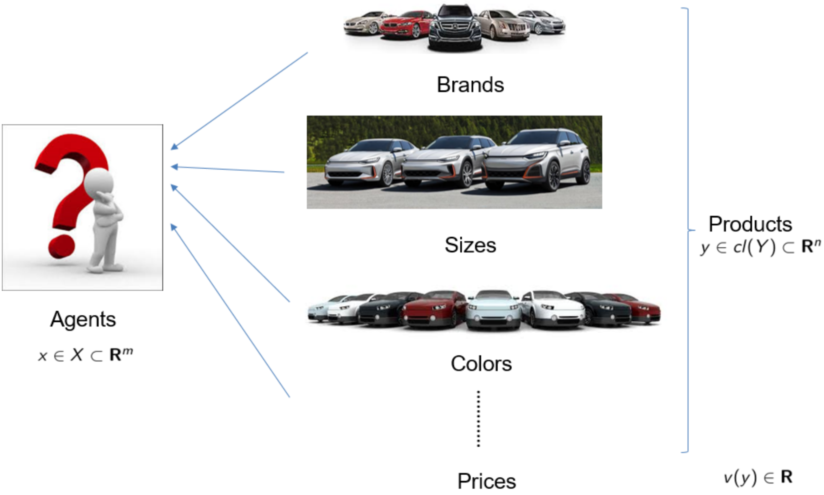
ON CONCAVITY OF THE PRINCIPAL'S PROFIT MAXIMIZATION FACING AGENTS WHO RESPOND NONLINEARLY TO PRICES

Shuangjian Zhang
This is joint work with my supervisor Robert J. McCann

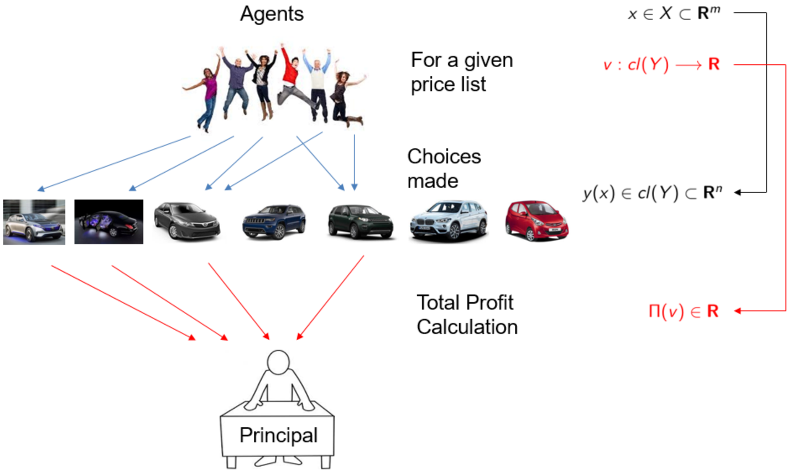
University of Toronto

April 11, 2017

Agents' problem



Principal's problem



Questions for Principal:

Can principal's total profit achieve its maximum by manipulating the price lists?

Questions for Principal:

Can principal's total profit achieve its maximum by manipulating the price lists?

If the maximizers exist, under what conditions will that be unique?

Questions for Principal:

Can principal's total profit achieve its maximum by manipulating the price lists?

If the maximizers exist, under what conditions will that be unique?

What is the structure of profit functional?

Main Result:

Identified certain hypotheses under which this maximization problem is **strictly concave** on a **convex** function space, where the maximizer is unique.

Mathematical framework of principal-agent problem on nonlinear pricing

monopolist(principal): produces and sells products $y \in cl(Y)$, at price $v(y) \in \mathbf{R}$, to be designed, where $Y \subset \mathbf{R}^n$.

Mathematical framework of principal-agent problem on nonlinear pricing

monopolist(principal): produces and sells products $y \in cl(Y)$, at price $v(y) \in \mathbf{R}$, to be designed, where $Y \subset \mathbf{R}^n$.

consumers(agents): $x \in X$, where $X \subset \mathbf{R}^m$, buys one of those products which maximize his benefit

$$u(x) := \max_{y \in cl(Y)} G(x, y, v(y)) \tag{1}$$

given benefit function $G(x, y, z) : X \times cl(Y) \times cl(Z) \longrightarrow \mathbf{R}$, denotes the benefit to agent x when he chooses product y at price z, where $Z = (\underline{z}, \bar{z}) \subset \bar{\mathbf{R}}$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$.

