If $Q_\rho(x_1,\dots,x_d;z) = Q(\beta_1 x_1,\dots,\beta_d x_d;\alpha z)$

then the weighting is central: $\rho_{s_1,s_2,\dots,s_d} = \alpha\, \beta_1^{s_1}\beta_2^{s_2}\cdots\beta_d^{s_d}$.

$\mathcal{S} =$ 

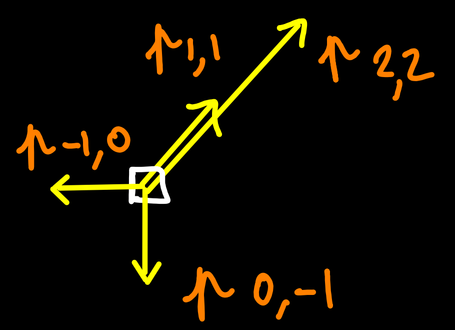
$$Q_\rho(x,y;z) = Q(\beta x, \gamma y; \alpha z)$$



THE CONJECTURE

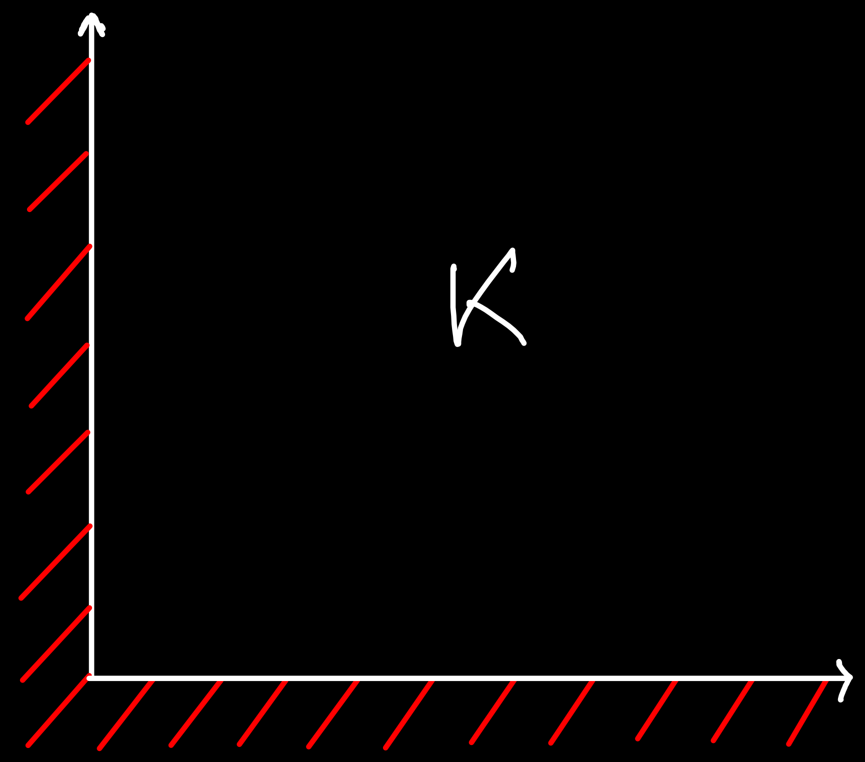
Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

$\mathcal{S} =$ 

$$Q_p(x, y; z) = Q(\beta x, \gamma y; \alpha z)$$

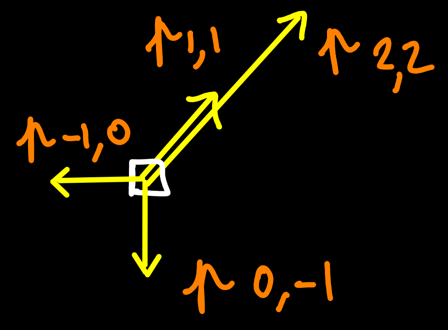
Coefficient of xyz: $p_{1,1} = \alpha \beta \gamma$



