Disjointness Graphs

János Pach

Rényi Institute, Budapest
and EPFL, Lausanne


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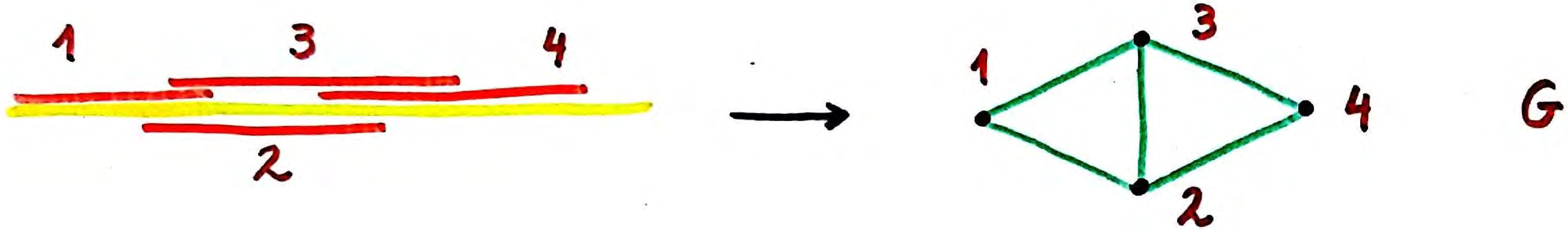
INTERSECTION GRAPHS



chromatic number $\chi(G)$ = minimum # of colors needed to color the vertices with no monochr. edge

clique number $\omega(G)$ = maximum # of vertices in a complete subgraph of G

G is perfect : $\chi(G') = \omega(G')$ for every induced subgraph $G' \subseteq G$

Theorem (Gallai, Hajós)
The intersection graph of any set of intervals along a line is perfect.

Theorem (Asplund - Grünbaum 1960)
The intersection graph of any set of axis-parallel rectangles in the plane satisfies $\chi(G) \le 8(\omega(G))^2$.

A class of graphs $\mathcal{G}$ is called χ-bounded if there exists a function f such that

$$\chi(G) \le f(\omega(G)) \quad \text{for every } G \in \mathcal{G}.$$

(Gyárfás - Lehel 1985)

Theorem (Pawlik - Kozik - Krawczyk - Lasoń - Micek - Trotter - Walczak 2014)

There exist triangle-free intersection graphs of segments in the plane with arbitrarily large chromatic numbers.

$\Longrightarrow$ segment intersection graphs are **not** χ - bounded

Definition

The disjointness graph of a system of objects (sets) is the complement of its intersection graph.

SEGMENT DISJOINTNESS GRAPHS

Theorem (Larman - Matoušek - P. - Törőcsik, 1994)
For the disjointness graph G of any system of segments in $\mathbb{R}^2$, we have $\chi(G) \le (\omega(G))^4$.

Corollary
For the disjointness graph G of any system of n segments in $\mathbb{R}^2$, we have $\max(\alpha(G), \omega(G)) \ge n^{1/5}$.

$$\alpha(G) \ge \frac{n}{\chi(G)} \ge \frac{n}{(\omega(G))^4}$$

Károlyi - P. - Tóth, 1997

Kynčl, 2012 $\quad \exists G$ with $\alpha(G), \omega(G) \le n^{0.405\ldots}$

SEGMENT DISJOINTNESS GRAPHS IN $\mathbb{R}^d$, $d > 2$

Theorem (P.-Tardos-Tóth 2017)

For the disjointness graph G of any system of segments in $\mathbb{R}^d$, $d > 2$, we have

$$\chi(G) \leq (\omega(G))^4 + (\omega(G))^3$$

Lemma

Given any system of segments in $\mathbb{R}^d$ ($d > 2$) that lie in k 2-dimensional planes $\Pi_1, \dots, \Pi_k$, their disjointness graph G satisfies

$$\chi(G) \leq (k-1)\,\omega(G) + (\omega(G))^4$$

DISJOINTNESS GRAPH

= complement of intersection graph

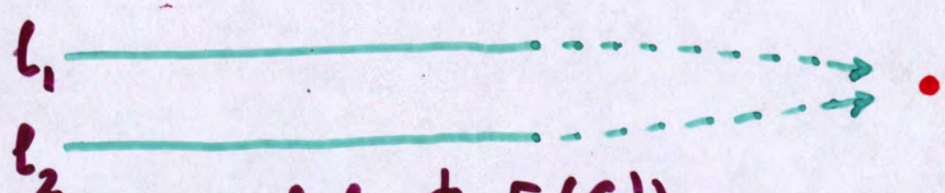
Theorem (P.-Tardos-Tóth 2017)