Mathematical framework of principal-agent problem on nonlinear pricing

monopolist(principal): produces and sells products $y \in cl(Y)$, at price $v(y) \in \mathbf{R}$, to be designed, where $Y \subset \mathbf{R}^n$.

consumers(agents): $x \in X$, where $X \subset \mathbf{R}^m$, buys one of those products which maximize his benefit

$$u(x) := \max_{y \in cl(Y)} G(x, y, v(y)) \tag{1}$$

given benefit function $G(x, y, z) : X \times cl(Y) \times cl(Z) \longrightarrow \mathbf{R}$, denotes the benefit to agent x when he chooses product y at price z, where $Z = (\underline{z}, \bar{z}) \subset \bar{\mathbf{R}}$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$.

e.g. $G(x, y, z) :=< x, y > -\sqrt{z}$, where $X, Y \subset \mathbf{R}^n$, $Z \subset [0, \infty]$ and $<,>$ is the Euclidean inner product in $\mathbf{R}^n$.

Principal-agent framework on nonlinear pricing (cont.)

distribution of agents: $d\mu(x)$ on X.

Principal-agent framework on nonlinear pricing (cont.)

distribution of agents: $d\mu(x)$ on X.

profit gained by monopolist:
$\Pi(v, y) = \int_X \pi(x, y(x), v(y(x)))d\mu(x)$,
given profit function $\pi(x, y, z) : X \times cl(Y) \times cl(Z) \longrightarrow \mathbf{R}$, which represents profit to the principal who sells product y to agent x at price z.

where $y(x)$ denotes that product y which agent x chooses to buy, while the function v represents a price list.
e.g. $\pi(x, y, z) = z - a(y)$, where $a(y)$ denotes the principal's cost of manufacturing y.

Principal-agent framework on nonlinear pricing (cont.)

distribution of agents: $d\mu(x)$ on X.

profit gained by monopolist:
$\Pi(v, y) = \int_X \pi(x, y(x), v(y(x)))d\mu(x)$,
given profit function $\pi(x, y, z) : X \times cl(Y) \times cl(Z) \longrightarrow \mathbf{R}$, which represents profit to the principal who sells product y to agent x at price z.

where $y(x)$ denotes that product y which agent x chooses to buy, while the function v represents a price list.
e.g. $\pi(x, y, z) = z - a(y)$, where $a(y)$ denotes the principal's cost of manufacturing y.

Question of monopolist: How to maximize her profit among all feasible pricing policies?

Constraints

Definition (incentive compatible)

A Borel map $x \in X \longmapsto (y(x), z(x)) \in cl(Y) \times cl(Z)$ of agents to (product, price) pairs is called incentive compatible if and only if $G(x, y(x), z(x)) \geq G(x, y(x'), z(x'))$ for all $x, x' \in X$.

Such a map offers agent x no incentive to pretend to be x'.

Definition (participation constraint)

There exists a function $u_\emptyset : X \longrightarrow \mathbf{R}$ such that the agents' utility u is bounded below by $u_\emptyset$, i.e. $u(x) \geq u_\emptyset(x)$, for all $x \in X$.

This constraint provides an outside option for each agent so that he can choose not to participate if the maximum utility gained from buying activity is less than $u_\emptyset(x)$.

principal's program

Proposition

The principal's program can be described as follows:

$$(P_0)\begin{cases} \sup \Pi(v,y) = \int_X \pi(x, y(x), v(y(x)))d\mu(x) \\ s.t. \\ x \in X \longmapsto (y(x), v(y(x))) \textit{ incentive compatible}; \\ u(x) := G(x, y(x), v(y(x))) \geq u_{\emptyset}(x) \textit{ for all } x \in X; \\ \pi(x, y(x), v(y(x))) \textit{ is measurable.} \end{cases} \quad (2)$$

Previous Results by Others

Existence: Mirrlees(1971), Spence(1974), ..., Rochet(1987), ..., Rochet-Choné (1998, [1]), Monteiro-Page(1998), ..., Carlier(2001, [2]), ..., Nöldeke & Samuelson (2015, [3]), etc.

- Mirrlees, Spence: one-dimensional
- Rochet-Choné, Monteiro-Page, Carlier: quasi-linear pricing models, multi-dimensional
- Nöldeke & Samuelson: nonlinear pricing model/nonlinear matching model, multi-dimensional

Previous Results by Others

Concavity and Stability: Figalli-Kim-McCann (FKM, 2011), etc.

FKM: quasilinear pricing model, total social welfare.

Previous Results by Others

Concavity and Stability: Figalli-Kim-McCann (FKM, 2011), etc.

FKM: quasilinear pricing model, total social welfare.

Economic Interpretation: Mirrlees(1971), Spence(1974),..., Rochet-Choné (1998, [1]), etc.