THE CONJECTURE

Conjecture If $Q_p(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

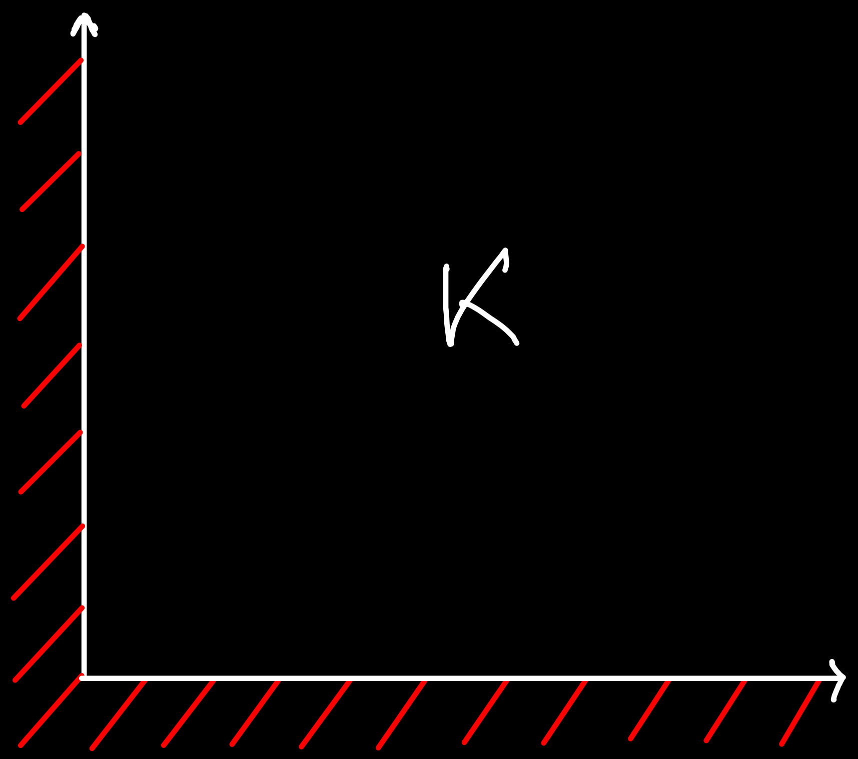
then the weighting is central: $p_{s_1, s_2, \dots, s_d} = \alpha \, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

$\mathcal{S} =$ 

$$\boxed{Q_p(x, y; z) = Q(\beta x, \gamma y; \alpha z)}$$

Coefficient of xyz: $p_{1,1} = \alpha \beta \gamma$

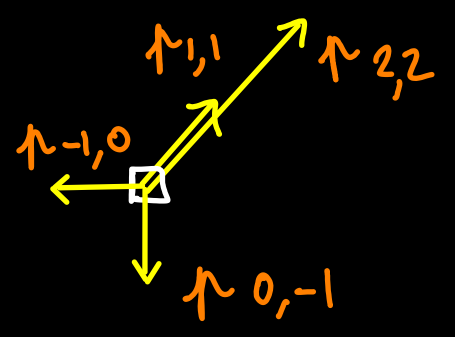
Coefficient of $x^2 y^2 z$: $p_{2,2} = \alpha \beta^2 \gamma^2$

