Cohesive behaviour arising in homogenization of Mumford-Shah type functionals

Marco Barchiesi

Università di Napoli “Federico II”

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Brittle and Cohesive-zone models

Setting: Ω represents the cross section of a cylindrical body in its reference configuration, while $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ represents the displacement (antiplane shear): $(x_1, x_2, x_3) \mapsto (x_1, x_2, u(x_1, x_2))$.

Brittle model (Griffith 1920)

$$MS(u) := \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \mathcal{H}^1(S_u)$$

Cohesive model (Barenblatt 1959, Dugdale 1960)

$$E(u) := \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_{S_u} g([u]) \, \mathrm{d}\mathcal{H}^1$$

$g([u])$:=energy per unit area spent to create a crack with opening

$[u] := u^+ - u^-$ (g nondecreasing, concave, bounded)

Aim of the work

- ⋆ Derive a cohesive-type model by homogenising a purely brittle composite whose components have different elastic moduli but the same toughness
- ⋆ Show that the cohesive-type model so obtained is not the "right one".

Γ-convergence

What is the Γ-convergence?
It is a tool to analyze the asymptotic behaviour of a sequence of minimum problems of the form

$$\mathrm{m}_k = \min\{F_k(u) \ : \ u \in \mathbb{U}\},$$

where

- $\mathbb{U}$ is a normed space;
- F_k is a sequence of functionals on $\mathbb{U}$.

Definition

We say that F_k Γ-converges to a functional F, if for every $u \in \mathbb{U}$ the following conditions are satisfied:

i) Liminf inequality: for every sequence u_k in $\mathbb{U}$ such that $u_k \to u$,

$$F(u) \leq \liminf_{k \to +\infty} F_k(u_k);$$

ii) Recovery sequence: there exists a sequence u_k in $\mathbb{U}$ such that $u_k \to u$ and

$$F(u) = \lim_{k \to +\infty} F_k(u_k).$$

Main Property

Let u_k be a minimum for F_k. If F_k Γ-converges to F and $u_k \to u$ in $\mathbb{U}$, then

- $F_k(u_k) \to F(u)$;
- u is a solution of the minimum problem

$$\mathrm{m} = \min\{F(u) \ : \ u \in \mathbb{U}\}.$$

Homogenization

We use this tool to describe composites, *i.e.*, structures constituted by two or more materials which are finely mixed at microscopic length scales.

Despite the high complexity of their microstructure, composites appear essentially as homogeneous at macroscopic length scale.

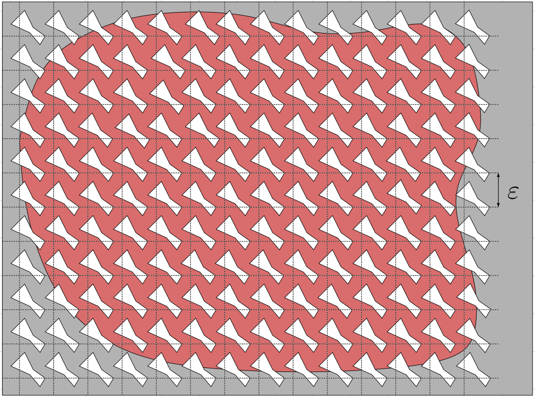
This suggests that their effective properties be a kind of average made on the respective properties of the constituents.

Homogenization: think a composite as a limit of a sequence of structures whose heterogeneities become finer and finer, and extract the effective property via the Γ-limit.

For instance, let

$$F_\varepsilon(u) = \alpha_1 \int_{\Omega\cap\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \alpha_2 \int_{\Omega\setminus\varepsilon P} |\nabla u|^2 \,\mathrm{d}x$$
$$+\beta_1 \mathcal{H}^1(S_u \cap \varepsilon P) + \beta_2 \mathcal{H}^1(S_u \setminus \varepsilon P),$$

where $u \in SBV^2(\Omega)$, $P \subset \mathbb{R}^2$ is a periodic set, and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are constants,



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where $u \in SBV^2(\Omega)$, $P \subset \mathbb{R}^2$ is a periodic set, and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are constants. Fixed a sequence $\varepsilon_k \to 0$, one gets as Γ-limit an integral/local functional

$$F(u) = \int_\Omega f(\nabla u) \, \mathrm{d}x + \int_{S_u} g(\nu_u) \, d\mathcal{H}^1.$$

Under standard growth conditions, homogenisation in SBV preserves independence of the amplitude $[u]$.

