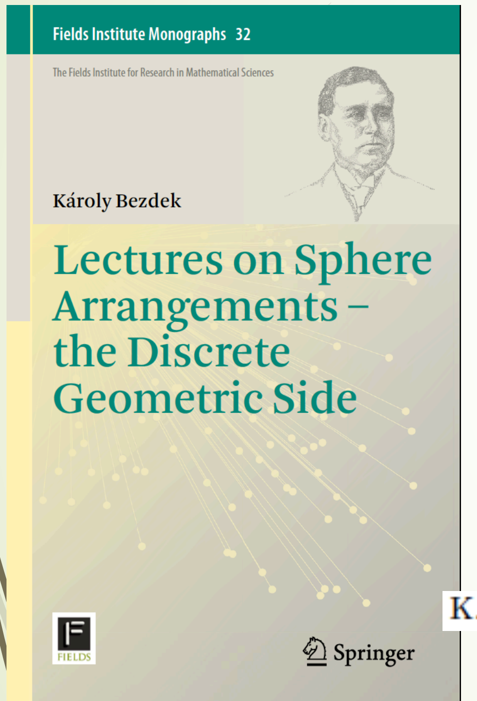
The Kneser-Poulsen conjecture revisited

Karoly Bezdek **Canada Research Chair**

Univ. of Calgary, Canada **and** Univ. of Pannonia, Hungary

Part I: On the status of the Kneser-Poulsen conjecture



3.1 The Kneser–Poulsen Conjecture

Recall that $\|\cdot\|$ denotes the standard Euclidean norm of the d-dimensional Euclidean space $\mathbb{E}^d$. So, if $\mathbf{p}_i, \mathbf{p}_j$ are two points in $\mathbb{E}^d$, then $\|\mathbf{p}_i - \mathbf{p}_j\|$ denotes the Euclidean distance between them. It is convenient to denote the (finite) point configuration consisting of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ in $\mathbb{E}^d$ by $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$. Now, if $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ are two configurations of N points in $\mathbb{E}^d$ such that for all $1 \leq i < j \leq N$ the inequality $\|\mathbf{q}_i - \mathbf{q}_j\| \leq \|\mathbf{p}_i - \mathbf{p}_j\|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. Finally, let $\mathbf{B}^d[\mathbf{p}_i, r_i]$ denote the (closed) d-dimensional ball centered at $\mathbf{p}_i$ with radius r_i in $\mathbb{E}^d$ and let $\mathrm{vol}_d(\cdot)$ represent the d-dimensional volume (Lebesgue measure) in $\mathbb{E}^d$.

K. Bezdek, *Lectures on Sphere Arrangements - the Discrete Geometric Side*, Fields Institute Monographs, Volume 32, Springer, New York, 2013

The Kneser-Poulsen conjecture (1954-1955):

158. E.T. Poulsen, Problem 10. Math. Scand. **2**, 346 (1954)

140. M. Kneser, Einige Bemerkungen über das Minkowskische Flächenmass. Arch. Math. **6**, 382–390 (1955)

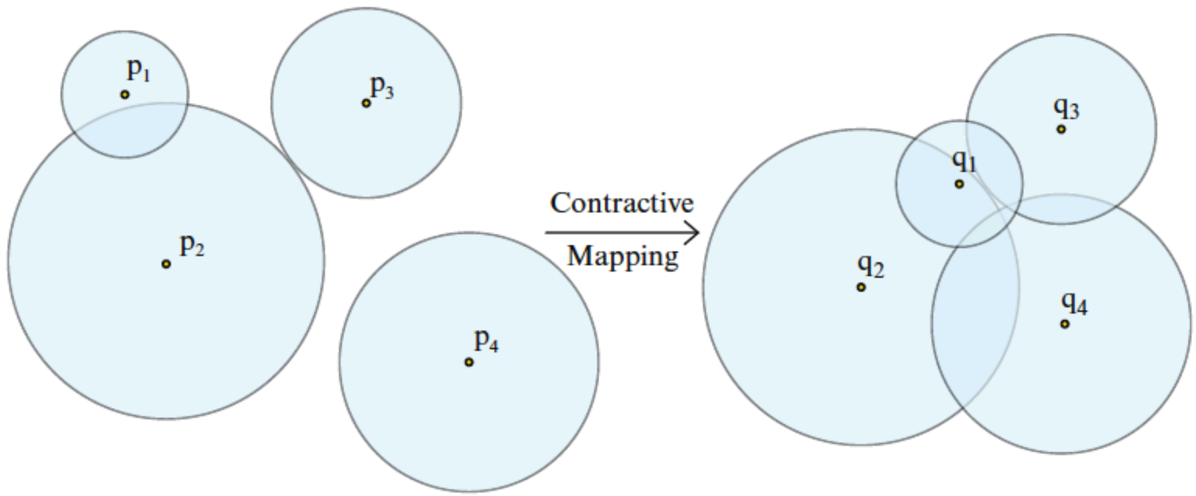


Fig. 3.1 The Kneser-Poulsen conjecture in $\mathbb{E}^2$

Martin Kneser (1928-2004)

Conjecture 3.1.1. If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^d$, then

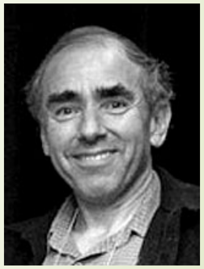
$$\mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]\right) \geq \mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r_i]\right).$$

A similar conjecture was proposed by Klee and Wagon [138] in 1991.

Conjecture 3.1.2. If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^d$, then

$$\mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]\right) \leq \mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r_i]\right).$$

138. V. Klee, V.S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*. MAA Dolciani Mathematical Expositions (Mathematical Association of America, Washington, DC, 1991)

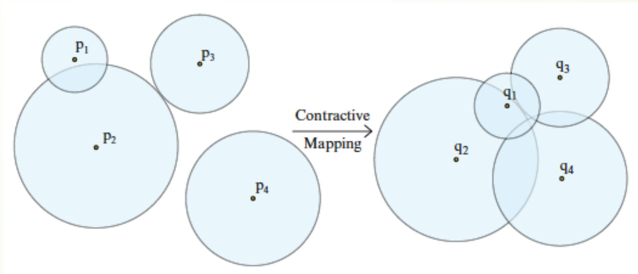


Victor Klee (1925 –2007)

The Kneser-Poulsen conjecture is proved for continuous contractions:

Now, if

$\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ are two configurations of N points in $\mathbb{E}^d$ such that for all $1 \leq i < j \leq N$ the inequality $\|\mathbf{q}_i - \mathbf{q}_j\| \leq \|\mathbf{p}_i - \mathbf{p}_j\|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. If $\mathbf{q}$ is a contraction of $\mathbf{p}$, then there may or may not be a continuous motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbb{E}^d$ for all $0 \leq t \leq 1$ and $1 \leq i \leq N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is monotone decreasing for all $1 \leq i < j \leq N$. When there is such a motion, we say that $\mathbf{q}$ is a *continuous contraction* of $\mathbf{p}$.