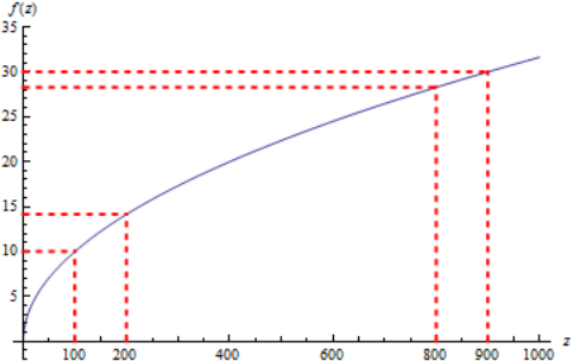
Nobel Economics Prizes: Mirrlees(1996), Spence(2001)

Why not quasi-linear?

- $G(x, y, z) = b(x, y) - z$, where utility G linearly depends on prices z.

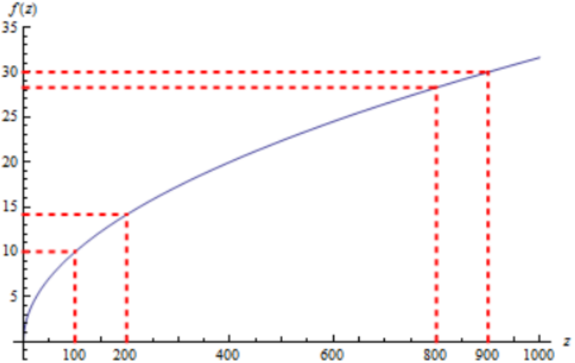
Why not quasi-linear?

- $G(x, y, z) = b(x, y) - z$, where utility G linearly depends on prices z.
- $G(x, y, z) = b(x, y) - \sqrt{z}$, non-constant marginal utilities.



Why not quasi-linear?

- $G(x, y, z) = b(x, y) - z$, where utility G linearly depends on prices z.
- $G(x, y, z) = b(x, y) - \sqrt{z}$, non-constant marginal utilities.



- $G(x, y, z) = b(x, y) - f(x, z)$, where agents have different sensitivities to the same price.

b-convexity

Definition (b-convexity)

A function $u : cl(X) \longrightarrow \mathbf{R}$ is called b-convex if $u = (u^{b^*})^b$ and $v : cl(Y) \longrightarrow \mathbf{R}$ is called b^*-convex if $v = (v^b)^{b^*}$, where

$$v^b(x) = \sup_{y \in cl(Y)} b(x,y) - v(y) \text{ and } u^{b^*}(y) = \sup_{x \in cl(X)} b(x,y) - u(x) \tag{3}$$

Taking $b(x,y) =< x, y >$, then b-convexity coincides with convexity.

b-convexity

Definition (*b*-convexity)

A function $u : cl(X) \longrightarrow \mathbf{R}$ is called b-convex if $u = (u^{b^*})^b$ and $v : cl(Y) \longrightarrow \mathbf{R}$ is called b^*-convex if $v = (v^b)^{b^*}$, where

$$v^b(x) = \sup_{y \in cl(Y)} b(x,y) - v(y) \text{ and } u^{b^*}(y) = \sup_{x \in cl(X)} b(x,y) - u(x) \tag{3}$$

Taking $b(x,y) =< x, y >$, then b-convexity coincides with convexity.

Definition (*b*-concavity)

A function $u : cl(X) \longrightarrow \mathbf{R}$ is called b-concave if $(-u)$ is $(-b)$-convex, i.e., $u = -((-u)^{(-b)^*})^{(-b)}$. And $v : cl(Y) \longrightarrow \mathbf{R}$ is called b^*-concave if $(-v)$ is $(-b)^*$-convex.

Hypotheses of FKM

Hypotheses ($G(x,y,z) = b(x,y) - z$, $m = n$)

(B0) $b \in C^4(cl(X \times Y))$, *where* $X, Y \in \mathbf{R}^n$ *are open and bounded;*

(B1) *(bi-twist) Both* $y \in cl(Y) \longmapsto D_x b(x_0, y)$ *and* $x \in cl(X) \longmapsto D_y b(x, y_0)$ *are diffeomorphisms onto their ranges, for each* $x_0 \in X$ *and* $y_0 \in Y$*, respectively;*

(B2) *(bi-convexity) Both ranges* $D_x b(x_0, Y)$ *and* $D_y b(X, y_0)$ *are convex subsets of* $\mathbf{R}^n$*, for each* $x_0 \in X$ *and* $y_0 \in Y$*, respectively;*

(B3) *(non-negative cross-curvature)*

$$\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{(s,t)=(0,0)} b(x(s), y(t)) \geq 0 \tag{4}$$

whenever either $s \in [-1,1] \longmapsto D_y b(x(s), y(0))$ *or* $t \in [-1,1] \longmapsto D_x b(x(0), y(t))$ *forms an affinely parameterized line segment.*

Result of FKM

Theorem (Figalli-Kim-McCann, 2011)

Suppose $m = n$, $G(x, y, z) = b(x, y) - z$, $\pi(x, y, z) = z - a(y)$, b satisfies $(\mathbf{B0} - \mathbf{B3})$ and the manufacturing cost $a : cl(Y) \longrightarrow \mathbf{R}$ is b^*-convex, then the principal's problem becomes a concave maximization over a convex set.

