Valuations on Lattice Polytopes

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joint work with Laura Silverstein

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Valuations on Convex Bodies

- $\mathcal{F}$ family of subsets of $\mathbb{R}^n$
  - $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
  - $\mathcal{P}^n$ space of convex polytopes in $\mathbb{R}^n$
- $\langle \mathcal{A}, + \rangle$ Abelian semigroup
- A function $\mathrm{Z} : \mathcal{F} \to \langle \mathcal{A}, + \rangle$ is a *valuation* $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{F}$ such that $K \cup L, K \cap L \in \mathcal{F}$.

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Examples

- $K \mapsto V_n(K)$ *n*-dimensional volume of K
- $K \mapsto \mathrm{L}(K)$ number of points in $K \cap \mathbb{Z}^n$
- $K \mapsto \int_K x \, \mathrm{d}x$ moment vector of K

Classification of Valuations on Convex Bodies

- **Real valuations**:
  Blaschke 1937, Hadwiger 1949, McMullen 1977, Klain 1995, Alesker 1998, Ludwig 1999, Ludwig & Reitzner 1999, Bernig & Fu 2011, Haberl & Parapatits 2014, . . .
- **Vector and tensor valuations**:
  Hadwiger & Schneider 1971, Schneider 1972, McMullen 1977, Alesker 1999, Ludwig 2002, Haberl & Parapatits 2016, Bernig & Hug 2017+, Ma & Zeng 2017+ . . .
- **Convex body valued and star body valued valuations:**
  Schneider 1974 (Minkowski endomorphisms), Ludwig 2002 (Minkowski valuations), Kiderlen 2006, Haberl & Ludwig 2006, Ludwig 2006, Schneider & Schuster 2006, Schuster 2007, Haberl 2009, Abardia & Bernig 2011, Wannerer 2011, Schuster & Wannerer 2012, Abardia 2012, Parapatits 2014, Li & Yuan & Leng 2015, Li & Leng 2016, . . .

The Hadwiger Classification Theorem 1952

Theorem

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a rigid motion invariant and continuous valuation*

$$\Longleftrightarrow$$

$\exists \;\; c_0, c_1, \dots, c_n \in \mathbb{R}$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

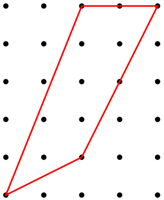
for every $K \in \mathcal{K}^n$.

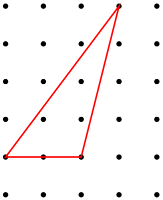
$V_0(K), \dots, V_n(K)$	intrinsic volumes of K
$V_n(K)$	n-dimensional volume of K
$2\, V_{n-1}(K) = S(K)$	surface area of K

Steiner formula: $V_n(K + s\, B^n) = \sum_{j=0}^{n} s^{n-j} v_{n-j}\, V_j(K)$

Lattice Polytopes

- P lattice polytope in $\mathbb{R}^n$
  $\iff P$ is the convex hull of finitely many points from $\mathbb{Z}^n$





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- Applications
  - Integer programming
  - Geometry of numbers
  - Combinatorics
  - Algebraic geometry (Newton polytope)

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$$x \mapsto \phi x$$

ϕ $n \times n$-matrix with integer coefficients and determinant 1

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$$\Longleftrightarrow$$

$\mathrm{Z}(\phi P) = \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL_n}(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

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- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{A}$ is translation invariant

$$\Longleftrightarrow$$

$$\mathrm{Z}(P + x) = \mathrm{Z}(P) \text{ for all } x \in \mathbb{Z}^n \text{ and } P \in \mathcal{P}(\mathbb{Z}^n)$$

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Question.

Classification of $\mathrm{SL_n}(\mathbb{Z})$ and translation invariant valuations on $\mathcal{P}(\mathbb{Z}^n)$.

