Causal optimal transport and its links to enlargement of filtrations and stochastic optimization problems

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Problem formulation

Problem: Given

- two filtrations $\mathcal{F} := (\mathcal{F}_t)_t \subset (\mathcal{G}_t)_t =: \mathcal{G}$ on a space of events Ω
- a probability measure $\mathbb{P}$
- X semimartingale in $(\Omega, \mathcal{F}, \mathbb{P})$

$\longrightarrow$ when is X going to remain a semimartingale in $(\Omega, \mathcal{G}, \mathbb{P})$?

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- Semimartingales are the processes for which classical stochastic integration works: $\int H dX$ (e.g. asset price proc.)
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Today: $X = B$ Brownian motion in its own filtration $\mathcal{F} \subset \mathcal{G}$:

- When is B semimartingale w.r.t. $\mathcal{G}$? $B_t = \tilde{B}_t + A_t$
- In particular, when is FV $\ll \mathcal{L}$? $B_t = \tilde{B}_t + \int_0^t a_s ds$

$\rightarrow$ We will answer via a specific kind of transport

From classical to causal transport

- **Classical (Monge-Kantorovich) transport problem:** Given two Polish probability spaces $(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$, "move the mass" from μ to ν minimizing the cost of transportation $c : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$

$$P := \inf \left\{ \mathbb{E}^{\pi}[c(x, y)] : \pi \in \Pi(\mu, \nu) \right\},$$

$\Pi(\mu, \nu)$: probability measures on $\mathcal{X} \times \mathcal{Y}$ with marginals μ and ν.

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- **Causal transport problem:** Given right-continuous filtrations $\mathcal{F}^{\mathcal{X}}=(\mathcal{F}^{\mathcal{X}}_t)_{t\in[0,T]}$ on $\mathcal{X}$, and $\mathcal{F}^{\mathcal{Y}}=(\mathcal{F}^{\mathcal{Y}}_t)_{t\in[0,T]}$ on $\mathcal{Y}$, $T<\infty$:

Definition (Yamada-Watanabe'71 criterion, Lassalle'13)

A transport plan $\pi\in\Pi(\mu,\nu)$ is called causal between $(\mathcal{X},\mathcal{F}^{\mathcal{X}},\mu)$ and $(\mathcal{Y},\mathcal{F}^{\mathcal{Y}},\nu)$ if, for all t and $D\in\mathcal{F}^{\mathcal{Y}}_t$, the map $\mathcal{X}\ni x\mapsto\pi^x(D)$ is measurable w.r.t. $\mathcal{F}^{\mathcal{X}}_t$ (π^x regular conditional kernel w.r.t. $\mathcal{X}$).

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$$P_C := \inf \left\{ \mathbb{E}^\pi[c(x, y)] : \pi \in \Pi^{\mathcal{F}^\mathcal{X}, \mathcal{F}^\mathcal{Y}}(\mu, \nu) := \Pi(\mu, \nu) \cap \text{causal} \right\}$$

Prominent example I: weak-solutions of SDEs

- $\mathcal{X} = \mathcal{Y} = \mathcal{C} := \mathcal{C}_0[0, T]$
- $\mathcal{F}$ right-continuous canonical filtration on $\mathcal{C}$

Example (Yamada-Watanabe'71)

Assume weak-existence of the solution to the SDE:

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt, \quad b, \sigma \text{ Borel measurable.}$$

$\Rightarrow$ $(B, Y)_{\#}\mathbb{P}$ causal plan between $(\mathcal{C}, \mathcal{F}, B_{\#}\mathbb{P})$ and $(\mathcal{C}, \mathcal{F}, Y_{\#}\mathbb{P})$

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- **Transport perspective:** from an observed trajectory of B, the mass can be split at each moment of time into Y only based on the information available up to that time.
- **Monge transport** $\iff$ **strong solution** $Y = F(B)$.