Discrete Comput Geom 20:449–461 (1998)

Discrete & Computational Geometry

On the Volume of the Union of Balls*

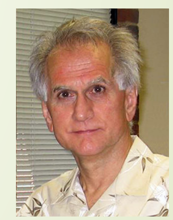
B. Csikós

Department of Geometry, Eötvös University,
Budapest, Rákóczi út 5, H-1088 Hungary
csb@ludens.elte.hu

Abstract. We prove that if some balls in the Euclidean space move continuously in such a way that the distances between their centers decrease, then the volume of their union cannot increase. The proof is based on a formula expressing the derivative of the volume of the union as a linear combination of the derivatives of the distances between the centers with nonnegative coefficients.

The following formula discovered by Csikós [84] proves Conjecture 3.1.1 as well as Conjecture 3.1.2 for continuous contractions in a straighforward way in any dimension. (Actually, the planar case of the Kneser–Poulsen conjecture under continuous contractions has been proved independently in [62, 71, 83], and [22].)

The planar case has been proved independently by:



62. B. Bollobás, Area of union of disks. Elem. Math. **23**, 60–61 (1968)

71. V. Capoyleas, On the area of the intersection of disks in the plane. Comput. Geom. **6**(6), 393–396 (1996)

Csikos's theorem:

Theorem 3.2.1. *Let $d \geq 2$ and let $\mathbf{p}(t), 0 \leq t \leq 1$ be a smooth motion of a point configuration in $\mathbb{E}^d$ such that for each t, the points of the configuration are pairwise distinct. Then*

$$\frac{d}{dt}\mathrm{vol}_d\left(\bigcup_{i=1}^{N}\mathbf{B}^d[\mathbf{p}_i(t), r_i]\right)$$

$$= \sum_{1 \leq i < j \leq N}\left(\frac{d}{dt}d_{ij}(t)\right)\cdot \mathrm{vol}_{d-1}\left(W_{ij}(\mathbf{p}_i(t), r_i)\right),$$

$$\frac{d}{dt}\mathrm{vol}_d\left(\bigcap_{i=1}^{N}\mathbf{B}^d[\mathbf{p}_i(t), r_i]\right)$$

$$= \sum_{1 \leq i < j \leq N} -\left(\frac{d}{dt}d_{ij}(t)\right)\cdot \mathrm{vol}_{d-1}\left(W^{ij}(\mathbf{p}_i(t), r_i)\right),$$

where $d_{ij}(t) = \|\mathbf{p}_i(t) - \mathbf{p}_i(t)\|$.

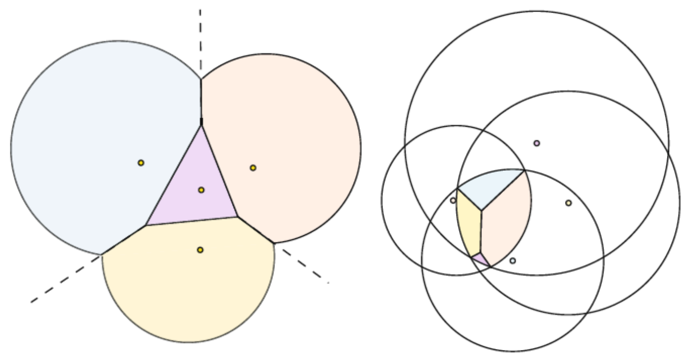


Fig. 3.2 The nearest (resp., farthest) point truncated Voronoi cell decomposition of the union (resp., intersection) of four disks in $\mathbb{E}^2$

The Kneser-Poulsen conjecture is proved in the plane:

J. reine angew. Math. **553** (2002), 221—236

Journal für die reine und angewandte Mathematik

Berlin · New York 2002

Pushing disks apart—the Kneser-Poulsen conjecture in the plane

By *Károly Bezdek* at Budapest and *Robert Connelly* at Ithaca

B-C Theorem (short form):

Theorem 3.3.1. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^2$, *then*

$$\mathrm{vol}_2\left(\bigcup_{i=1}^{N} \mathbf{B}^2[\mathbf{p}_i, r_i]\right) \geq \mathrm{vol}_2\left(\bigcup_{i=1}^{N} \mathbf{B}^2[\mathbf{q}_i, r_i]\right);$$

moreover,

$$\mathrm{vol}_2\left(\bigcap_{i=1}^{N} \mathbf{B}^2[\mathbf{p}_i, r_i]\right) \leq \mathrm{vol}_2\left(\bigcap_{i=1}^{N} \mathbf{B}^2[\mathbf{q}_i, r_i]\right).$$

B-C theorem (extended form):

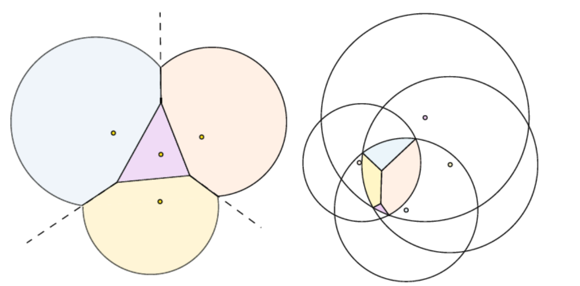


Fig. 3.2 The nearest (resp., farthest) point truncated Voronoi cell decomposition of the union (resp., intersection) of four disks in $\mathbb{E}^2$

Theorem 3.3.2. *Consider N moving closed d-dimensional balls $\mathbf{B}^d[\mathbf{p}_i(t), r_i]$ with $1 \leq i \leq N, 0 \leq t \leq 1$ in $\mathbb{E}^d, d \geq 2$. If $F_i(t)$ is the contribution of the ith ball to the boundary of the union $\bigcup_{i=1}^N \mathbf{B}^d[\mathbf{p}_i(t), r_i]$ (resp., of the intersection $\bigcap_{i=1}^N \mathbf{B}^d[\mathbf{p}_i(t), r_i]$), then*

$$\sum_{1\leq i\leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i(t))$$

decreases (resp., increases) in t under any analytic contraction $\mathbf{p}(t)$ of the center points, where $0 \leq t \leq 1$ and $\operatorname{svol}_{d-1}(\ldots)$ refers to the relevant $(d-1)$-dimensional surface volume.