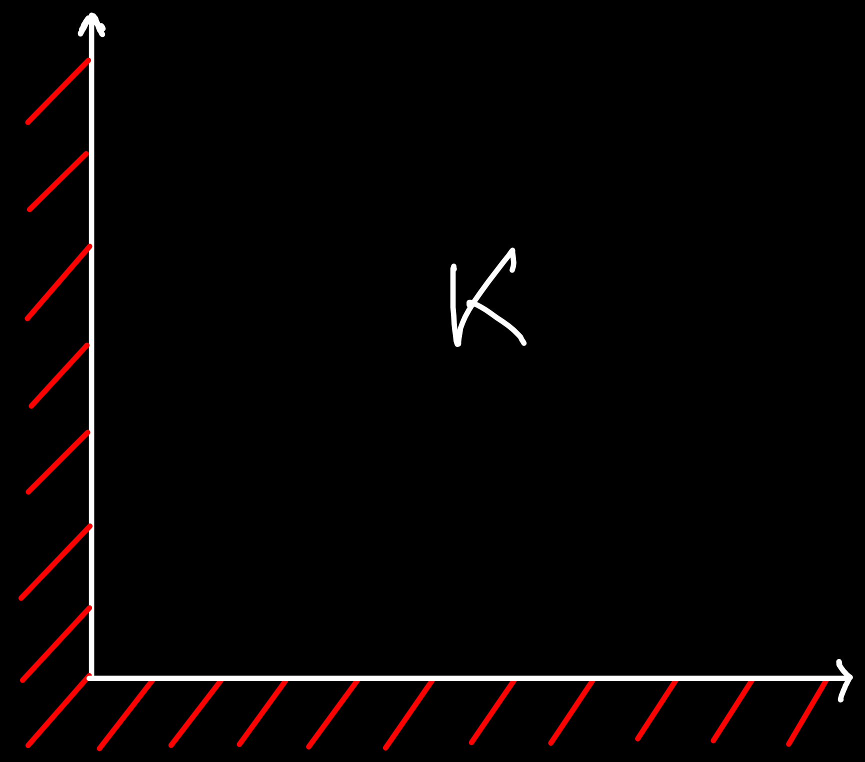


THE CONJECTURE

<u>Conjecture</u> If $Q_\rho(x_1,\dots,x_d;z) = Q(\beta_1 x_1,\dots,\beta_d x_d;\alpha z)$

then the weighting is central: $\rho_{s_1,s_2,\dots,s_d} = \alpha\,\beta_1^{s_1}\beta_2^{s_2}\cdots\beta_d^{s_d}$.

$\mathcal{S} =$ 



$$Q_\rho(x,y;z) = Q(\beta x,\gamma y;\alpha z)$$

Coefficient of xyz: $\rho_{1,1} = \alpha\beta\gamma$

Coefficient of x^2y^2z: $\rho_{2,2} = \alpha\beta^2\gamma^2$

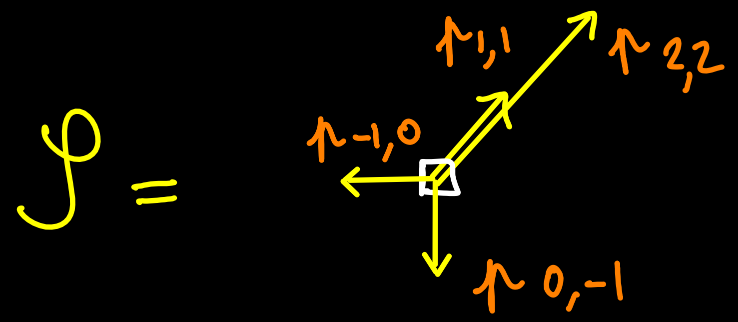
Coefficient of xz^2: $\rho_{1,1}\rho_{0,-1} = \alpha^2\beta$

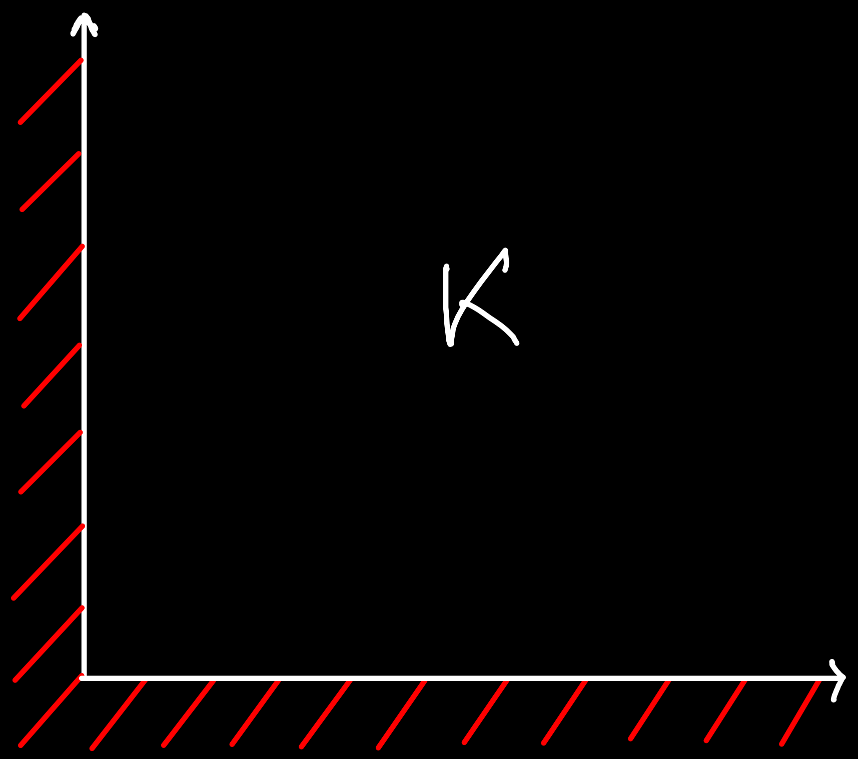
$\Rightarrow \rho_{0,-1} = \alpha\gamma^{-1}$