Prominent example II: filtration enlargement

- $\mathcal{X} = \mathcal{Y} = \mathcal{C}$, and $\mathcal{F}^{\mathcal{X}} = \mathcal{F}$ as above.
- $\mathcal{F}^{\mathcal{Y}} = \mathcal{G}$ obtained as enlargement of $\mathcal{F}$ with $G(W) = (G_t(W))_t$ (W coordinate process on $\mathcal{C}$):

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \mathcal{G}^0_{t+\epsilon}, \quad \mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(\{G_s, s \leq t\}).$$

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Example

Let B be a Brownian motion on $(\Omega, \mathcal{F}^B = B^{-1}(\mathcal{F}), \mathbb{P})$, which remains a semimartingale w.r.t. $\mathcal{F}^{B,G} = B^{-1}(\mathcal{G})$, with

$$dB_t = d\tilde{B}_t + dA_t.$$

$\Rightarrow$ $(\tilde{B}, B)_{\#}\mathbb{P}$ is a causal plan between $(\mathcal{C}, \mathcal{F}, \gamma)$ and $(\mathcal{C}, \mathcal{G}, \gamma)$

where $\gamma =$ Wiener measure on $\mathcal{C}$

Characterizations of causality

Remark. For a probability measure $\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$, TFAE:

- π is a causal transport plan w.r.t. $\mathcal{F}^{\mathcal{X}}$ and $\mathcal{F}^{\mathcal{Y}}$;
- $\pi\left(\mathcal{X} \times D_t | \mathcal{F}_t^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}\right) = \pi\left(\mathcal{X} \times D_t | \mathcal{F}_T^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}\right)$, $\forall t, D_t \in \mathcal{F}_t^{\mathcal{Y}}$;
- $\{\emptyset, \mathcal{X}\} \otimes \mathcal{F}_t^{\mathcal{Y}}$ conditionally independent from $\mathcal{F}_T^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}$ given $\mathcal{F}_t^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}$ w.r.t. π, for all t;
- $\mathcal{H}$-hypothesis between $\mathcal{F}^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}$ and $\mathcal{F}^{\mathcal{X}} \otimes \mathcal{F}^{\mathcal{Y}}$ w.r.t. π (all sq.integrable $\mathcal{F}^{\mathcal{X}} \otimes \{\emptyset, \mathcal{Y}\}$-mart. remain $\mathcal{F}^{\mathcal{X}} \otimes \mathcal{F}^{\mathcal{Y}}$-mart.).

Recall our questions

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- Brownian bridge: $dB_t = d\tilde{B}_t + \frac{B_T - B_t}{T-t}dt$
- Initial enlargement under Jacod's condition
- Progressive enlargement with a random time (Jeulin-Yor's formula)
- Enlargement with $J_t := \inf_{s\geq t} R_s$, where $dR_t = \frac{1}{R_t}dt + dB_t$: $dB_t = d\tilde{B}_t + 2dJ_t - \frac{1}{R_t}dt$

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→ In particular, when does it have an absolutely continuous finite variation part? $\left(B_t = \tilde{B}_t + \int_0^t a_s ds\right)$

Semimartingale preservation

Notations. $(\omega,\overline{\omega})$: generic element in $\mathcal{C}\times\mathcal{C}$, $\gamma =$ Wiener measure, $V_t(Z)$: total variation of a process/path Z up to time t.

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Theorem

For any fixed anticipation G, TFAE:

i. *any process B which is Brownian motion on some $(\Omega, \mathcal{F}^B, \mathbb{P})$, remains a semimartingale in the enlarged filtration $\mathcal{F}^{B,G}$;*

ii. *for some $\nu \sim \gamma$, the following causal transport problem is finite*

$$\inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\nu)} \mathbb{E}^\pi[V_T(\overline{\omega} - \omega)].$$

Optimal transport $\hat{\pi} := (\xi, id)_{\#}\nu$, where $\xi_t(\overline{\omega}) := \overline{\omega}_t - A_t(\overline{\omega})$, with A $(\pi, \{\emptyset, \mathcal{C}\} \times \mathcal{G})$-dual pr.pr. of $(\overline{\omega}_t - \omega_t)$, for any π with finite cost.

The absolutely continuous case

→ In order to characterize the absolutely continuous case, the total variation will be replaced by the following type of costs

$$c_\rho(\omega, \overline{\omega}) := \int_0^T \rho(\dot{\overbrace{\overline{\omega}_t - \omega_t}})dt,$$

where $\rho : \mathbb{R} \to \mathbb{R}_+$ is convex, even, $\rho(0) = 0$ and $\rho(+\infty) = +\infty$.