Theorem 3.3.3. *Let the centers of the closed d-dimensional balls $\mathbf{B}^d[\mathbf{p}_i, r_i]$, $1 \leq i \leq N$ lie in the $(d-2)$-dimensional affine subspace L of $\mathbb{E}^d, d \geq 3$. If F_i stands for the contribution of the ith ball to the boundary of the union $\bigcup_{i=1}^N \mathbf{B}^d[\mathbf{p}_i, r_i]$ (resp., of the intersection $\bigcap_{i=1}^N \mathbf{B}^d[\mathbf{p}_i, r_i]$), then*

$$\operatorname{vol}_{d-2}\left(\bigcup_{i=1}^N \mathbf{B}^{d-2}[\mathbf{p}_i, r_i]\right) = \frac{1}{2\pi} \sum_{1\leq i\leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i)$$

$$\left(resp.,\ \operatorname{vol}_{d-2}\left(\bigcap_{i=1}^N \mathbf{B}^{d-2}[\mathbf{p}_i, r_i]\right) = \frac{1}{2\pi} \sum_{1\leq i\leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i)\right),$$

where $\mathbf{B}^{d-2}[\mathbf{p}_i, r_i] = \mathbf{B}^d[\mathbf{p}_i, r_i] \cap L, 1 \leq i \leq N$.

Theorem 3.3.4. *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^d, d \geq 1$, then there is an analytic contraction $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t)), 0 \leq t \leq 1$ in $\mathbb{E}^{2d}$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$.*

Two corollaries of the B-C theorem:

Theorem 3.3.5. *Let* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *be two point configurations in* $\mathbb{E}^d$ *such that* $\mathbf{q}$ *is a piecewise-analytic contraction of* $\mathbf{p}$ *in* $\mathbb{E}^{d+2}$. *Then the conclusions of Conjecture 3.1.1 as well as Conjecture 3.1.2 hold in* $\mathbb{E}^d$.

The following generalizes a result of Gromov in [108], who proved it in the case $N \leq d + 1$.

Theorem 3.3.6. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is an arbitrary contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$ *and* $N \leq d + 3$, *then both Conjectures 3.1.1 and 3.1.2 hold.*

As a next step it would be natural to investigate the case $N = d + 4$.

108. M. Gromov, Monotonicity of the volume of intersections of balls, in *Geometrical Aspects of Functional Analysis*, ed. by J. Lindenstrauss, V.D. Milman. Springer Lecture Notes, vol. 1267, (Springer, New York, 1987), pp. 1–4

Gorbovickis's theorem (the Kneser-Poulsen conjecture for large equal radii):

I. Gorbovickis, Strict Kneser–Poulsen conjecture for large radii. Geom. Dedic. **162**, 95–107 (2013)

Theorem 3.5.3. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$, $d \geq 2$, *then there exists* $r_0 > 0$ *such that for any* $r \geq r_0$,

$$\mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r]\right) \geq \mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r]\right)$$

and

$$\mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r]\right) \leq \mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r]\right),$$

and if the point configurations $\mathbf{q}$ *and* $\mathbf{p}$ *are not congruent, then the inequality is strict.*

The Kneser-Poulsen conjecture for hemispheres in spherical space:

Discrete Comput Geom 32:101–106 (2004)
DOI: 10.1007/s00454-004-0831-1

Discrete & Computational Geometry

The Kneser–Poulsen Conjecture for Spherical Polytopes*

Károly Bezdek[1] and Robert Connelly[2]

[1]Department of Mathematics and Statistics, University of Calgary,
2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4
bezdek@math.ucalgary.ca

[2]Department of Mathematics, Malott Hall, Cornell University,
Ithaca, NY 14853, USA
connelly@math.cornell.edu

Theorem 3.4.1. *If a finite set of closed d-dimensional balls of radius $\frac{\pi}{2}$ (i.e., of closed hemispheres) in the d-dimensional spherical space $\mathbb{S}^d$, $d \geq 2$ is rearranged so that the (spherical) distance between each pair of centers does not increase, then the (spherical) d-dimensional volume of the intersection does not decrease and the (spherical) d-dimensional volume of the union does not increase.*

Part II: The Kneser-Poulsen conjecture is proved for uniform contractions of congruent balls

THE KNESER–POULSEN CONJECTURE FOR SPECIAL CONTRACTIONS

KÁROLY BEZDEK AND MÁRTON NASZÓDI

ABSTRACT. The Kneser–Poulsen Conjecture states that if the centers of a family of N unit balls in $\mathbb{E}^d$ is contracted, then the volume of the union (resp., intersection) does not increase (resp., decrease). We consider two types of special contractions.

First, a *uniform contraction* is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. We obtain that a uniform contraction of the centers does not decrease the volume of the intersection of the balls, provided that $N \geq (1+\sqrt{2})^d$. Our result extends to intrinsic volumes. We prove a similar result concerning the volume of the union.

Second, a *strong contraction* is a contraction in each coordinate. We show that the conjecture holds for strong contractions. In fact, the result extends to arbitrary unconditional bodies in the place of balls.

arXiv:1701.05074v2 [math.MG] 26 Feb 2017

Intrinsic volume:

Let $K \subset \mathbf{E^d}$ be a convex body (i.e. a compact convex set with nonempty interior in $\mathbf{E^d}$). Let ω_i denote the i-dimensional volume of the unit i-ball, $0 \leq i \leq d$. Then the *intrinsic i-volume* $V_i(K)$ of K can be defined via Steiner's formula

$$\mathrm{Vol}_d(K + \rho B^d) = \sum_{i=0}^{d} \omega_i V_{d-i}(K)\rho^i,$$

where $\rho > 0$ is an arbitrary positive real number and ρB^d denotes the closed ball of radius ρ centered at the origin $\mathbf{o}$ of $\mathbf{E^d}$ and $K + \rho B^d$ denotes the vector sum of the convex bodies K and ρB^d with d-dimensional volume $\mathrm{Vol}_d(K + \rho B^d)$. It is well-known (see for example [4]) that $\mathrm{Vol}_d(K)$ is the d-dimensional volume of K, $2\mathrm{Vol}_{d-1}(K)$ is the surface area of K and $\frac{2\omega_{d-1}}{d\omega_d}\mathrm{V}_1(K)$ is equal to the mean width of K. (Moreover, $V_0(K) = 1$.)

Notations:

We denote the Euclidean norm of a vector p in the d-dimensional Euclidean space $\mathbb{E}^d$ by $|p| := \sqrt{\langle p, p\rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. For a positive integer N, we use $[N] = \{1, 2, \ldots, N\}$. Let $A \subset \mathbb{E}^d$ be a set, and $k \in [d]$. We denote the k–th intrinsic volume of A by $V_k(A)$; in particular, $V_d(A)$ is the d–dimensional volume. The closed Euclidean ball of radius ρ centered at $p \in \mathbb{E}^d$ is denoted by $\mathbf{B}[p, \rho] := \{q \in \mathbb{E}^d \ : \ |p - q| \leq \rho\}$, its volume is $\rho^d \kappa_d$, where $\kappa_d := V_d(\mathbf{B}[o, 1])$. For a set $X \subset \mathbb{E}^d$, the intersection of balls of radius ρ around the points in X is $\mathbf{B}[X, \rho] := \cap_{x \in X} \mathbf{B}[x, \rho]$; when ρ is omitted, then $\rho = 1$. The *circumradius* $\mathrm{cr}(X)$ of X is the radius of the smallest ball containing X. Clearly, $\mathbf{B}[X, \rho]$ is empty, if, and only if, $\mathrm{cr}(X) > \rho$. We denote the unit sphere centered at the origin $o \in \mathbb{E}^d$ by $\mathbb{S}^{d-1} := \{u \in \mathbb{E}^d \ : \ |u| = 1\}$.