Hypotheses

Hypotheses

(C0) *(Strictly Monotonicity)* $G(x, y, z)$ *is strictly decreasing in* z*, for any* $(x, y) \in X \times cl(Y)$*,* $z \in cl(Z)$*;*

(C0) is automatically satisfied when $G(x, y, z) = b(x, y) - z$.

Hypotheses

Hypotheses

(C0) *(Strictly Monotonicity)* $G(x,y,z)$ *is strictly decreasing in* z*, for any* $(x,y) \in X \times cl(Y)$*,* $z \in cl(Z)$*;*

(C0) is automatically satisfied when $G(x,y,z) = b(x,y) - z$.

Definition ($H(x,y,\cdot) = G^{-1}(x,y,\cdot)$)

For all $x \in X, y \in cl(Y), u \in G(x,y,cl(Z))$, define $H(x,y,u) := z$ where z satisfies $G(x,y,z) = u$.

Hypotheses (cont.)

Hypotheses (cont.)

(C1) *$G \in C^1(cl(X \times Y \times Z))$, where $X \in \mathbf{R}^m$, $Y \in \mathbf{R}^n$ are open and bounded and $Z = (\underline{z}, \bar{z})$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$;*

(B0) *$b \in C^4(cl(X \times Y))$, where $X, Y \in \mathbf{R}^n$ are open and bounded;*

(C2) *(twist) The map $(y, z) \in cl(Y \times Z) \longmapsto (G_x, G)(x_0, y, z)$ is homeomorphism onto its range, for each $x_0 \in X$;*

(B1) *(bi-twist) Both $y \in cl(Y) \longmapsto D_x b(x_0, y)$ and $x \in cl(X) \longmapsto D_y b(x, y_0)$ are diffeomorphisms onto their ranges, for each $x_0 \in X$ and $y_0 \in Y$;*

The first part of **(B1)** implies **(C2)**, in the quasilinear case.

G-convexity, G-subdifferentiability

Definition (G-convexity)

A function $u \in C^0(X)$ is called G-convex in X, if for each $x_0 \in X$, there exists $y_0 \in cl(Y)$, and $z_0 \in cl(Z)$ such that $u(x_0) = G(x_0, y_0, z_0)$, and $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$.

For $G(x, y, z) = b(x, y) - z$, G-convexity coincides with b-convexity.

G-convexity, G-subdifferentiability

Definition (G-convexity)

A function $u \in C^0(X)$ is called G-convex in X, if for each $x_0 \in X$, there exists $y_0 \in cl(Y)$, and $z_0 \in cl(Z)$ such that $u(x_0) = G(x_0, y_0, z_0)$, and $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$.

For $G(x, y, z) = b(x, y) - z$, G-convexity coincides with b-convexity.

Definition (G-subdifferential)

The G-subdifferential of a function $u(x)$ is defined by

$$\partial^G u(x) := \{y \in Y | u(x') \geq G(x', y, H(x, y, u(x))), \forall x' \in X\}$$

A function u is said to be G-subdifferentiable at x if and only if $\partial^G u(x) \neq \emptyset$.

For $G(x, y, z) = < x, y > -z$, G-subdifferential coincides with subdifferential.

Proposition (1)

A function $u : X \to \mathbf{R}$ is G-convex if and only if it is G-subdifferentiable everywhere.

Proposition (1)

A function $u : X \to \mathbf{R}$ is G-convex if and only if it is G-subdifferentiable everywhere.

Proposition (2)

Let (y, z) be a pair of functions from X to $\bar{Y} \times \bar{Z}$, then it represents an incentive compatible contract if and only if $u(\cdot) := G(\cdot, y(\cdot), z(\cdot))$ is G-convex on X and $y(x) \in \partial^G u(x)$ for each $x \in X$.