THE CONJECTURE

Conjecture If $Q_\rho(x_1,\dots,x_d;z) = Q(\beta_1 x_1,\dots,\beta_d x_d;\alpha z)$

then the weighting is central: $\rho_{s_1,s_2,\dots,s_d} = \alpha\,\beta_1^{s_1}\beta_2^{s_2}\cdots\beta_d^{s_d}$.





$$Q_\rho(x,y;z) = Q(\beta x,\gamma y;\alpha z)$$

Coefficient of xyz : $\rho_{1,1} = \alpha\beta\gamma$

Coefficient of x^2y^2z : $\rho_{2,2} = \alpha\beta^2\gamma^2$

Coefficient of xz^2 : $\rho_{1,1}\rho_{0,-1} = \alpha^2\beta$

$\Rightarrow \rho_{0,-1} = \alpha\gamma^{-1}$

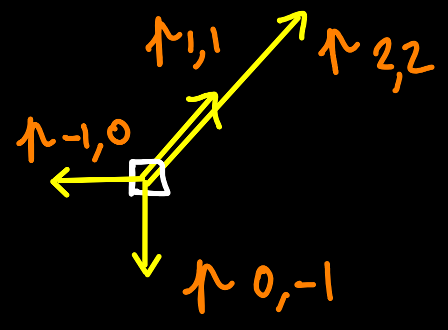
Coefficient of yz^2 : $\rho_{-1,0} = \alpha\beta^{-1}$

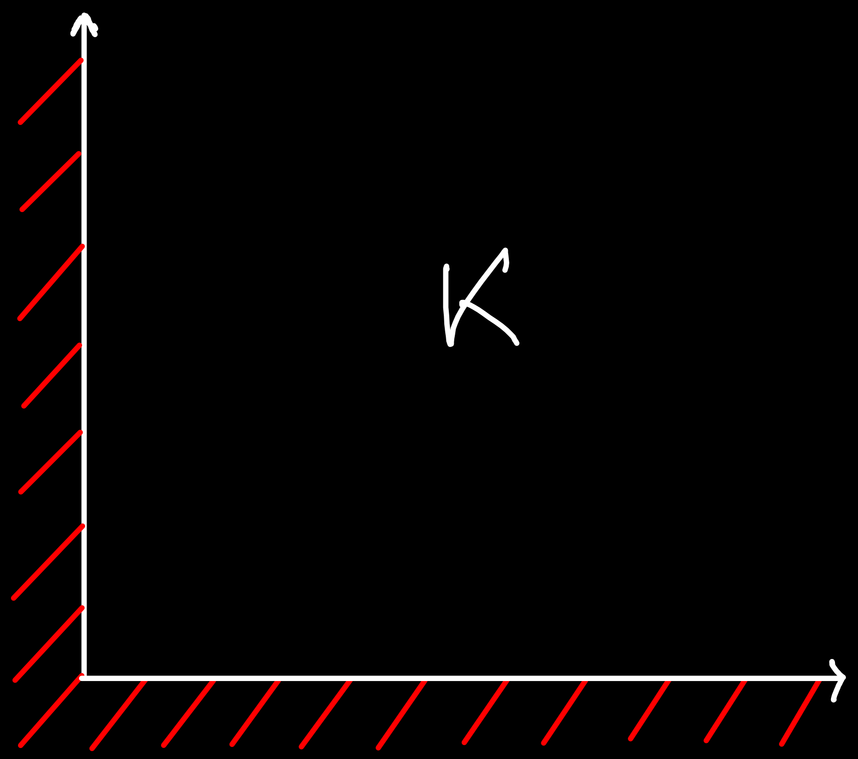
THE CONJECTURE

<u>Conjecture</u> If $Q_\rho(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$

then the weighting is central: $\rho_{s_1, s_2, \ldots, s_d} = \alpha\, \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