Definition (uniform contraction):

16

It is convenient to denote the (finite) point configuration consisting of N points $p_1, p_2, \ldots, p_N$ in $\mathbb{E}^d$ by $\mathbf{p} = (p_1, \ldots, p_N)$, also considered as a point in $\mathbb{E}^{d\times N}$.

We say that $\mathbf{q} \in \mathbb{E}^{d\times N}$ is a *uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with separating value* $\lambda > 0$, if

(UC) $$|q_i - q_j| \le \lambda \le |p_i - p_j| \text{ for all } i, j \in [N], i \ne j.$$

Main theorem:

Theorem 1.1. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$, *and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0, 2]$. *If* $N \ge \left(1+\sqrt{2}\right)^d$ *then*

(1) $$\mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[p_i]\right) \le \mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[q_i]\right).$$

Theorem 1.2. *Let* $d, N \in \mathbb{Z}^+$, *and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0, 2]$. *If* $N \ge (1+2d^3)^d$ *then*

(2) $$\mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[p_i]\right) \ge \mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[q_i]\right).$$

Theorem 1.1. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$, *and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0,2]$. *If* $N \geq \left(1+\sqrt{2}\right)^d$ *then*

$$\mathrm{V}_k\left(\bigcap_{i=1}^{N}\mathbf{B}[p_i]\right) \leq \mathrm{V}_k\left(\bigcap_{i=1}^{N}\mathbf{B}[q_i]\right). \tag{1}$$

2. Proof of Theorem 1.1

Theorem 1.1 clearly follows from the following

Theorem 2.1. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$, *and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0,2]$. *If*

(a) $N \geq \left(1+\frac{2}{\lambda}\right)^d$,

or

(b) $\lambda \leq \sqrt{2}$ *and* $N \geq \left(1+\sqrt{\frac{2d}{d+1}}\right)^d$,

then (1) *holds.*

Definition:

In this section, we prove Theorem 2.1. We may consider a point configuration $\mathbf{p} \in \mathbb{E}^{d\times N}$ as a subset of $\mathbb{E}^d$, and thus, we may use the notation $\mathbf{B}[\mathbf{p}] = \bigcap_{i\in N} \mathbf{B}[p_i]$. We define two quantities that arise naturally. For $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda \in (0, 2]$, let

$$f_k(d, N, \lambda) := \min \left\{ \mathrm{V}_k \left(\mathbf{B}[\mathbf{q}] \right) \ : \ \mathbf{q} \in \mathbb{E}^{d\times N}, |q_i - q_j| \le \lambda \text{ for all } i, j \in [N], i \ne j \right\},$$

and

$$g_k(d, N, \lambda) := \max \left\{ \mathrm{V}_k \left(\mathbf{B}[\mathbf{p}] \right) \ : \ \mathbf{p} \in \mathbb{E}^{d\times N}, |p_i - p_j| \ge \lambda \text{ for all } i, j \in [N], i \ne j \right\}.$$

In this paper, for simplicity, the maximum of the empty set is zero.

Clearly, to establish Theorem 2.1, it will be sufficient to show that $f_k \ge g_k$ with the parameters satisfying the assumption of the theorem.

Two easy estimates:

Lemma 2.2. *Let $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda \in (0, \sqrt{2}]$. Then*

$$f_k(d, N, \lambda) \geq \left(1 - \sqrt{\frac{2d}{d+1}\frac{\lambda}{2}}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right). \tag{5}$$

Proof of Lemma 2.2. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a point configuration in the definition of f_k. Then Jung's theorem [Jun01, DGK63] implies that the circumradius of the set $\{q_i\}$ in $\mathbb{E}^d$ is at most $\sqrt{\frac{2d}{d+1}\frac{\lambda}{2}}$. It follows that $\mathbf{B}[\mathbf{q}]$ contains a ball of radius $1 - \sqrt{\frac{2d}{d+1}\frac{\lambda}{2}}$. By the monotonicity (with respect to containment) and the degree-k homogeneity of V_k, the proof of the Lemma is complete. □

Lemma 2.3. *Let $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda > 0$.*

$$\text{If } N\left(\frac{\lambda}{2}\right)^d \geq \left(1 + \frac{\lambda}{2}\right)^d, \text{ then } g_k(d, N, \lambda) = 0. \tag{6}$$

Proof of Lemma 2.3. Let $\mathbf{p} \in \mathbb{E}^{d\times N}$ be such that $|p_i - p_j| \geq \lambda$ for all $i, j \in [N], i \neq j$. The balls of radius $\lambda/2$ centered at the points $\{p_i\}$ form a packing. By the assumption, taking volume yields that the circumradius of the set $\{p_i\}$ is at least one. Hence, $\mathbf{B}[\mathbf{p}]$ is a singleton or empty. □

Definition:

Let X be a non-empty subset of $\mathbb{E}^d$ with $\mathrm{cr}(X) \leq \rho$. For $\rho > 0$, the *ρ-spindle convex hull* of X is defined as

$$\mathrm{conv}_\rho(X) := \mathbf{B}[\mathbf{B}[X,\rho],\rho].$$

It is not hard to see that

$$\mathbf{B}[X,\rho] = \mathbf{B}[\mathrm{conv}_\rho(X),\rho]. \tag{7}$$

We say that X is *ρ-spindle convex*, if $X = \mathrm{conv}_\rho(X)$.

An additive Blaschke-Santalo type inequality:

Theorem 2.4. *Let $Y \subset \mathbb{E}^d$ be a ρ-spindle convex set with $\rho > 0$, and , $k \in [d]$. Then*

$$\mathrm{V}_k(Y)^{1/k} + \mathrm{V}_k(\mathbf{B}[Y,\rho])^{1/k} \leq \rho \mathrm{V}_k(\mathbf{B}[o])^{1/k}. \tag{8}$$

Motivated by [FKV16] we observe that Theorem 2.4 clearly follows from the following proposition combined with the Brunn–Minkowski theorem for intrinsic volumes, cf. [Gar02, equation (74)].

Proposition 2.5. *Let $Y \subset \mathbb{E}^d$ be a ρ-spindle convex set with $\rho > 0$. Then*

$$Y - \mathbf{B}[Y,\rho] = \mathbf{B}[o,\rho].$$

Remark:

We will need the following fact later, the proof is an exercise for the reader.

$$\mathbf{B}[\mathbf{q}] = \mathbf{B}\left[\bigcup_{i=1}^{N} \mathbf{B}[q_i, \mu], 1 + \mu\right], \tag{9}$$

for any $\mathbf{q} \in \mathbb{E}^{d \times N}$ and $\mu > 0$.