Aim of the work

Without standar growth condition the situation is different. In particular it is possible to obtain functionals having a cohesive behaviour

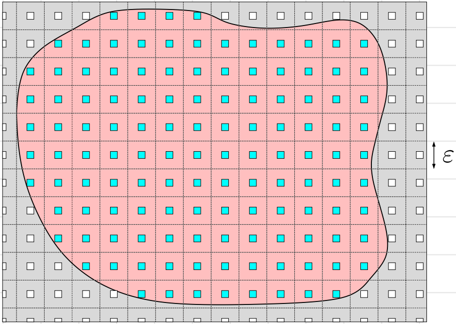
$$F(u) = \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{S_u} g([u], \nu_u)\, d\mathcal{H}^1.$$

Surprisingly, this happens with a very simple functional: the previous one with $\alpha_1 = \beta_1 = \beta_2 = 1$ and $\alpha_2 = \varepsilon$ (different elastic moduli but the same toughness), and a basic geometry P.

The key: because the "softening factor", at microscopic level it is possible to approximate a pure jump with a stretch.

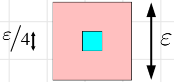
A strange phenomenon: the jump set of a recovery sequence strongly depends on the amplitude of the jump of the limit.

Brittle materials with soft inclusions



$\Omega \subset \mathbb{R}^2$ open, bounded

$P =$ union of cells $Q_1 \setminus Q_{\frac{1}{4}}$,
open, connected, periodic



$\Omega \cap \varepsilon P =$ stiff matrix

$\Omega \setminus \varepsilon P =$ soft inclusions

$$F_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV^2(\Omega)$$

As $\varepsilon \to 0$ we determine the macroscopic behaviour, via Γ-convergence (w.r.t. s-L^1)

Homogenisation result

$$F_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV^2(\Omega)$$

Theorem (B.-Lazzaroni-Zeppieri, SIAM J. Math. Anal. 2016)

Given $\varepsilon_k \to 0$, up to subsequences $F_{\varepsilon_k} \xrightarrow{\Gamma} F$ with

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{S_u} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^1 \qquad \textit{for } u \in GSBV^2(\Omega)$$

- *f is the quadratic form given by a standard cell formula;*
- *$g(\cdot,\nu)$ is nondecreasing, $g(-t,-\nu) = g(t,\nu)$, and*

$$\min\{\tfrac{3}{4} + c\,t^2,\, 1\} \le g(t,e_i) \le \min\{\tfrac{3}{4} + \sqrt{2}\,t\,,\, 1\} \qquad \textit{for } i = 1,2$$

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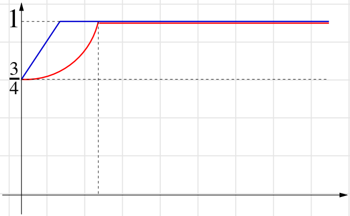
Remarks on the limit model

Homogenised functional

$$F(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{S_u} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^1 \qquad \text{for } u \in GSBV^2(\Omega)$$

$$\min\{\tfrac{3}{4} + c\,t^2,\, 1\} \le g(t,e_i) \le \min\{\tfrac{3}{4} + \sqrt{2}\,t\,,\, 1\} \qquad \text{for } i = 1,2$$

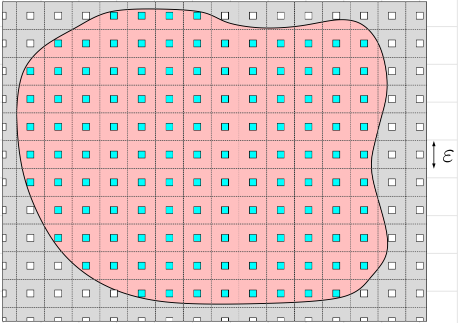
- $g(0,e_i) > 0$ activation threshold
- $g(t,e_i) = 1$ for large $|t|$