$\mathcal{S} =$





$$Q_\rho(x, y; z) = Q(\beta x, \gamma y; \alpha z)$$

Coefficient of xyz : $\rho_{1,1} = \alpha \beta \gamma$ ✓

Coefficient of $x^2 y^2 z$: $\rho_{2,2} = \alpha \beta^2 \gamma^2$ ✓

Coefficient of $x z^2$: $\rho_{1,1}\, \rho_{0,-1} = \alpha^2 \beta$

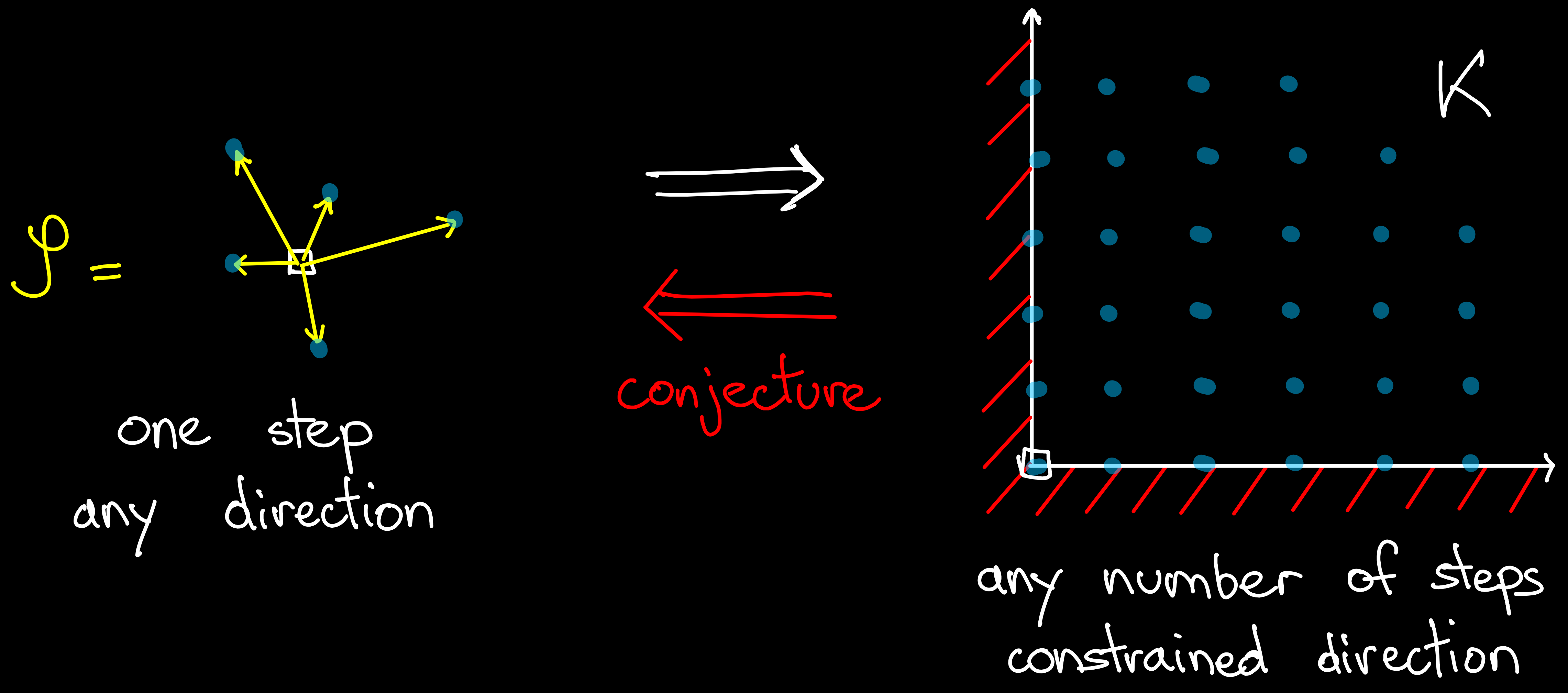
$\Rightarrow \rho_{0,-1} = \alpha \gamma^{-1}$ ✓