A non-trivial bound on g_k:

Lemma 2.6. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$ *and* $\lambda \in (0, \sqrt{2}]$. *Then*

$$g_k(d, N, \lambda) \leq \max\left\{0, \left(1 - \left(N^{1/d} - 1\right)\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right)\right\}. \tag{10}$$

Proof of Lemma 2.6 (Key Lemma):

Let $\mathbf{p} \in \mathbb{E}^{d\times N}$ be such that $|p_i - p_j| \geq \lambda$ for all $i, j \in [N], i \neq j$. We will assume that $\operatorname{cr}(\mathbf{p}) \leq 1$, otherwise, $\mathbf{B}[\mathbf{p}] = \emptyset$, and there is nothing to prove.

To denote the union of non-overlapping (that is, interior-disjoint) convex sets, we use the $\bigsqcup$ operator.

Using (9) with $\mu = \lambda/2$, we obtain

$$\mathrm{V}_k(\mathbf{B}[\mathbf{p}]) = \mathrm{V}_k\left(\mathbf{B}\left[\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i, \frac{\lambda}{2}\right], 1+\frac{\lambda}{2}\right]\right) \quad = \quad (\text{using } \operatorname{cr}(\mathbf{p}) \leq 1, \text{and } (7))$$

$$\mathrm{V}_k\left(\mathbf{B}\left[\operatorname{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right), 1+\frac{\lambda}{2}\right]\right) \leq \qquad (\text{by } (8))$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k(\mathbf{B}[o])^{1/k} - \mathrm{V}_k\left(\operatorname{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right)^{1/k}\right]^k \leq$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k(\mathbf{B}[o])^{1/k} - \frac{\lambda}{2}N^{1/d}\mathrm{V}_k(\mathbf{B}[o])^{1/k}\right]^k,$$

where, in the last step, we used the following.

We have

$$\mathrm{V}_d\left(\mathrm{conv}_{1+\lambda/2}\left(\bigcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right)\right)\geq \mathrm{V}_d\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right).$$

Thus, by a general form of the isoperimetric inequality (cf. [Sch14, Section 7.4.]) stating that among all convex bodies of given (positive) volume precisely the balls have the smallest $k-$th intrinsic volume for $k=1,\ldots,d-1$, we have

$$\mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right)\right)\geq \mathrm{V}_k\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right).$$

Finally, (10) follows. □

[Sch14] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, expanded, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183

Theorem 2.1. *Let $d, N \in \mathbb{Z}^+, k \in [d]$, and let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with some separating value $\lambda \in (0,2]$. If*

(a) $N \geq \left(1+\frac{2}{\lambda}\right)^d$,

or

(b) $\lambda \leq \sqrt{2}$ *and* $N \geq \left(1+\sqrt{\frac{2d}{d+1}}\right)^d$,

then (1) *holds.*

2.4. **Proof of Theorem 2.1.** (a) follows from Lemma 2.3. To prove (b), we assume that $\lambda \leq \sqrt{2}$.

By (5), we have

$$\left(\frac{f_k(d,N,\lambda)}{V_k\left(\mathbf{B}[o]\right)}\right)^{1/k} \geq 1-\sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}. \tag{11}$$

On the other hand, (10) yields that either $g_k(d,N,\lambda)=0$, or

$$\left(\frac{g_k(d,N,\lambda)}{V_k\left(\mathbf{B}[o]\right)}\right)^{1/k} \leq 1-\left(N^{1/d}-1\right)\frac{\lambda}{2}. \tag{12}$$

Comparing (11) and (12) completes the proof of (b), and thus, the proof of Theorem 2.1.

Part III: From r-dual sets to uniform contractions

From r-dual sets to uniform contractions *

Károly Bezdek†

Abstract

Let $\mathbb{M}^d$ denote the d-dimensional Euclidean, hyperbolic, or spherical space. The r-dual set of given set in $\mathbb{M}^d$ is the intersection of balls of radii r centered at the points of the given set. In this paper we prove that for any set of given volume in $\mathbb{M}^d$ the volume of the r-dual set becomes maximal if the set is a ball. As an application we prove the following. The Kneser–Poulsen Conjecture states that if the centers of a family of N congruent balls in Euclidean d-space is contracted, then the volume of the intersection does not decrease. A uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. We prove the Kneser–Poulsen conjecture for uniform contractions (with N sufficiently large) in $\mathbb{M}^d$.

arXiv:1704.08290v1 [math.MG] 26 Apr 2017

r-dual sets in $\mathbf{M}^d$:

Let $\mathbb{M}^d$, $d > 1$ denote the d-dimensional Euclidean, hyperbolic, or spherical space, i.e., one of the simply connected complete Riemannian manifolds of constant sectional curvature. Since simply connected complete space forms, the sectional curvature of which have the same sign are similar, we may assume without loss of generality that the sectional curvature κ of $\mathbb{M}^d$ is $0, -1$, or 1. Let $\mathbf{R}_+$ denote the set of positive real numbers for $\kappa \leq 0$ and the half-open interval $(0, \frac{\pi}{2}]$ for $\kappa = 1$. Let $\operatorname{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y})$ stand for the geodesic distance between the points $\mathbf{x} \in \mathbb{M}^d$ and $\mathbf{y} \in \mathbb{M}^d$. Furthermore, let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$ denote the closed d-dimensional ball with center $\mathbf{x} \in \mathbb{M}^d$ and radius $r \in \mathbf{R}_+$ in $\mathbb{M}^d$, i.e., let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r] := \{\mathbf{y} \in \mathbb{M}^d \,|\operatorname{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y}) \leq r\}$. Now, we are ready to introduce the central notion of this paper.

Definition 1. *For a set $X \subseteq \mathbb{M}^d$, $d > 1$ and $r \in \mathbf{R}_+$ let the r-dual set X^r of X be defined by $X^r := \bigcap_{\mathbf{x} \in X} \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. If the interior* $\operatorname{int}(X^r) \neq \emptyset$, *then we call X^r the r-dual body of X.*

r-dual sets satisfy some basic identities such as

$$((X^r)^r))^r = X^r \text{ and } (X \cup Y)^r = X^r \cap Y^r,$$

which hold for any $X \subseteq \mathbb{M}^d$ and $Y \subseteq \mathbb{M}^d$. Clearly, also monotonicity holds namely, $X \subseteq Y \subseteq \mathbb{M}^d$ implies $Y^r \subseteq X^r$. Thus, there is a good deal of similarity between r-dual sets and polar sets (resp., spherical polar sets) in $\mathbb{E}^d$ (resp., $\mathbb{S}^d$). In this paper we explore further this similarity by investigating a volumetric relation between X^r and X in $\mathbb{M}^d$. For this reason let $V_{\mathbb{M}^d}(\cdot)$ denote the Lebesgue measure in $\mathbb{M}^d$, to which we are going to refer as volume in $\mathbb{M}^d$.

Gao-Hug-Schneider theorem in S^d :

for any convex body of given volume in $\mathbb{S}^d$ the volume of the spherical polar body becomes maximal if the convex body is a ball.