Cohesive-type behaviour: g depending nontrivially on $[u]$, constant for large $|[u]|$

A simpler model: perforated domains

$$F_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV^2(\Omega)$$



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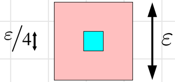
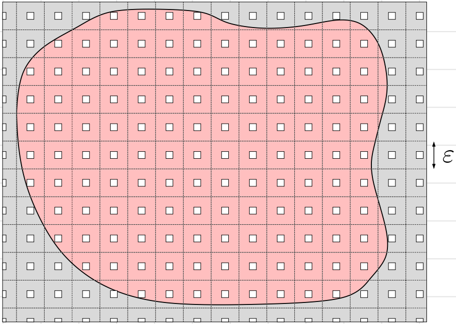


$\Omega \cap \varepsilon P =$ stiff matrix

$\Omega \setminus \varepsilon P =$ soft inclusions

A simpler model: perforated domains

$$\hat{F}_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(S_u \cap \Omega \cap \varepsilon P) \quad \text{for } u \in SBV^2(\Omega \cap \Omega \cap \varepsilon P)$$



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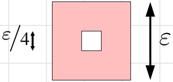

$\Omega \cap \varepsilon P$ = brittle domain

$\Omega \setminus \varepsilon P$ = perforation

A simpler model: perforated domains

$$\hat{F}_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u \cap \Omega \cap \varepsilon P) \qquad \text{for } u \in SBV^2(\Omega \cap \Omega \cap \varepsilon P)$$

See Focardi-Gelli-Ponsiglione 2009, Cagnetti-Scardia 2011, B.-Focardi 2011.

Theorem

$$\textit{For } \varepsilon \to 0\,, \quad \hat{F}_\varepsilon \xrightarrow{\Gamma} \hat{F}(u) := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{S_u} \hat{g}(\nu_u)\,\mathrm{d}\mathcal{H}^1$$

$$\textit{where} \quad \begin{cases} c\,|\xi|^2 \le f(\xi) \le \mathcal{L}^2(Q\cap P)|\xi|^2 \\ c \le \hat{g}(\nu) \le \mathcal{L}^2(Q \cap P) \end{cases} \implies \quad c\, MS \le \hat{F}$$

$\hat{F}_\varepsilon \leq F_\varepsilon \leq MS \quad \Longrightarrow \quad c\, MS \leq \Gamma\text{-}\liminf F_\varepsilon \leq \Gamma\text{-}\limsup F_\varepsilon \leq MS$,
Good estimate: there is integral representation of the Γ-limit of F_ε.

Since

$$\hat{g}(e_2) = \hat{F}(u, Q),$$

the value of $\hat{g}(e_2)$ is simply given by the best way to "approximate in energy" $u := \chi_{(-1/2,1/2)\times(0,1/2)}$, so

$$\hat{g}(e_2) = \frac{3}{4}.$$

Therefore $g(t,\nu) \geq \hat{g}(\nu)$ in a sharp way:

$$\min\{\tfrac{3}{4} + c\,t^2,\, 1\} \leq g(t, e_2) \leq \min\{\tfrac{3}{4} + \sqrt{2}\,t\,,\, 1\}$$