Coefficient of $y z^2$: $\rho_{-1,0} = \alpha \beta^{-1}$ ✓

THE CONJECTURE

<u>Conjecture</u> If $Q_\rho(x_1,\dots,x_d;z) = Q(\beta_1 x_1,\dots,\beta_d x_d;\alpha z)$

then the weighting is central: $\rho_{s_1,s_2,\dots,s_d} = \alpha\,\beta_1^{s_1}\beta_2^{s_2}\cdots\beta_d^{s_d}$.

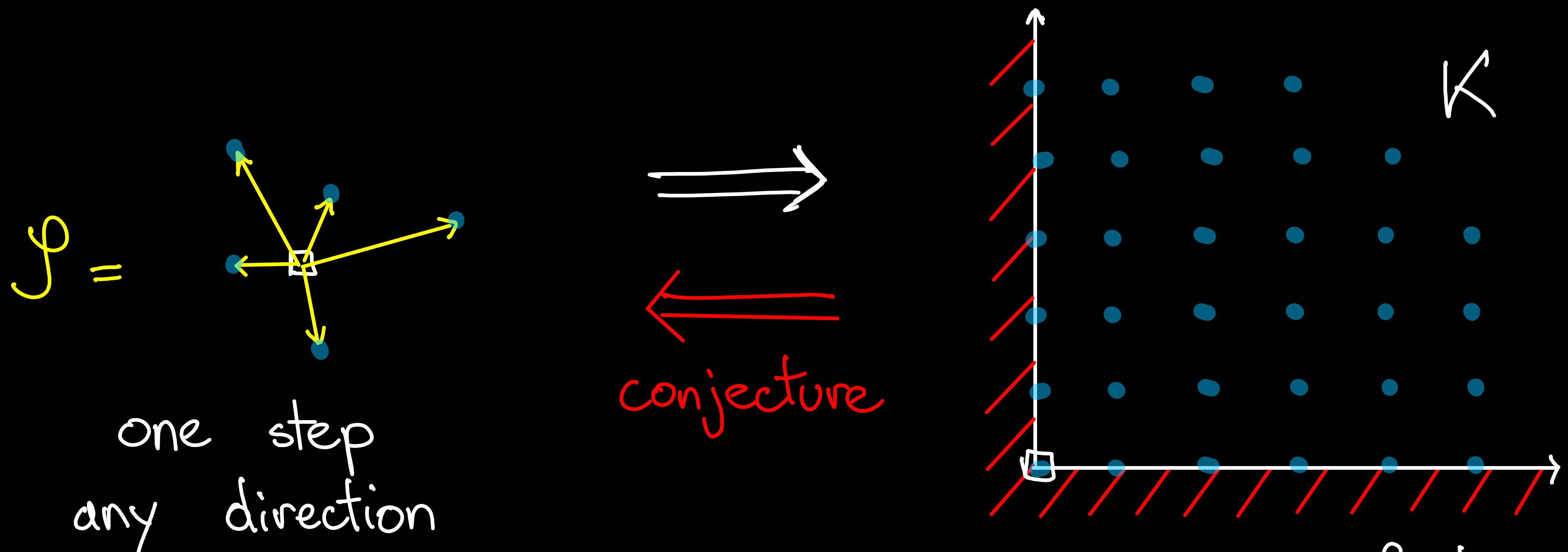


THE CONJECTURE

<ins>Conjecture</ins> If $Q_\rho(x_1, \dots, x_d; z) = Q(\beta_1 x_1, \dots, \beta_d x_d; \alpha z)$

then the weighting is central: $\rho_{s_1, s_2, \dots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

True for:
- small 2D-models
- models where there exists a step in $\mathring{K}$ [Tarrago] ...



one step
any direction

any number of steps
constrained direction

THE CONJECTURE

Conjecture If $Q_{\mu}(x_1, \ldots, x_d; z) = Q(\beta_1 x_1, \ldots, \beta_d x_d; \alpha z)$

then the weighting is central: $\mu_{s_1, s_2, \ldots, s_d} = \alpha \beta_1^{s_1} \beta_2^{s_2} \cdots \beta_d^{s_d}$.