F. Gao, D. Hug, and R. Schneider, Intrinsic volumes and polar sets in spherical space, *Math. Notae* **41** (2003), 159–176.

A Blaschke-Santalo-type inequality for r-duality in M^d:

Theorem 1. *Let $A \subseteq \mathbb{M}^d$, $d > 1$ be a compact set of volume $V_{\mathbb{M}^d}(A) > 0$ and $r \in \mathbf{R}_+$. If $B \subseteq \mathbb{M}^d$ is a ball with $V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(B)$, then $V_{\mathbb{M}^d}(A^r) \leq V_{\mathbb{M}^d}(B^r)$.*

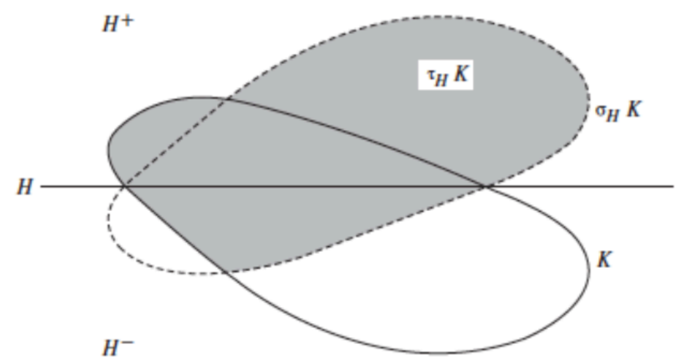
Note that the Gao–Hug–Schneider theorem is a special case of Theorem 1 namely, when $\mathbb{M}^d = \mathbb{S}^d$ and $r = \frac{\pi}{2}$. As this theorem of [10] is often called a spherical counterpart of the Blaschke–Santaló inequality, one may refer to Theorem 1 as a Blaschke–Santaló-type inequality for r-duality in $\mathbb{M}^d$.

Proof of Theorem 1

We adapt the two-point symmetrization method of the proof of the Gao-Hug-Schneider theorem from [10].

[10] F. Gao, D. Hug, and R. Schneider, Intrinsic volumes and polar sets in spherical space, *Math. Notae* 41 (2003), 159–176.

Two-point symmetrization:



Definition 3. *Let H be a hyperplane in $\mathbb{M}^d$ with an orientation, which determines H^+ and H^- the two closed halfspaces bounded by H in $\mathbb{M}^d$, $d > 1$. Let σ_H denote the reflection about H in $\mathbb{M}^d$. If $K \subseteq \mathbb{M}^d$, then the two-point symmetrization τ_H with respect to H transforms K into the set*

$$\tau_H K := (K \cap \sigma_H K) \cup \big((K \cup \sigma_H K) \cap H^+\big)\,.$$

If $K_H := K \cap \sigma_H K$ stands for the H-symmetric core of K, then we call

$$\tau_H K = K_H \cup \big((K \cap H^+) \setminus K_H\big) \cup \sigma_H \big((K \cap H^-) \setminus K_H\big) \tag{1}$$

the canonical decomposition of $\tau_H K$.

Remark 3. *The canonical decomposition of $\tau_H K$ is a disjoint decomposition of $\tau_H K$, which easily implies that two-point symmetrization preserves volume.*

Definition 4. *Let $K \subset \mathbb{M}^d$, $d > 1$ and $r \in \mathbf{R}_+$. Then the r-convex hull $\mathrm{conv}_r K$ of K is defined by*

$$\mathrm{conv}_r K := \bigcap \{\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r] \mid K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]\}.$$

Moreover, let the r-convex hull of $\mathbb{M}^d$ be $\mathbb{M}^d$. Furthermore, we say that $K \subseteq \mathbb{M}^d$ is an r-convex set if $K = \mathrm{conv}_r K$.

Lemma 4. *If $K \subseteq \mathbb{M}^d$, $d > 1$ and $r \in \mathbf{R}_+$, then*

$$K^r = (\mathrm{conv}_r K)^r.$$

Lemma 5. *If $K \subseteq \mathbb{M}^d$, $d > 1$ and $r \in \mathbf{R}_+$, then*

$$\tau_H(K^r) \subseteq (\mathrm{conv}_r(\tau_H K))^r .$$

Proof. Lemma 4 implies that $(\mathrm{conv}_r(\tau_H K))^r = (\tau_H K)^r$ and so, it is sufficient to prove that $\tau_H(K^r) \subseteq (\tau_H K)^r$. For this we need to show that if $\mathbf{x} \in \tau_H(K^r)$, then $\mathbf{x} \in (\tau_H K)^r$, i.e.,

$$\tau_H K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]. \tag{3}$$

Remark 3 implies that

$$\tau_H(K^r) = (K^r)_H \cup \left((K^r \cap H^+) \setminus (K^r)_H\right) \cup \sigma_H \left((K^r \cap H^-) \setminus (K^r)_H\right)$$

is a disjoint decomposition of $\tau_H(K^r)$ with $(K^r)_H = K^r \cap \sigma_H(K^r)$. Thus, either $\mathbf{x} \in (K^r)_H$ (Case 1), or $\mathbf{x} \in (K^r \cap H^+) \setminus (K^r)_H$ (Case 2), or $\mathbf{x} \in \sigma_H\left((K^r \cap H^-) \setminus (K^r)_H\right)$ (Case 3). In all three cases we use (1) for the proof of (3).

Finishing the proof of Theorem 1:

Now, we are ready to prove Theorem 1. To avoid any trivial case we may assume that $V_{\mathbb{M}^d}(A^r) > 0$ for $A \subseteq \mathbb{M}^d$ with $a := V_{\mathbb{M}^d}(A) > 0$. In fact, our goal is to maximize the volume $V_{\mathbb{M}^d}(A^r)$ for compact sets $A \subseteq \mathbb{M}^d$ of given volume $V_{\mathbb{M}^d}(A) = a > 0$ and for given $d > 1$ and $r \in \mathbf{R}_+$. As according to Lemma 4 we have $A^r = (\mathrm{conv}_r A)^r$ with $A \subseteq \mathrm{conv}_r A$, it follows from the monotonicity of $V_{\mathbb{M}^d}((\cdot)^r)$ in a straightforward way that for the proof of Theorem 1 it is sufficient to maximize the volume $V_{\mathbb{M}^d}(A^r)$ for r-convex sets $A \subseteq \mathbb{M}^d$ of given volume $V_{\mathbb{M}^d}(A) = a$ with given d and r. Next, consider the extremal family $\mathcal{E}_{a,r,d}$ of r-convex sets $A \subseteq \mathbb{M}^d$ with $V_{\mathbb{M}^d}(A) = a$ and maximal $V_{\mathbb{M}^d}(A^r)$ for given a, d and r. By standard arguments, $\mathcal{E}_{a,r,d} \neq \emptyset$.