Restate Principal's problem

Proposition (Reformulation of Principal's problem)

Assume hypotheses $(\mathbf{C0}-\mathbf{C2})$, $\pi \in C^0(cl(X)\times cl(Y)\times cl(Z))$, $\bar{z} < +\infty$ *and* $\mu \ll \mathcal{L}^m$. *Then the principal's problem* (P_0) *is equivalent to*

$$(P)\begin{cases}\max \tilde{\Pi}(u,y) = \int_X \pi(x, y(x), H(x, y(x), u(x)))d\mu(x)\\ \text{among } G\text{-convex } u \text{ with}\\ u(x) \geq u_\emptyset(x) \text{ and } y(x) \in \partial^G u(x) \text{ for all } x \in X.\end{cases} \quad (5)$$

Further reformulation of Principal's functional

By (C2), the optimal choice $y(x)$ of Lebesgue almost every agent $x \in X$ is uniquely determined by u. For $x \in \mathrm{dom} Du$, let $y(x, u(x), Du(x))$ be the unique solution y of the system

$$u(x) = G(x, y, z), \quad Du(x) = D_x G(x, y, z). \tag{6}$$

Proposition (Reformulation of Principal's problem)

Assume hypotheses $(\mathbf{C0} - \mathbf{C2})$, $\pi \in C^0(cl(X \times Y \times Z))$, $\bar{z} < +\infty$ *and* $\mu \ll \mathcal{L}^m$. *Then the principal's problem* (P_0) *can be rewritten as maximizing a functional depending only on the agents' utility u:*

$$\mathbf{\Pi}(u) := \int_X \pi(x, y(x, u(x), Du(x)), H(x, y(x, u(x), Du(x)), u(x))) d\mu(x)$$

on the space

$$U_\emptyset := \{u : X \longrightarrow \mathbf{R} | u \text{ is } G\text{-convex and } u \geq u_\emptyset\}.$$

Hypotheses

Hypotheses (cont.)

(C3) *(convexity) The set* $(G_x, G)(x_0, cl(Y \times Z)) \subset \mathbf{R}^{m+1}$ *is convex, for each* $x_0 \in X$*;*

(B2) *(bi-convexity) Both ranges* $D_x b(x_0, Y)$ *and* $D_y b(X, y_0)$ *are convex subsets of* $\mathbf{R}^n$*, for each* $x_0 \in X$ *and* $y_0 \in Y$*, respectively;*

G-segment

Definition (G-segment)

For each $x_0 \in X$ and $(y_0, z_0), (y_1, z_1) \in cl(Y \times Z)$, define $(y_t, z_t) \in cl(Y \times Z)$ such that the following equation holds for each $t \in [0, 1]$:

$$(G_x, G)(x_0, y_t, z_t) = (1 - t)(G_x, G)(x_0, y_0, z_0) + t(G_x, G)(x_0, y_1, z_1) \quad (7)$$

By (C2) and (C3), (y_t, z_t) is uniquely determined by (7). We call $t \in [0, 1] \longmapsto (x_0, y_t, z_t)$ the G-segment connecting (x_0, y_0, z_0) and (x_0, y_1, z_1).

Hypotheses (cont.)

Hypotheses (cont.)

(C4) *For each $x, x_0 \in X$, assume $t \in [0,1] \longmapsto G(x, y_t, z_t)$ is convex along all G-segments (x_0, y_t, z_t);*

(B3) *(non-negative cross-curvature)*

$$\frac{\partial^4}{\partial s^2 \partial t^2}\Big|_{(s,t)=(0,0)} b(x(s), y(t)) \geq 0 \tag{8}$$

whenever either $s \in [-1,1] \longmapsto D_y b(x(s), y(0))$ or $t \in [-1,1] \longmapsto D_x b(x(0), y(t))$ forms an affinely parameterized line segment;

Convexity of underlying space/Concavity of the functional

Theorem 1 (McCann-Z., 2017)

Under assumptions $(\mathbf{C1}-\mathbf{C4})$, $U_{\emptyset}$ *is convex.*

Convexity of underlying space/Concavity of the functional

Theorem 1 (McCann-Z., 2017)

Under assumptions $(\mathbf{C1} - \mathbf{C4})$, $U_\emptyset$ *is convex.*

Theorem 2 (McCann-Z., 2017)

Assuming $(\mathbf{C0} - \mathbf{C4})$, *and* $\mu \ll \mathcal{L}^m$, *the following statements are equivalent:*

(i) $t \in [0,1] \longmapsto \pi(x, y_t(x), z_t(x))$ *is concave along all G-segments* $(x, y_t(x), z_t(x))$ *whose endpoints satisfy* $\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x), z_1(x))\} \geq u_\emptyset(x)$;

(ii) $\mathbf{\Pi}(u)$ *is concave in* $\mathcal{U}_\emptyset$.

Remarks

Remark 1 (Convexity of Principal's functional)

Under the same hypotheses as in Theorem 2, (i) remains equivalent to (ii) when both occurences of concavity are replaced by convexity; (i) implies to (ii) when both occurences of concavity are replaced by strictly concavity or strictly convexity, respectively.