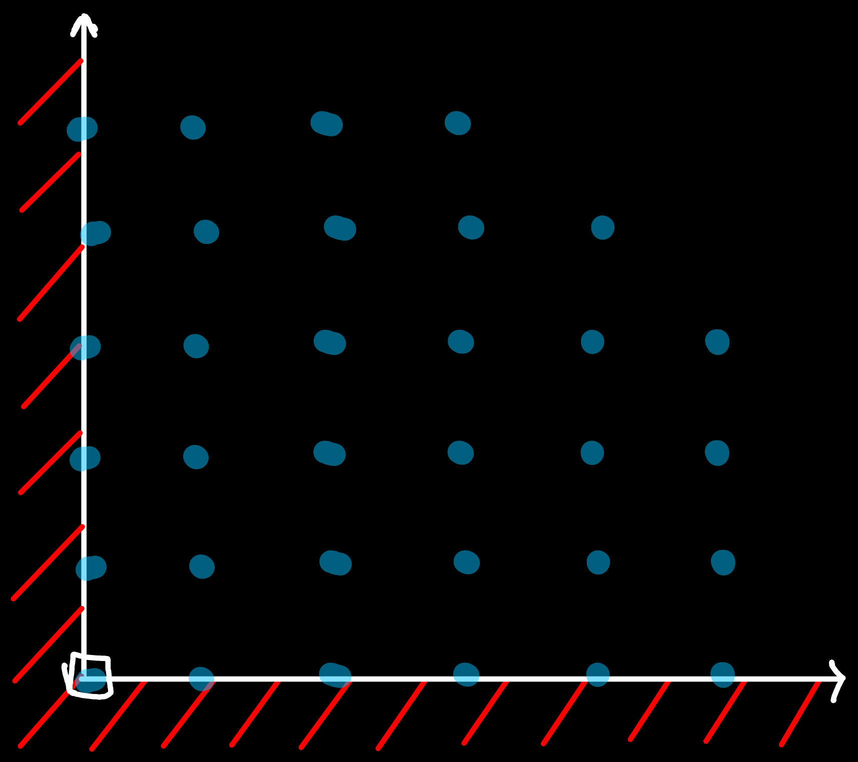
A monstrosity

$$\mathcal{S} = \left\{ \begin{array}{l} (1,0,0,0),(0,1,0,0), \\ (0,0,1,0),(0,0,0,1), \\ (1,1,-1,0),(1,1,0,-1), \\ (1,-1,1,0),(1,0,1,-1), \\ (1,-1,0,1),(1,0,-1,1), \\ (-1,1,1,0),(0,1,1,-1), \\ (-1,0,1,1),(0,-1,1,1), \\ (-1,1,0,1),(0,1,-1,1) \end{array} \right\}$$

$\Longrightarrow$

$\Longleftarrow$ conjecture

$K = \mathbb{R}^4_{\geq 0}$



CONCLUSION

→ good framework to understand transitions for asymptotic behaviors

→ an accessible conjecture on lattice walks ...

CONCLUSION

→ good framework to understand transitions for asymptotic behaviors

→ an accessible conjecture on lattice walks …
which may require algebra, analysis or combinatorics?

CONCLUSION

→ good framework to understand transitions for asymptotic behaviors

→ an accessible conjecture on lattice ✓ walks ✓ ...
which may require algebra ✓, analysis ✓ or combinatorics ✓ ?

BINGO!

CONCLUSION

→ good framework to understand transitions for asymptotic behaviors

→ an accessible conjecture on lattice ✓ walks ✓ ...
which may require algebra ✓, analysis ✓ or combinatorics ✓?

BINGO!