Lemma 6. *The extremal family $\mathcal{E}_{a,r,d}$ is closed under two-point symmetrization.*

Proof. Let $A \in \mathcal{E}_{a,r,d}$ be an arbitrary extremal set and consider $\tau_H A$ for an arbitrary hyperplane H in $\mathbb{M}^d$. Lemmas 4 and 5 imply that $\tau_H(A^r) \subseteq (\mathrm{conv}_r(\tau_H A))^r = (\tau_H A)^r$ and therefore

$$V_{\mathbb{M}^d}(A^r) = V_{\mathbb{M}^d}(\tau_H(A^r)) \leq V_{\mathbb{M}^d}((\mathrm{conv}_r(\tau_H A))^r) = V_{\mathbb{M}^d}((\tau_H A)^r). \quad (6)$$

Here $\tau_H A \subseteq \mathrm{conv}_r(\tau_H A)$ implying that

$$a = V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(\tau_H A) \leq V_{\mathbb{M}^d}(\mathrm{conv}_r(\tau_H A)) . \quad (7)$$

[10] F. Gao, D. Hug, and R. Schneider, Intrinsic volumes and polar sets in spherical space, *Math. Notae* **41** (2003), 159–176.

We finish the proof of Theorem 1 by adapting an argument from [10]. Namely, we are going to show that $B \in \mathcal{E}_{a,r,d}$, where $B \subseteq \mathbb{M}^d$ is a ball with $a = V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(B)$. By a standard argument there exists an r-convex set $C \in \mathcal{E}_{a,r,d}$ for which $V_{\mathbb{M}^d}(B \cap C)$ is maximal. Suppose that $B \neq C$. As $a = V_{\mathbb{M}^d}(B) = V_{\mathbb{M}^d}(C)$ therefore there are congruent balls $C_1 \subseteq C \setminus B$ and $C_2 \subseteq B \setminus C$. Let H be the hyperplane in $\mathbb{M}^d$ with an orientation, which determines H^+ and H^- the two closed halfspaces bounded by H in $\mathbb{M}^d$, $d > 1$ such that $\sigma_H C_1 = C_2$ with $C_1 \subset H^-$. Clearly, $V_{\mathbb{M}^d}(B \cap \tau_H C) > V_{\mathbb{M}^d}(B \cap C)$ moreover, Lemma 6 implies that $\tau_H C \in \mathcal{E}_{a,r,d}$, a contradiction. Thus, $B = C \in \mathcal{E}_{a,r,d}$, finishing the proof of Theorem 1.

This completes the proof of Theorem 1.

Uniform contractions in $\mathbf{M^d}$:

Definition 2. *We say that the (labeled) point set* $\{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{M}^d$ *is a uniform contraction of the (labeled) point set* $\{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{M}^d$ *with separating value* $\lambda > 0$ *in* $\mathbb{M}^d$, $d > 1$ *if*

$$\mathrm{dist}_{\mathbb{M}^d}(\mathbf{q}_i, \mathbf{q}_j) \le \lambda \le \mathrm{dist}_{\mathbb{M}^d}(\mathbf{p}_i, \mathbf{p}_j)$$

holds for all $1 \le i < j \le N$.

[5] K. Bezdek and M. Naszódi, The Kneser-Poulsen conjecture for special contractions, arXiv:1701.05074 [math.MG], 18 January, 2017.

Now, recall the following recent theorem of the author and Naszódi [5] : Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1$ and $0 < \lambda \le \sqrt{2}\delta$. If $Q := \{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{E}^d$ is a uniform contraction of $P := \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ with separating value λ in $\mathbb{E}^d$ and $N \ge (1 + \sqrt{2})^d$, then $V_{\mathbb{E}^d}(P^\delta) \le V_{\mathbb{E}^d}(Q^\delta)$. As it is explained in [5], this proves the Kneser–Poulsen conjecture for uniform contractions.

The Kneser-Poulsen conjecture for uniform contractions in $\mathbf{M}^d$:

Theorem 2.

(i) Let $d \in \mathbb{Z}$ *and* $\delta, \lambda \in \mathbb{R}$ *be given such that* $d > 1$ *and* $0 < \lambda \leq \sqrt{2}\delta$. *If* $Q := \{\mathbf{q}_1, \ldots, \mathbf{q}_N\} \subset \mathbb{E}^d$ *is a uniform contraction of* $P := \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with separating value* λ *in* $\mathbb{E}^d$ *and* $N \geq (1+\sqrt{2})^d$, *then* $V_{\mathbb{E}^d}(P^\delta) < V_{\mathbb{E}^d}(Q^\delta)$.

(ii) Let $d \in \mathbb{Z}$ *and* $\delta, \lambda \in \mathbb{R}$ *be given such that* $d > 1, 0 < \delta < \frac{\pi}{2}$, *and* $0 < \lambda < \min\left\{\frac{2\sqrt{2}}{\pi}\delta, \pi - 2\delta\right\}$. *If* $Q := \{\mathbf{q}_1, \ldots, \mathbf{q}_N\} \subset \mathbb{S}^d$ *is a uniform contraction of* $P := \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \subset \mathbb{S}^d$ *with separating value* λ *in* $\mathbb{S}^d$ *and* $N \geq 2ed\pi^{d-1}\left(\frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)^d$, *then* $V_{\mathbb{S}^d}(P^\delta) < V_{\mathbb{S}^d}(Q^\delta)$.

(iii) Let $d, k \in \mathbb{Z}$ *and* $\delta, \lambda \in \mathbb{R}$ *be given such* $d > 1, k > 0$ *and* $0 < \frac{\sinh k}{\sqrt{2k}}\lambda \leq \delta < k$. *If* $Q := \{\mathbf{q}_1, \ldots, \mathbf{q}_N\} \subset \mathbb{H}^d$ *is a uniform contraction of* $P := \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \subset \mathbb{H}^d$ *with separating value* λ *in* $\mathbb{H}^d$ *and* $N \geq (\frac{\sinh 2k}{2k})^{d-1}(\frac{\sqrt{2}\sinh k}{k} + 1)^d$, *then* $V_{\mathbb{H}^d}(P^\delta) < V_{\mathbb{H}^d}(Q^\delta)$.

[5] K. Bezdek and M. Naszódi, The Kneser-Poulsen conjecture for special contractions, arXiv:1701.05074 [math.MG], 18 January, 2017.

Following [5], our proof is based on estimates of the following functionals.