The Betke & Kneser Theorem 1985

Theorem

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists \; c_0, c_1, \dots, c_n \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_0 \,\mathrm{L}_0(P) + \cdots + c_n \,\mathrm{L}_n(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

$\mathrm{L}_0(P), \dots, \mathrm{L}_n(P)$ coefficients of the Ehrhart polynomial

Ehrhart Polynomial

$\mathrm{L}(P)$ number of points in $P \cap \mathbb{Z}^n$ for $P \in \mathcal{P}(\mathbb{Z}^n)$
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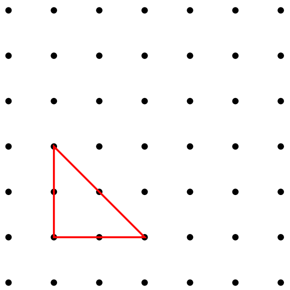

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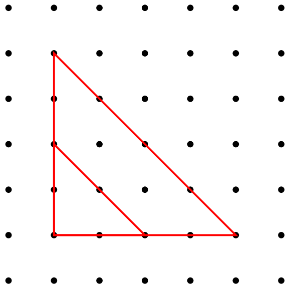

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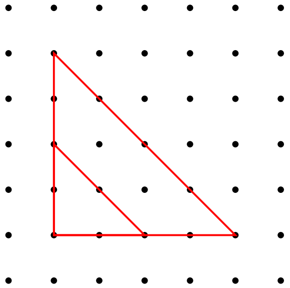

$$\mathrm{L}(kP) = \sum_{i=0}^{n} \mathrm{L}_i(P)\, k^i \quad \text{for } k \in \mathbb{N}$$

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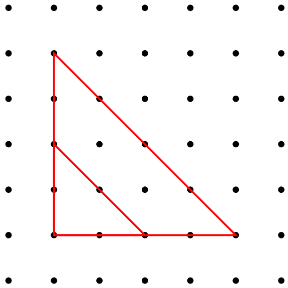

$$L(kP) = \sum_{i=0}^{n} L_i(P)\, k^i \quad \text{for } k \in \mathbb{N}$$

- $L_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}$ is $SL_n(\mathbb{Z})$ and translation invariant and homogeneous of degree i

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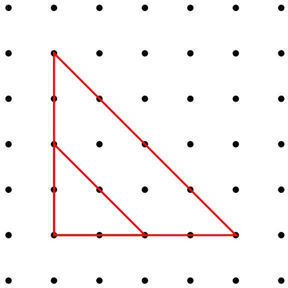

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- Eugène Ehrhart 1962
- Ehrhart Theory

Classification Theorems

Theorem (Betke & Kneser)

$\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

$$\iff$$

$\exists \;\; c_0, c_1, \dots, c_n \in \mathbb{R}$:

$$\mathrm{Z} = c_0\,\mathrm{L}_0 + \dots + c_n\,\mathrm{L}_n$$

Theorem (Hadwiger)

$\mathrm{Z}: \mathcal{K}^n \to \mathbb{R}$ *is a rigid motion invariant and continuous valuation*

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Theorem (Blaschke)

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{R})$ *and translation invariant, continuous valuation*

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Minkowski Valuations

- $\mathcal{F}$ family of subsets of $\mathbb{R}^n$
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  - $\mathcal{K}^n$ space of convex bodies in $\mathbb{R}^n$
  - $\mathcal{P}^n$ space of convex polytopes in $\mathbb{R}^n$
  - $\mathcal{P}(\mathbb{Z}^n)$ space of lattice polytopes in $\mathbb{R}^n$
- $\langle \mathcal{K}^n, + \rangle$ convex bodies with Minkowski addition

  $K + L = \{x + y : x \in K, y \in L\}$ Minkowski sum of K and L
- A function $\operatorname{Z} : \mathcal{F} \to \langle \mathcal{K}^n, + \rangle$ is a **Minkowski valuation** $\iff$

$$\operatorname{Z} K + \operatorname{Z} L = \operatorname{Z}(K \cup L) + \operatorname{Z}(K \cap L)$$

for all $K, L \in \mathcal{F}$ such that $K \cup L, K \cap L \in \mathcal{F}$.

Classification of Minkowski Valuations

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$$\mathrm{Z}(K + x) = \mathrm{Z}\, K$$

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- $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is $\mathrm{SL_n}(\mathbb{R})$ equivariant $\iff$

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- Corresponding definitions for $\operatorname{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}^n$

Classification of Minkowski Valuations

Theorem (L.: TAMS 2005)

$\mathrm{Z} : \mathcal{P}^n \to \langle \mathcal{K}^n, + \rangle$ *is an* $\mathrm{SL_n}(\mathbb{R})$ *equivariant and translation invariant valuation*

$$\iff$$

$\exists\ c \geqslant 0$:

$$\mathrm{Z}\,P = c(P + (-P))$$

for every $P \in \mathcal{P}^n$.