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Large crack-opening

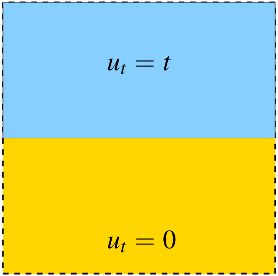
To prove:

$$g(t, e_2) \leq \min\{\tfrac{3}{4} + \sqrt{2}\, t\,,\ 1\}$$

$$g(t, e_2) = F(u_t, Q) \leq MS(u_t, Q) = 1$$

$$F_{\varepsilon_k}(u_k, Q) = 1$$

where $\quad u_t = u_k = t\, \chi_{Q^+}$

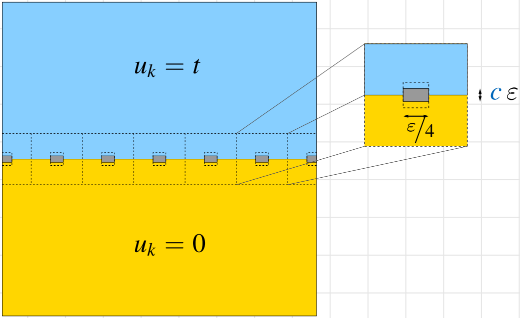


$\leadsto$ The "pure jump" is optimal for large values of t

Small crack-opening: bridging mechanism

To prove:

$$g(t,e_2) \le \min\left\{ \tfrac{3}{4} + \sqrt{2}\,t\,,\ 1 \right\}$$



Optimising in c leads to $c = \frac{t}{2\sqrt{2}}$, hence $F_{\varepsilon_k}(u_k) \simeq \frac{3}{4} + \sqrt{2}\,t$

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To prove:

$$g(t, e_2) \leq \min\left\{ \tfrac{3}{4} + \sqrt{2}\, t \,,\ 1 \right\}$$

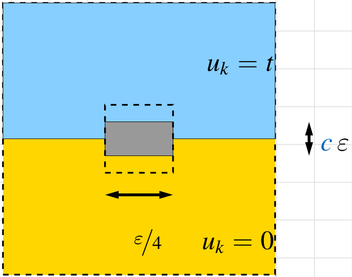
Cost of an affine transition in the grey region

+ Cost of the jumps

summed up over all interfacial cells

$$F_{\varepsilon_k}(u_k) \simeq \frac{1}{\varepsilon_k}\left(\varepsilon_k \left(\frac{t}{c\,\varepsilon_k} \right)^2 \frac{c}{4}\, \varepsilon_k^2 + 2c\,\varepsilon_k + \frac{3}{4}\,\varepsilon_k \right)$$

$$= \frac{t^2}{4c} + 2c + \frac{3}{4}$$



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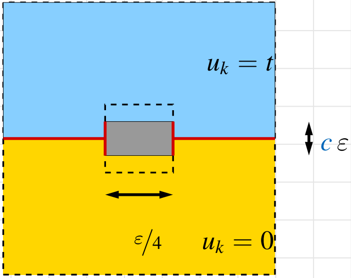

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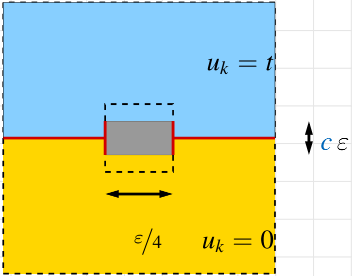

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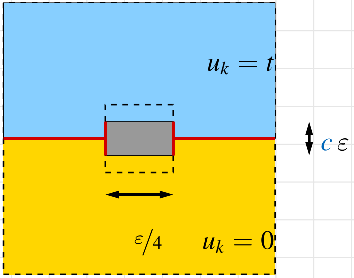

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Cohesive behaviour: Lower bound

$$g(t,e_2) = F(u_t,Q) = \inf\left\{ \liminf F_{\varepsilon_k}(u_k,Q) \colon u_k \to u_t \right\}$$

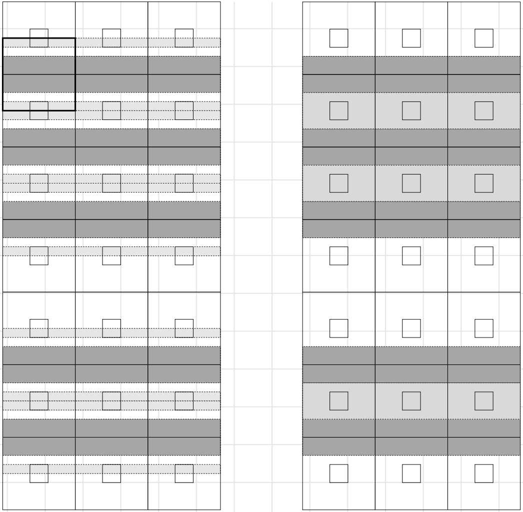
To prove:

$$\liminf F_{\varepsilon_k}(u_k,Q) \geq \min\left\{ \tfrac{3}{4} + c\,t^2\,,\ 1 \right\} \qquad \forall\, u_k \to u_t$$