Remarks

Remark 1 (Convexity of Principal's functional)

Under the same hypotheses as in Theorem 2, (i) remains equivalent to (ii) when both occurences of concavity are replaced by convexity; (i) implies to (ii) when both occurences of concavity are replaced by strictly concavity or strictly convexity, respectively.

Remark 2 (Uniqueness)

Theorem 1 and Theorem 2 [strict version] together imply uniqueness of principal's maximization problem.

Hypotheses (cont.)

Define $\bar{G}(\bar{x}, \bar{y}) = \bar{G}(x, x_0, y, z) := x_0 G(x, y, z)$, where $\bar{x} = (x, x_0)$, $\bar{y} = (y, z)$ and $x_0 \in X_0$, where $X_0 \subset (-\infty, 0)$ is an open bounded interval containing -1.

Hypotheses (cont.)

Define $\bar{G}(\bar{x}, \bar{y}) = \bar{G}(x, x_0, y, z) := x_0 G(x, y, z)$, where $\bar{x} = (x, x_0)$, $\bar{y} = (y, z)$ and $x_0 \in X_0$, where $X_0 \subset (-\infty, 0)$ is an open bounded interval containing -1.

Hypotheses (cont.)

(C5) *(non-degeneracy)* $G \in C^2(cl(X \times Y \times Z))$, *and* $D_{\bar{x},\bar{y}}(\bar{G})(x, -1, y, z)$ *has full rank, for each* $(x, y, z) \in X \times Y \times Z$.

(B1) *(bi-twist) Both* $y \in cl(Y) \longmapsto D_x b(x_0, y)$ *and* $x \in cl(X) \longmapsto D_y b(x, y_0)$ *are diffeomorphisms onto their ranges, for each* $x_0 \in X$ *and* $y_0 \in Y$;

Characterizing concavity of functional in the smooth case

Assuming $(C5)$, we denote $(D^2_{\bar{x},\bar{y}}\bar{G})^{-1}$ the left inverse of $D_{\bar{x},\bar{y}}(\bar{G})(x,-1,y,z)$.

Characterizing concavity of functional in the smooth case

Assuming $(C5)$, we denote $(D^2_{\bar{x},\bar{y}}\bar{G})^{-1}$ the left inverse of $D_{\bar{x},\bar{y}}(\bar{G})(x,-1,y,z)$.

Remark 3 (Characterizing concavity of principal's profit in the smooth case)

If $G \in C^3(cl(X \times Y \times Z))$ satisfies $(\mathbf{C0} - \mathbf{C5})$, $\pi \in C^2(cl(X \times Y \times Z))$ and $\mu \ll \mathcal{L}^m$, then the following statements are equivalent:

(i) *concavity of $t \in [0,1] \longmapsto \pi(x, y_t(x), z_t(x))$ along all G-segments $(x, y_t(x), z_t(x))$;*

(ii) *non-positive definiteness of $(D^2_{\bar{y}\bar{y}}\pi - D_{\bar{y}}\pi(D^2_{\bar{x},\bar{y}}\bar{G})^{-1}D^3_{\bar{x},\bar{y}\bar{y}}\bar{G})\big|_{x_0=-1}$ on $\mathbf{R}^{n+1}$.*

(iii) *concavity of $\bar{y}_1 \longmapsto \pi(x,\bar{y}_1) - D_{\bar{y}}\pi(x,\bar{y})\cdot (D^2_{\bar{x},\bar{y}}\bar{G}(x,-1,\bar{y}))^{-1}\cdot D_{\bar{x}}\bar{G}(x,-1,\bar{y}_1)$ at $\bar{y}_1 = \bar{y}$, for any $(x,\bar{y}) \in X \times Y \times Z$.*

Comparison

Corollary 1 (Concavity of principal's objective with her utility not depending on agents' types)

If $G \in C^3(cl(X \times Y \times Z))$ *satisfies* $(\mathbf{C0} - \mathbf{C5})$, $\pi = \pi(y, z) \in C^2(cl(Y \times Z))$ *is* $(\bar{G})^*$*-concave and* $\mu \ll \mathcal{L}^m$, *then* $\mathbf{\Pi}$ *is concave.*

Comparison

Corollary 1 (Concavity of principal's objective with her utility not depending on agents' types)

If $G \in C^3(cl(X \times Y \times Z))$ satisfies $(\mathbf{C0} - \mathbf{C5})$, $\pi = \pi(y, z) \in C^2(cl(Y \times Z))$ is $(\bar{G})^*$-concave and $\mu \ll \mathcal{L}^m$, then $\mathbf{\Pi}$ is concave.