Given a system of lines in $\mathbb{R}^d$ (or in the projective space $\mathbb{P}^d$), let G (and G', resp.) denote their disjointness graph. For any $d=3$,

(i) $\chi(G) \leq (\omega(G))^3$

(ii) $\chi(G') \leq (\omega(G'))^2$

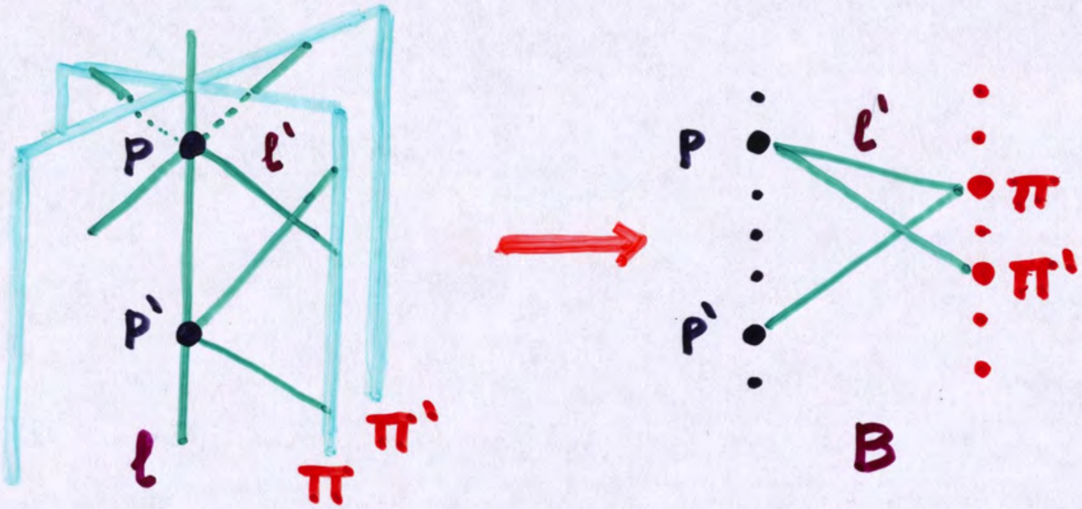
Clearly, $G' \subseteq G$



$\ell_1 \ell_2 \notin E(G')$

PROOF OF $\chi(G) \leq (\omega(G))^3$

- If G has an isolated vertex ℓ, $\chi(G) \leq (\omega(G))^2$



bipartite multigraph

By König-Hall theorem, $\nu(B) = \tau(B)$

max matching min vertex cover

$$\nu(B) \leq \omega(G) \longrightarrow \tau(B) \leq \omega(G)$$

If $\ell' = p\pi \in E(B)$ is covered by $\begin{cases} p & \text{color it by } p \\ \pi & \text{color it by} \leq \omega(G) \text{ new colors (perfect!)} \end{cases}$

PROOF OF $\chi(G) \leq (\omega(G))^3$ - GENERAL CASE

- $C \subseteq V(G)$ induces a maximal clique in G, $|C| = \omega(G)$

  $\longrightarrow$ for every $v \in V(G) - C$, there exists $c \in C : cv \notin E(G)$

- $V(G) = \bigcup_{c \in C} V_c$, where c is an isolated vertex in $G[V_c]$

Thus,
$$\begin{aligned}
\chi(G) &\leq \sum_{c \in C} \chi(G[V_c]) \\
&\leq \sum_{c \in C} (\omega(G[V_c]))^2 \\
&\leq \sum_{c \in C} (\omega(G))^2 \\
&= (\omega(G))^3
\end{aligned}$$

Problem 1

Is the family of disjointness graphs of polygonal paths of length 2 χ-bounded?

Problem 2

Is the family of intersection graphs of lines in $\mathbb{R}^3$ χ-bounded?