Reconfiguration of Dominating sets

Ruth Haas, U. Hawaii
Karen Seyffarth, U. Calgary

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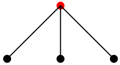
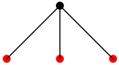
Dominating sets



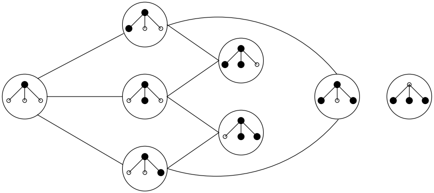
$\gamma(K_{1,3}) = 1$



$\Gamma(K_{1,3}) = 1$

$S \subset V(G)$ is a *dominating set* of G if and only if every vertex of $V(G) \backslash S$ is adjacent to a vertex of S.
The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G.
upper domination number , $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G.

The k-dominating graph



$D_3(K_{1,3})$.

$D_k(G)$, vertices are dominating sets with cardinality $\leqslant k$; two vertices of $D_k(G)$

Reconfiguration rule: addition or deletion of a single vertex.

Others models for domination reconfiguration also of interest. E.g., Subramaniam, Sridharan, and Fricke; Hedetniemi, Hedetniemi, Hutson,

γ–graph

- Only γ sets
- Token jumping.

The k-dominating graph

First question: find $d_0(G)$ the least value of k for which $D_k(G)$ is connected for all $k \geqslant d_0(G)$.
First results:

(i) $d_0(G) \geqslant \Gamma(G) + 1$, if $E(G)$ is non-empty.
(any Γ set is isolated)

(ii) $d_0(G) \leqslant |V(G)|$.

The k-dominating graph

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(ii) $d_0(G) \leqslant |V(G)|$.

(iii) $d_0(G) \leqslant \gamma(G) + \Gamma(G)$.

In (H&S 2014) gave classes of graphs for which $d_0(G) = \Gamma(G) + 1$ (bipartite graphs, chordal graphs)

Suzuki, Mouawad and Nishimura have shown that

Theorem

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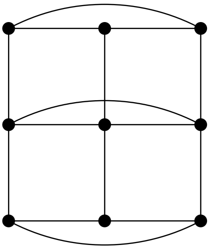
And, that sometimes $d_0(G) > \Gamma(G) + 1$.



Alikhani, Fatehi and Klavzar considered which graphs can be $D_k(G)$. They showed:

Theorem

If $V(G) \geqslant 2$ and $G \cong D_k(G)$, then $k = 2$ and $G = K_{1,n-1}$ for some $n \geqslant 4$.

Theorem

For a fixed r there exist only a finite number of r-regular, connected dominating graphs of connected graphs.

new results

Today we show

- All independent dominating sets are in the same connected component of $D_{\Gamma+1}(G)$
- If G is both perfect and irredundant perfect then $d_0(G) = \Gamma(G) + 1$.
- For certain classes of well-covered graphs, $d_0(G) = \Gamma(G) + 1$.

more notation, basics

- If dominating sets S and T of G are in the same component of $D_k(G)$. Then for all $m \geqslant k$, $D_k(G)$ is an induced subgraph of $D_m(G)$, and hence S and T are in the same component of $D_m(G)$.
- Write $A \leftrightarrow B$ if there is a path in $D_k(G)$ joining A and B.

Independent dominating sets

$S \subseteq G$ is a maximal independent set of G if and only if S is an independent dominating set of G.

Thus, $\alpha(G) \leqslant \Gamma(G)$.

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Theorem (H&S)

Let T_1 and T_2 be independent dominating sets of a graph G. Then $T_1 \leftrightarrow T_2$ in $D_{\alpha+1}(G)$, and hence in $D_{\Gamma+1}(G)$.

Proof that all independent dominating sets in same component

$\forall v \in V(G)$, let $\mathcal{S}_v$ be the set of maximal independent sets of G that contain v. Note $\mathcal{S}_v \neq \emptyset$.

Show

(i) Each $\mathcal{S}_v$ is connected (by induction on α).

(ii) If $\mathcal{S}_v \cap \mathcal{S}_u \neq \emptyset$ then these are in same connected component.

(iii) $\mathcal{S}_v \cap \mathcal{S}_u = \emptyset$ then these are in same connected component.

(i) Show by induction $\mathcal{S}_v$ is connected.

Lemma

S is a maximal independent set if and only if $S \backslash \{v\}$ is a maximal independent set of $G - N[v]$.