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Theorem (Böröczky & L.: JEMS 2016+)

$\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \langle \mathcal{K}^n, + \rangle$ is an $\mathrm{SL_n}(\mathbb{Z})$ equivariant and translation invariant valuation

$$\iff$$

$\exists \; a, b \geqslant 0:$

$$\mathrm{Z}\, P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Discrete Moment Vectors

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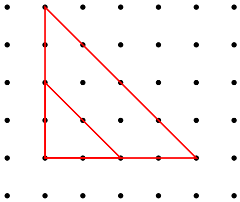

$$\ell(kP) = \sum_{i=1}^{n+1} \ell_i(P)\, k^i$$

(Ehrhart polynomial, McMullen 1977)

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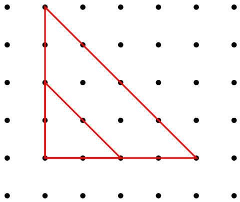

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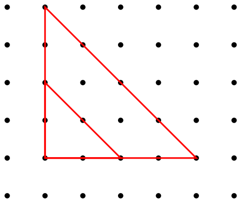

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- $\ell_{n+1}(P) = \int_P x \,\mathrm{d}x = m_{n+1}(P)$ moment vector of $P \in \mathcal{P}(\mathbb{Z}^n)$
- Böröczky & L.: JEMS 2016+

Classification of Vector Valuations

Theorem (L. & Silverstein 2017+)

$\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant, translation covariant valuation*

$$\iff$$

$\exists\ c_1, \dots, c_{n+1} \in \mathbb{R}: \quad \mathrm{Z} = c_1 \ell_1 + \dots + c_{n+1}\, \ell_{n+1}$

- $\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is translation covariant $\iff \exists\, \mathrm{Z}^0 : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}:$
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$\operatorname{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a rotation equivariant, translation covariant, continuous valuation*

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Theorem (Ludwig 2002; Haberl & Parapatits: AJM 2016)

$\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{R}^n$ *is an* $\mathrm{SL_n}(\mathbb{R})$ *equivariant and measurable valuation*

$$\Longleftrightarrow$$

$\exists\ c \in \mathbb{R}: \quad \mathrm{Z} = c\, m_{n+1}$

Classification of Tensor Valuations

Theorem (Alesker: Annals of Mathematics 1999)

$Z : \mathcal{K}^n \to \mathbb{T}^r$ *is a rotation equivariant, translation covariant, continuous valuation*

$$\Longleftrightarrow$$

Z *is a linear combination of* $Q^l\, \Phi_k^{m,s}$ *with* $2l + m + s = r$.

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- $\mathbb{T}^r$ symmetric tensors of rank r in $\mathbb{R}^n$
- $\Phi_k^{m,s}(K) = \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} x^m\, u^s\, d\Theta_k(K, (x, u))$ Minkowski tensors McMullen 1997
- $\Theta_k(K, \cdot)$ k-th generalized curvature measure, Q metric tensor

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- $\Theta_k(K,\cdot)$ k-th generalized curvature measure, Q metric tensor
- $\mathrm{M}^r(K) = \frac{1}{r!}\int_K x^r\, dx$ moment tensor
- Steiner formula:

$$\mathrm{M}^r(K + s\,B^n) = \sum_{j=1}^{n+r} s^{n+1-j}\, v_{n+1-j} \sum_{k\in\mathbb{N}} \Phi_{j-r+k}^{r-k,k}(K)$$

Translation Covariance

$$\mathrm{M}^r(K+y) = \sum_{j=0}^{r} \mathrm{M}^{r-j}(K)\frac{y^j}{j!}$$

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$$\mathrm{Z}(K+y) = \sum_{j=0}^{r} \mathrm{Z}^{r-j}(K)\frac{y^j}{j!}$$

for all $y \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$.

Translation Covariance

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Question.

Classification of $\mathrm{SL_n}(\mathbb{Z})$ equivariant and translation covariant tensor valuations on $\mathcal{P}(\mathbb{Z}^n)$.