Theorem (Figalli-Kim-McCann, 2011; $m = n$, $G(x, y, z) = b(x, y) - z$, $\pi(x, y, z) = z - a(y)$)

If b satisfies $(\mathbf{B0} - \mathbf{B3})$ and the manufacturing cost $a : cl(Y) \longrightarrow \mathbf{R}$ is b^*-convex, then the principal's problem becomes a concave maximization over a convex set.

A shaper result

Proposition 1 (Concavity of principal's objective with her utility not depending on agents' types 2)

Suppose $G \in C^3(cl(X \times Y \times Z))$ *satisfies* $(C0 - C5)$, $\pi \in C^2(cl(Y \times Z))$, $\mu \ll \mathcal{L}^m$, *and there exists a set* $J \subset cl(X)$ *such that for each* $\bar{y} \in Y \times Z$, $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(cl(J), \bar{y})$, *then the following statements are equivalent:*

(i) $\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x, \bar{y})$ *is non-positive definite whenever* $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$, *for each* $(x, \bar{y}) \in cl(J) \times Y \times Z$;

(ii) $\mathbf{\Pi}$ *is concave.*

A shaper result

Proposition 1 (Concavity of principal's objective with her utility not depending on agents' types 2)

Suppose $G \in C^3(cl(X \times Y \times Z))$ *satisfies* $(C0 - C5)$, $\pi \in C^2(cl(Y \times Z))$, $\mu \ll \mathcal{L}^m$, *and there exists a set* $J \subset cl(X)$ *such that for each* $\bar{y} \in Y \times Z$, $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(cl(J), \bar{y})$, *then the following statements are equivalent:*

(i) $\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x, \bar{y})$ *is non-positive definite whenever* $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$, *for each* $(x, \bar{y}) \in cl(J) \times Y \times Z$;

(ii) $\mathbf{\Pi}$ *is concave.*

Remark 4

Under the same hypotheses, if $\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x, \bar{y})$ *is negative definite whenever* $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$, *for each* $(x, \bar{y}) \in cl(J) \times Y \times Z$, *then statement* $\mathbf{\Pi}$ *is strictly concave.*

A shaper result

Definition ($(-G)$-concavity)

A function $\pi : cl(Y \times Z) \longrightarrow \mathbf{R} \cup \{+\infty\}$, not identically $+\infty$, is said to be $(-G)$-concave if there exists $J \subset cl(X)$, such that $\pi(\bar{y}) = \inf_{x \in cl(J)} -G(x, \bar{y})$, for all $\bar{y} \in cl(Y \times Z)$.

Corollary 2

Suppose $G \in C^3(cl(X \times Y \times Z))$ satisfies $(C0 - C5)$, $\pi \in C^2(cl(Y \times Z))$, $\mu \ll \mathcal{L}^m$, if π is $(-G)$-concave, i.e., there exists $J \in cl(X)$ such that $\pi(\bar{y}) = \min_{x \in cl(J)} -G(x, \bar{y})$ for each $\bar{y} \in cl(Y \times Z)$, and the equation $(\pi + G)_{\bar{y}}(x, \bar{y}) = 0$ has unique solution $x \in cl(J)$ for each $\bar{y} \in Y \times Z$, then $\mathbf{\Pi}$ is concave.

Example 1

Example (Nonlinear yet homogeneous sensitivity of agents to prices)

Take $\pi(x,y,z) = z - a(y)$, $G(x,y,z) = b(x,y) - f(z)$, satisfying $(\mathbf{C0} - \mathbf{C5})$, $G \in C^3(cl(X \times Y \times Z))$, $\pi \in C^2(cl(X \times Y \times Z))$ and $\mu \ll \mathcal{L}^m$.

1. If $f(z)$ is convex in $cl(Z)$, then $\mathbf{\Pi}(u)$ is concave if and only if there exist $\varepsilon \geq 0$ such that each $(x,y,z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy
$$\left\{a_{kj}(y) - \frac{b_{,kj}(x,y)}{f'(z)} + \left(\frac{b_{,l}(x,y)}{f'(z)} - a_l(y)\right)[b_{i,l}(x,y)]^{-1} b_{i,kj}(x,y)\right\}\xi^k\xi^j \geq \varepsilon|\xi|^2.$$
2. If in addition, $f'' > 0$ and $\varepsilon > 0$, then $\mathbf{\Pi}(u)$ is strictly concave.