So $\{S \backslash \{v\} \mid S \in \mathcal{S}_v\}$ is the set of all independent dominating sets of $G - N[v]$.

Lemma

For any graph G and any $v \in V(G)$, $\Gamma(G - N[v]) < \Gamma(G)$ and $\alpha(G - N[v]) < \alpha(G)$.

So, $\alpha(G - N[v]) < \alpha(G)$.

To show $T_1 \leftrightarrow T_2$ in $D_{\alpha+1}$

If T_1, T_2 max indep in G then by lemmas:
$(T_1 \backslash \{v\}) \leftrightarrow (T_2 \backslash \{v\})$ in $D_{\alpha(G-N[v])+1}(G - N[v])$.

$$T_1 \backslash \{v\}, A_1, A_2, \ldots, A_k, T_2 \backslash \{v\}$$

in $D_{\alpha(G-N[v])+1}(G - N[v])$

$$T_1, A_1 \cup \{v\}, A_2 \cup \{v\} \ldots, A_k \cup \{v\}, T_2$$

is a path in $D_{\alpha+1}$.

So all sets of $\mathcal{S}_v$ are in the same component of $D_{\alpha+1}(G)$.

(ii) $\mathcal{S}_u \cap \mathcal{S}_v \neq \varnothing$

If $\mathcal{S}_u \cap \mathcal{S}_v \neq \varnothing$, then there exists a maximal independent set containing both u and v.

Thus all the the sets of $\mathcal{S}_v$ and $\mathcal{S}_u$ are in the same connected component of $D_{\alpha+1}$.

(iii) $\mathcal{S}_u \cap \mathcal{S}_v = \emptyset$

Suppose $T_1 \cap T_2 = \emptyset$, with $u \in T_1$ and $v \in T_2$. (If non-empty both in $\mathcal{S}_w$, some w)
If there is a path in $\overline{G}$ joining u and v, say

$$u, x_1, x_2, \ldots, x_k, v,$$

then there exist maximal independent sets $S_1 \in \mathcal{S}_u \cap \mathcal{S}_{x_1}$, $S_i \in \mathcal{S}_{x_{i-1}} \cap \mathcal{S}_{x_i}$ for $2 \leqslant i \leqslant k$, and $S_{k+1} \in \mathcal{S}_k \cap \mathcal{S}_v$, such that

$$T_1 \leftrightarrow S_1, S_1 \leftrightarrow S_2, \ldots, S_k \leftrightarrow S_{k+1}, S_{k+1} \leftrightarrow T_2$$

in $D_{\alpha+1}(G)$. Thus $T_1 \leftrightarrow T_2$ in $D_{\alpha+1}(G)$.

(iii') $\mathcal{S}_u \cap \mathcal{S}_v = \emptyset$, continued

Suppose $T_1 \cap T_2 = \emptyset$, with $u \in T_1$ and $v \in T_2$.
If there is no path in $\overline{G}$ joining u and v, then u and v are in different components of $\overline{G}$.

Lemma

If $\overline{G}$ is disconnected, and $u, v \in V(G)$ are in different components of $\overline{G}$, then $\{u, v\}$ is a dominating set of G and hence $\gamma(G) \leqslant 2$.

So,

$$T_1 \leftrightarrow T_1 \cup \{v\} \leftrightarrow \{u, v\} \leftrightarrow T_2 \cup \{u\} \leftrightarrow T_2$$

from T_1 to T_2 in $D_{\alpha+1}(G)$.

Thus $\mathcal{S}_u$, $\mathcal{S}_v$ are in same connected component in this case too.

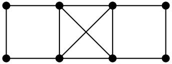
Theorem

For any graph G, $d_0(G) \leqslant \Gamma(G) + \alpha(G) - 1$. Furthermore, if G is triangle free, then $d_0(G) \leqslant \Gamma(G) + \alpha(G) - 2$.

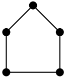
Irredundant perfect graphs

- $S \subseteq V(G)$ is an *irredundant* set if every $s \in S$ has a private neighbour.
- $\mathrm{ir(G)}$ and $\mathrm{IR(G)}$, are the cardinalities of the smallest and largest maximal irredundant sets of G.
- The clique cover number $\overline{\chi}(G)$, is the minimum number of cliques in a clique cover of G.
- $\alpha(G) \leqslant \Gamma(G) \leqslant \mathrm{IR(G)}$,
- $\alpha(G) \leqslant \overline{\chi}(G)$.