Discrete Moment Tensors

$$\mathrm{L}^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} \underbrace{x \odot \cdots \odot x}_{r} = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r \qquad \text{discrete moment tensor of } P$$

Discrete Moment Tensors

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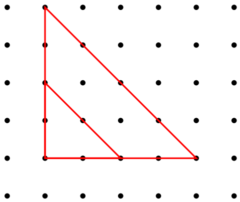

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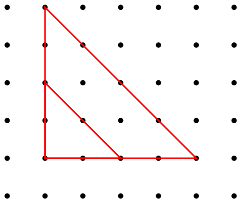

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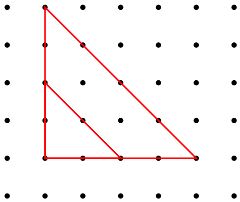

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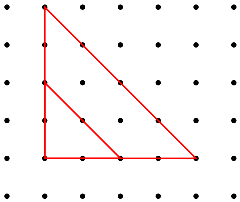

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- $\mathrm{L}_{n+r}^r(P) = \dfrac{1}{r!} \displaystyle\int_P x^r \,\mathrm{d}x = \mathrm{M}^r(P)$ moment tensor of rank r of P

Discrete Moment Tensors

$$\mathrm{L}^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r \quad \text{discrete moment tensor of } P$$

- $\mathrm{L}^r(P)(e_1[r]) = \dfrac{1}{r!} \displaystyle\sum_{x \in P \cap \mathbb{Z}^n} (x \cdot e_1)^r$
- $\mathrm{L}^r(k\,[0, e_1])(e_1[r]) = \dfrac{1}{r!} \displaystyle\sum_{i=1}^{k} i^r$

$$= \frac{1}{r+1} \sum_{l=0}^{r} (-1)^l \binom{r+1}{l} B_l k^{r+1-l}$$

Faulhaber's sum

Johann Faulhaber
(1580 - 1635)

Classification of Tensor Valuations

Theorem (L. & Silverstein 2017+)

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^2$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant, translation covariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, \dots, c_{n+2} \in \mathbb{R}$*:*

$$\mathrm{Z} = c_1\, \mathrm{L}_1^2 + \cdots + c_{n+2}\, \mathrm{L}_{n+2}^2$$

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Theorem (Alesker: Annals of Mathematics 1999)

$Z : \mathcal{K}^n \to \mathbb{T}^2$ *is a rotation equivariant, translation covariant, continuous valuation*

$$\iff$$

Z *is a linear combination of* $Q^l\, \Phi_k^{m,s}$ *with* $2l + m + s = 2$.

- Dimension of space of such valuations for $r = 2$ is $3n + 1$.

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Theorem (L.: DMJ 2003; Haberl & Parapatits: 2017+)

$\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{T}^2$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *equivariant and measurable valuation*

$$\iff$$

$\exists c_1, c_2 \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_1\,\mathrm{M}^2(P) + c_2\,\mathrm{M}^{0,2}(P^*)$$

for every $P \in \mathcal{P}^n_{(0)}$.

- $\mathrm{M}^{0,2}(P^*)$ LYZ tensor of P^* (Lutwak, Yang, Zhang: DMJ 2000)

Classification of Tensor Valuations

Theorem (L. & Silverstein 2017+)

For $1 \leqslant r \leqslant 8$, *a function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant and translation covariant valuation*

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- $M^{0,r}(K) = \int_{\mathbb{S}^{n-1}} u^r \, dS_r(K, u)$

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New Tensor Valuations

- New $\mathrm{SL}_2(\mathbb{Z})$ equivariant, translation invariant tensor valuation for $n = 2$ and $r = 9$:

$$\mathrm{N}^9(T_2) = \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2) \odot \mathrm{L}_1^3(T_2)$$

where $T_2 = [0, e_1, e_2]$

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$P = (\phi_1 T_2 + x_1) \sqcup \cdots \sqcup (\phi_m T_2 + x_m)$ with $\phi_i \in \mathrm{SL}_2(\mathbb{Z})$ and $x_i \in \mathbb{Z}^2$

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- For $n = 2$ and $r \geqslant 9$ odd:

$$\mathrm{L}_1^{s_1}(T_2) \odot \cdots \odot \mathrm{L}_1^{s_k}(T_2)$$

with $s_1 + \cdots + s_k = r$ and $s_i \geqslant 3$ odd

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