Example 2

Example (Inhomogeneous sensitivity of agents to prices)

Take $\pi(x,y,z) = z - a(y)$, $G(x,y,z) = b(x,y) - f(x,z)$, satisfying $(\mathbf{C0} - \mathbf{C5})$, $G \in C^3(cl(X \times Y \times Z))$, $\pi \in C^2(cl(X \times Y \times Z))$ and $\mu \ll \mathcal{L}^m$. Suppose $D_{x,y}b(x,y)$ has full rank for each $(x,y) \in X \times Y$, and $1 - (f_z)^{-1} b_{,\beta}(b_{\alpha,\beta})^{-1} f_{\alpha,z} \neq 0$, for all $(x,y,z) \in X \times Y \times Z$.

1. If $(x,y,z) \longmapsto h(x,y,z) := a_l(b_{i,l})^{-1} f_{i,zz} + \frac{[a_\beta(b_{\alpha,\beta})^{-1} f_{\alpha,z} - 1][b_{,l}(b_{i,l})^{-1} f_{i,zz} - f_{zz}]}{f_z - b_{,\beta}(b_{\alpha,\beta})^{-1} f_{\alpha,z}}$ ≥ 0, then $\mathbf{\Pi}(u)$ is concave if and only if there exist $\varepsilon \geq 0$ such that each $(x,y,z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy
$$\left\{ a_{kj} - a_l(b_{i,l})^{-1} b_{i,kj} + \frac{\left[a_\beta(b_{\alpha,\beta})^{-1} f_{\alpha,z} - 1\right]\left[b_{,kj} - b_{,l}(b_{i,l})^{-1} b_{i,kj}\right]}{f_z - b_{,\beta}(b_{\alpha,\beta})^{-1} f_{\alpha,z}} \right\} \xi^k \xi^j \geq \varepsilon |\xi|^2.$$
2. If in addition, $h > 0$ and $\varepsilon > 0$, then $\mathbf{\Pi}(u)$ is strictly concave.

Example 3

Example (Zero sum transactions)

Take $\pi(x, y, z) + G(x, y, z) = 0$, satisfying $(\mathbf{C0} - \mathbf{C4})$ and $\mu \ll \mathcal{L}^m$, which means the monopolist's profit in each transaction coincides exactly with the agent's loss. Then $\mathbf{\Pi}(u)$ is linear.

Hypotheses (cont.)

Hypotheses (cont.)

(C6) *(twist) For each* $(y_0, z_0) \in cl(Y \times Z)$*, the map* $x \in X \longmapsto \frac{G_y}{G_z}(x, y_0, z_0)$ *is one-to-one;*

(C7) *(convexity) Its range* $\frac{G_y}{G_z}(X, y_0, z_0)$ *is convex, for each* $(y_0, z_0) \in cl(Y \times Z)$*;*

(B1) *(bi-twist) Both* $y \in cl(Y) \longmapsto D_x b(x_0, y)$ *and* $x \in cl(X) \longmapsto D_y b(x, y_0)$ *are diffeomorphisms onto their ranges, for each* $x_0 \in X$ *and* $y_0 \in Y$*, respectively;*

(B2) *(bi-convexity) Both ranges* $D_x b(x_0, Y)$ *and* $D_y b(X, y_0)$ *are convex subsets of* $\mathbf{R}^n$*, for each* $x_0 \in X$ *and* $y_0 \in Y$*, respectively;*

Equivalence of (B3) and (C4)

Proposition 2 ((C4): (B3)-like hypothesis)

Assuming $m = n$, $G \in C^4(cl(X \times Y \times Z))$ satisfying $(\mathbf{C0} - \mathbf{C3}, \mathbf{C5} - \mathbf{C7})$, then $(\mathbf{C4})$ is equivalent to: (non-positive cross-curvature) For any given curve $x_s \in X$ connecting x_0 and x_1, and any curve $(y_t, z_t) \in cl(Y \times Z)$ connecting (y_0, z_0) and (y_1, z_1), we have

$$\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{(s,t)=(s_0,t_0)} \leq 0, \qquad (9)$$

whenever either of the two curves $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ and $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment.

- Rochet, Jean-Charles and Choné, Philippe, *Ironing, sweeping, and multidimensional screening.* Econometrica (1998), pp. 783–826.
- Carlier, Guillaume, *A general existence result for the principal-agent problem with adverse selection.* Journal of Mathematical Economics **35** (2001), no. 1, pp. 129–150.
- Noldeke, Georg and Samuelson, Larry, *The Implementation Duality.* Cowles Foundation Discussion Paper (2015).
- Trudinger, Neil S, *On the local theory of prescribed Jacobian equations.* Discrete Contin. Dyn. Syst. **34** (2014), no. 4, pp. 1663-1681.
- Figalli, Alessio and Kim, Young-Heon and McCann, Robert J, *When is multidimensional screening a convex program?* Journal of Economic Theory **146** (2011), no. 2, pp. 454–478.

Thank you!