Note that $\overline{\chi}(G)$ may be larger or smaller than $\Gamma(G)$.



$\Gamma(G) = 4 > \overline{\chi}(G) = 3$



$\overline{\chi}(C_5) = 3 > \Gamma(C_5) = 2$

If S is an independent set and $\mathcal{C}$ is a clique cover and $|S| = |\mathcal{C}|$, then

$$\alpha(G) = |S| = |\mathcal{C}| = \overline{\chi}(G).$$

- G is *perfect* if $\alpha(H) = \overline{\chi}(H)$ for all induced subgraphs H of G.
- G is *irredundant perfect* if and only if $\alpha(H) = \mathrm{IR}(H)$ for all induced subgraphs H of G.

The following theorem holds for all graphs that are both perfect and irredundant perfect (including all strongly perfect graphs), but it also holds slightly more generally.

Theorem (H&S)

Let G be a graph with $\alpha(G) = \overline{\chi}(G) = \Gamma(G)$, and $\alpha(H) = \Gamma(H)$ for all induced subgraphs H of G. Then $d_0(G) = \Gamma(G) + 1$.

Well covered and well dominated

Definition (Plummer)

G is *well-covered* if every maximal independent set has the same cardinality, namely $\alpha(G)$.

Definition (Finbow, Hartnell and Nowakowski)

G is *well-dominated* if every minimal dominating set has the same cardinality, namely $\gamma(G) = \Gamma(G)$

Since every maximal independent set of a graph is a dominating set, every well-dominated graph is necessarily well-covered; hence if G is well-dominated, $\alpha(G) = \Gamma(G)$.

Families of well-covered graphs

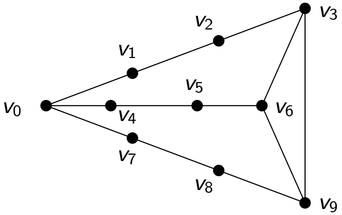
A graph G is in the family $\mathcal{L}$ if there exists $\{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ so that for each i, the subgraph induced by $N[x_i]$ is isomorphic to a complete graph and $\{N[x_i] \mid 1 \leqslant i \leqslant k\}$ is a partition of $V(G)$. We say that the set $\{x_1, x_2, \ldots x_k\}$ is a *kernel* of G. Note that a kernel of G is a maximal independent set of G.

Lemma

If $G \in \mathcal{L}$, then G is well-dominated and hence well-covered.

Theorem

If $G \in \mathcal{L}$ then $d_0(G) = \Gamma(G) + 1$.



Theorem (Finbow, Hartnell, Nowakowski)

A graph G is connected, well-covered and contains neither C_4 nor C_5 as a subgraph if and only if $G \in \mathcal{L}$ has kernel $\{x_1, \ldots, x_k\}$ in which the subgraph induced by $N[x_i]$ is isomorphic to K_1, K_2 or K_3; or G is isomorphic to C_7 or T_{10}.

Theorem (H&S)

If G is a connected well-covered graph containing neither C_4 nor C_5 as a subgraph, then $d_0(G) = \Gamma(G) + 1$.

Claw Free and well covered

A *basic chain* is a graph $\mathcal{L}$ with additional properties.

Theorem (Whitehead)

Let G be a connected well-covered claw free graph with no 4-cycle. Then G is either a basic chain or isomorphic to one of K_1, C_5 or C_7.

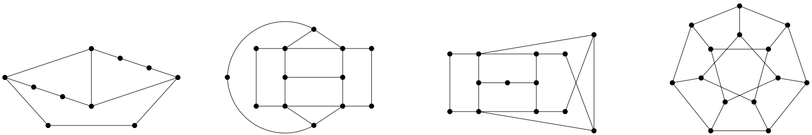
Theorem

Let G be a non-trivial, connected, well-covered, claw free graph with no 4-cycle. Then $d_0(G) = \Gamma(G) + 1$.

Well-covered graphs of girth at least five

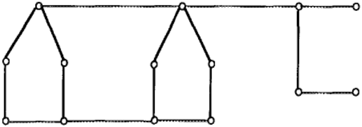
Theorem (Finbow, Hartnell and Nowakowski)

If G is a connected, well-covered graph of girth at least five, then $G \in \mathcal{PC}$ or G is isomorphic to one of six exceptional graphs: $K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}$.