Definition 5. *Let*

$$f_{\mathbb{M}^d}(N,\lambda,\delta) := \min\{V_{\mathbb{M}^d}(Q^\delta) \mid Q := \{\mathbf{q}_1,\dots,\mathbf{q}_N\} \subset \mathbb{M}^d,\ \mathrm{dist}_{\mathbb{M}^d}(\mathbf{q}_i,\mathbf{q}_j) \le \lambda \text{ for all } 1 \le i < j \le N\} \tag{8}$$

and

$$g_{\mathbb{M}^d}(N,\lambda,\delta) := \max\{V_{\mathbb{M}^d}(P^\delta) \mid P := \{\mathbf{p}_1,\dots,\mathbf{p}_N\} \subset \mathbb{M}^d,\ \lambda \le \mathrm{dist}_{\mathbb{M}^d}(\mathbf{p}_i,\mathbf{p}_j) \text{ for all } 1 \le i < j \le N\} \tag{9}$$

Definition 6. *The circumradius* $\mathrm{cr}X$ *of the set* $X \subseteq \mathbb{M}^d$, $d > 1$ *is defined by*

$$\mathrm{cr}X := \inf\{r \mid X \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathrm{x}, r]\}.$$

3.2 Proof of (ii) in Theorem 2

First, we lower bound (8). Let $R := \mathrm{cr}Q$. Then Jung's theorem ([7]) yields $\sin R \leq \sqrt{\frac{2d}{d+1}} \sin \frac{\lambda}{2}$. By assumption $0 < \lambda < \frac{\pi}{2}$ and so,

$$0 \leq \frac{2}{\pi} R < \sin R \leq \sqrt{\frac{2d}{d+1}} \sin \frac{\lambda}{2} < \sqrt{\frac{2d}{d+1}} \frac{\lambda}{2} < \frac{1}{\sqrt{2}} \lambda$$

implying that $0 \leq R < \frac{\pi}{2\sqrt{2}}\lambda$. Thus, $\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{x}, \delta - \frac{\pi}{2\sqrt{2}}\lambda\right] \subset Q^{\delta}$ for some $\mathbf{x} \in \mathbb{S}^d$. (We note that by assumption $\delta - \frac{\pi}{2\sqrt{2}}\lambda > 0$.) As a result we get that

$$f_{\mathbb{S}^d}(N, \lambda, \delta) > V_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{x}, \delta - \frac{\pi}{2\sqrt{2}}\lambda\right]\right). \tag{16}$$

Second, we upper bound (9). It follows in a straightforward way that

$$P^{\delta} = \left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{S}^d} \left[\mathbf{p}_i, \frac{\lambda}{2} \right] \right)^{\delta + \frac{\lambda}{2}}, \tag{17}$$

where the balls $\mathbf{B}_{\mathbb{S}^d}[\mathbf{p}_1, \frac{\lambda}{2}], \ldots, \mathbf{B}_{\mathbb{S}^d}[\mathbf{p}_N, \frac{\lambda}{2}]$ are pairwise non-overlapping in $\mathbb{S}^d$. Thus,

$$V_{\mathbb{S}^d} \left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{S}^d} \left[\mathbf{p}_i, \frac{\lambda}{2} \right] \right) = N V_{\mathbb{S}^d} \left(\mathbf{B}_{\mathbb{S}^d} \left[\mathbf{p}_1, \frac{\lambda}{2} \right] \right). \tag{18}$$

Let $\mu > 0$ be chosen such that

$$N V_{\mathbb{S}^d} \left(\mathbf{B}_{\mathbb{S}^d} \left[\mathbf{p}_1, \frac{\lambda}{2} \right] \right) = V_{\mathbb{S}^d} \left(\mathbf{B}_{\mathbb{S}^d} \left[\mathbf{p}_1, \mu \right] \right). \tag{19}$$

Proposition 7. *If* $0 < \mu < \frac{\pi}{2}$, *then* $\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}} N^{\frac{1}{d}}\lambda < \mu$.

Proof. One can rewrite (19) using the integral representation of volume of balls in $\mathbb{S}^d$ ([6]) as follows:

$$Nd\omega_d \int_{\frac{\pi}{2}-\frac{\lambda}{2}}^{\frac{\pi}{2}} (\cos t)^{d-1}dt = d\omega_d \int_{\frac{\pi}{2}-\mu}^{\frac{\pi}{2}} (\cos t)^{d-1}dt,$$

where $\omega_d := V_{\mathbb{E}^d}(\mathbf{B}_{\mathbb{E}^d}[\mathbf{x}, 1])$, $\mathbf{x} \in \mathbb{E}^d$. Then Lemma 4.7 of [4] yields the following chain of inequalities in a rather straightforward way:

$$\frac{N}{2ed\pi^{d-1}}\lambda^d < \frac{N}{ed}\frac{\lambda}{2}\left(\sin\frac{\lambda}{2}\right)^{d-1} \leq N\int_{\frac{\pi}{2}-\frac{\lambda}{2}}^{\frac{\pi}{2}} (\cos t)^{d-1}dt = \int_{\frac{\pi}{2}-\mu}^{\frac{\pi}{2}} (\cos t)^{d-1}dt \leq \mu(\sin\mu)^{d-1} \leq \mu^d.$$

From this the claim follows. □

Lemma 4.7 *For every* $\delta \in (0, \pi/2)$ *and every* $n \geq 1$ *one has*

$$\frac{\delta\,(\sin\delta)^n}{e(n+1)} \leq \int_{\pi/2-\delta}^{\pi/2} (\cos t)^n \,\mathrm{d}t \leq \delta\,(\sin\delta)^n.$$

[4] K. Bezdek and A. E. Litvak, Packing convex bodies by cylinders, *Discrete Comput. Geom.* **55/3** (2016), 725–738.

Now Theorem 1, (17), (18), and (19) imply in a straightforward way that

$$V_{\mathbb{S}^d}\left(P^\delta\right) = V_{\mathbb{S}^d}\left(\left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta+\frac{\lambda}{2}}\right) \leq V_{\mathbb{S}^d}\left(\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \mu\right]\right)^{\delta+\frac{\lambda}{2}}\right) \tag{20}$$

Clearly, $(\mathbf{B}_{\mathbb{S}^d}[\mathbf{p}_1, \mu])^{\delta+\frac{\lambda}{2}} = \mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta + \frac{\lambda}{2} - \mu\right]$ (with the usual convention that if $\delta + \frac{\lambda}{2} - \mu < 0$, then $\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta + \frac{\lambda}{2} - \mu\right] = \emptyset$). By assumption $0 < \delta + \frac{\lambda}{2} < \frac{\pi}{2}$ and so, if $\delta + \frac{\lambda}{2} - \mu \geq 0$, then necessarily $0 < \mu < \frac{\pi}{2}$. Thus, Proposition 7 and (20) yield

$$g_{\mathbb{S}^d}(N, \lambda, \delta) \leq V_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta - \left(\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}} N^{\frac{1}{d}} - \frac{1}{2}\right)\lambda\right]\right) \tag{21}$$

(with $V_{\mathbb{S}^d}(\emptyset) = 0$). As $N \geq 2ed\pi^{d-1}\left(\frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)^d$ therefore $\left(\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}} N^{\frac{1}{d}} - \frac{1}{2}\right)\lambda \geq \frac{\pi}{2\sqrt{2}}\lambda$ and so, (16) and (21) yield $g_{\mathbb{S}^d}(N, \lambda, \delta) < f_{\mathbb{S}^d}(N, \lambda, \delta)$, finishing the proof of (ii) in Theorem 2.