→ For such cost functions, the causal transport problem is over transports π under which $\overline{\omega} - \omega \ll \mathcal{L}$.

The absolutely continuous case

Theorem

For any fixed anticipation G, TFAE:

i. *any process B which is Brownian motion on some $(\Omega, \mathcal{F}^B, \mathbb{P})$, remains a semimartingale in $\mathcal{F}^{B,G}$, with decomposition*

$$dB_t = d\tilde{B}_t + \alpha_t(B)dt;$$

ii. *for some $\nu \sim \gamma$, and some ρ as above (eqv., for $\rho = |\,.\,|$), the following causal transport problem is finite*

$$\inf_{\pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\nu)} \mathbb{E}^\pi \left[c_\rho \right].$$

Optimal transport $\hat{\pi} := (\xi, id)_\# \nu$, where $\xi_t(\overline{\omega}) := \overline{\omega}_t - \int_0^t a_s(\overline{\omega})ds$, a is $(\pi, \{\emptyset, C\} \times \mathcal{G})$-pr.pr. of $\widehat{\overline{\omega}_t - \omega_t}$, for any π with finite cost.

Cameron-Martin cost

Consider the case $\rho(x) = x^2 \Rightarrow c_\rho(\omega, \overline{\omega}) = |\overline{\omega} - \omega|_H^2$.

- If $P_C = \inf \left\{ \mathbb{E}^\pi[c_\rho] : \pi \in \Pi^{\mathcal{F},\mathcal{G}}(\gamma,\gamma) \right\} < \infty$, then
  - $dB_t = d\tilde{B}_t + \alpha_t(B)dt$, with α square integrable;
  - $P_C = \mathbb{E}^\gamma \left[\int_0^T \alpha_t^2 dt \right]$.

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  - $dB_t = d\tilde{B}_t + \alpha_t(B)dt$, with α square integrable;
  - $P_C = \mathbb{E}^\gamma \left[\int_0^T \alpha_t^2 dt \right]$.

- If $\mathcal{G} = \mathcal{F}$ and $\nu \ll \gamma$, then $\inf \left\{ \mathbb{E}^\pi [c_\rho] : \pi \in \Pi^{\mathcal{F},\mathcal{F}}(\gamma, \nu) \right\} < \infty$,

$$\begin{aligned} 2H(\nu|\gamma) &= \inf \left\{ \mathbb{E}^\pi [|\overline{\omega} - \omega|_H^2] : \pi \in \Pi^{\mathcal{F},\mathcal{F}}(\gamma, \nu) \right\} \\ &\geq \inf \left\{ \mathbb{E}^\pi [|\overline{\omega} - \omega|_H^2] : \pi \in \Pi(\gamma, \nu) \right\} \\ &= d_H^2(\gamma, \nu) \end{aligned}$$

Wasserstein distance between γ and ν w.r.t. the CM space.
($\Rightarrow$ Talagrand's inequality for Gaussian measures)

Extensions

Our results have natural extensions in two directions:

→ **Multidimensional processes.**

→ **General continuous semimartingales:** for non-Brownian processes, generalization of the definition of causality:

$$\mathbb{E}^{\pi}[(\omega_t - \omega_s) f_s(\overline{\omega})] = 0, \qquad 0 \leq s < t \leq T,\ f_s \in L^{\infty}(\mathcal{C}, \mathcal{G}_s, \nu),$$

which leads to analogous results.

In particular, if X continuous semimartingale which remains a semimartingale in the enlarged filtration $\mathcal{F}^{X,G}$, with $X = \widetilde{X} + N$ ⇒ the transport plan $(\widetilde{X}, X)_{\#}\mathbb{P}$ satisfies the condition above.

Conclusions

- We imposed the causal constraint on transport plans $\rightarrow$ causal optimal transport problem (time matters!).
- With **cost function = total variation**, we used the causal optimal transport problem to characterize the preservation of semimartingale property in enlarged filtrations.

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- With **cost function = total variation**, we used the causal optimal transport problem to characterize the preservation of semimartingale property in enlarged filtrations.
- With the **same cost function**, the causal optimal transport problem can be used to estimate the value of additional information for classical stochastic optimization problems.
- In analogy to classical optimal transport: attainability of causal optimal transport problem, and duality results.

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THANK YOU FOR YOUR ATTENTION!