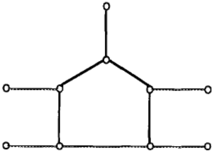


P_{10} P_{13} Q_{13} P_{14}

$\mathcal{PC}$ graphs



(a)



(b)

$V(G) = \mathcal{P} \cup \mathcal{C}$

$\mathcal{P}$ incident to pendant edges, and those form a matching.

$\mathcal{C}$ set of 5-cycles, adjacent vertices can not both have degree greater than two.

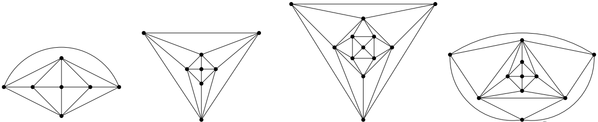
Theorem (H&S)

If G is a non-trivial, connected, well-covered graph of girth at least five, then $d_0(G) = \Gamma(G) + 1$.

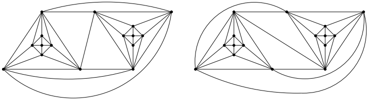
Well-covered plane triangulations

Theorem (Finbow, Hartnell, Nowakowski, Plummer)

A plane triangulation G is well-covered if and only if $G \in \mathcal{K}^+$ *or* $G \in \{K_3, R_6, R_7, R_8, R_{12}, R_8 \bigcirc K_3, R_8 \bigcirc R_8\}$.

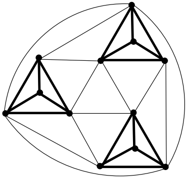


R_7 R_8 R_{12} $R_8 \bigcirc K_3$



The two non-isomorphic versions of $R_8 \bigcirc R_8$.

The Well covered plane triangulations $\mathcal{K}^+$



Construct a graph $G \in \mathcal{K}^+$ as follows:

Begin with a plane triangulation T from the family $\mathcal{L}$, where T has kernel $\{q_{10}, q_{20}, \ldots, q_{\mu 0}\}$, and q_{i0} has degree three in T, $1 \leqslant i \leqslant \mu$.

In each face of T that is *not* incident with a kernel vertex do one of the following: (i)nothing, (ii)O-join a triangle, or (iii)O-join a copy of R_8.

Theorem (H&S)

If G is a well-covered triangulation of the plane, then $d_0(G) = \Gamma(G) + 1$.

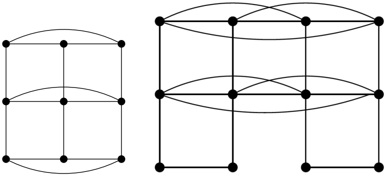
- (Finbow and van Bommel) Most graphs in $\mathcal{K}^+$ are not well-dominated.
- This makes proof for $G \in \mathcal{K}^+$ more complex.
- A maximal independent set of G has one vertex from each K_4, one vertex from each O-joined triangle, and two vertices from each O-joined R_8.
- Other minimal dominating sets might use a vertex from the original triangulation to dominate a vertex in an O-joined triangle or $R - 8$. And, may not use all kernel vertices.

Idea of proof:

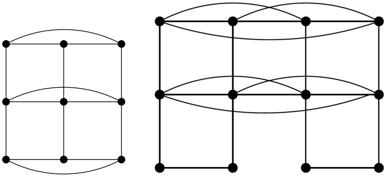
For S is a minimal dominating set, consider the connected component of $D_\Gamma(G)$ containing S.

Find the member of the component that uses the least non-kernel vertices and then show that number has to be 0.

On the other hand the graphs below are well covered but $d_0(G) = \Gamma(G) + 2$.



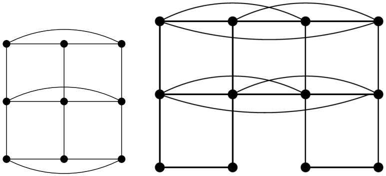
On the other hand the graphs below are well covered but $d_0(G) = \Gamma(G) + 2$.



Open:

1. Characterize graphs for which $d_0(G) = \Gamma(G) + 1$

On the other hand the graphs below are well covered but $d_0(G) = \Gamma(G) + 2$.



Open:

1. Characterize graphs for which $d_0(G) = \Gamma(G) + 1$
2. Are there any graph for which $d_0(G) > \Gamma(G) + 2$.