Strategy: Modify u_k obtaining a new sequence w_k such that

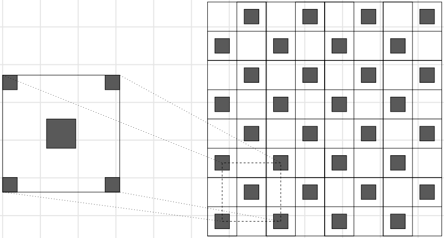
- ⋆ $w_k \to u_t = \lim u_k$
- ⋆ $\liminf F_{\varepsilon_k}(w_k,Q) \leq \liminf F_{\varepsilon_k}(u_k,Q)$
- ⋆ w_k is ε_k-periodic and symmetric in the first variable
- ⋆ w_k is piecewise affine outside a horizontal layer L_k of thickness $\simeq \varepsilon_k$
- ⋆ the energy of w_k essentially concentrates in L_k and

$$\liminf F_{\varepsilon_k}(w_k,L_k) \geq \min\left\{ \tfrac{3}{4} + c\,t^2\,,\ 1 \right\}$$



(a) (b)

Brittle materials with soft inclusions II



$\Omega \subset \mathbb{R}^2$ open, bounded

$P =$ open, connected, periodic

$\Omega \cap \varepsilon P =$ the stiff matrix, white

$\Omega \setminus \varepsilon P =$ the soft inclusions, gray.

$$F_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2 \, \mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2 \, \mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV^2(\Omega)$$

As $\varepsilon \to 0$ we determine the macroscopic behaviour, via Γ-convergence (w.r.t. s-L^1)

Homogenisation result II

$$F_\varepsilon(u) := \int_{\Omega\cap\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \varepsilon \int_{\Omega\setminus\varepsilon P} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV^2(\Omega)$$

Theorem

Given $\varepsilon_k \to 0$, up to subsequences $F_{\varepsilon_k} \xrightarrow{\Gamma} F$ with

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- $g(t,e_2) \le \frac{1}{\sqrt{2}} + 2\sqrt{2}\,t$.
- $g(t,e_2) = 1$ *for large* t.

Perforated domains again

$$\hat{g}(e_2) = \frac{1}{\sqrt{2}}$$

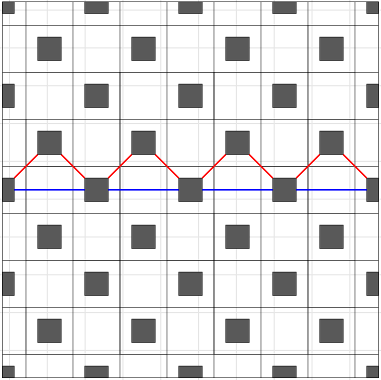
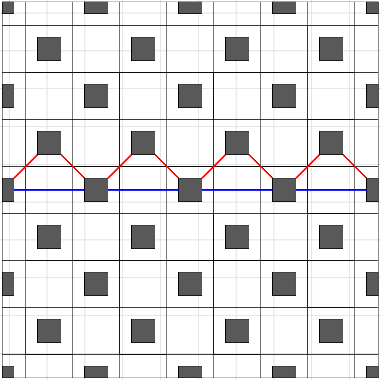


Figure: In red the "zig-zag" configuration.

Perforated domains again

Note that a zig-zag configuration is shorter than a straight line.



Small crack-opening: bridging mechanism

Proof that

$$g(t, e_2) \le \frac{1}{\sqrt{2}} + 2\sqrt{2}\,t$$

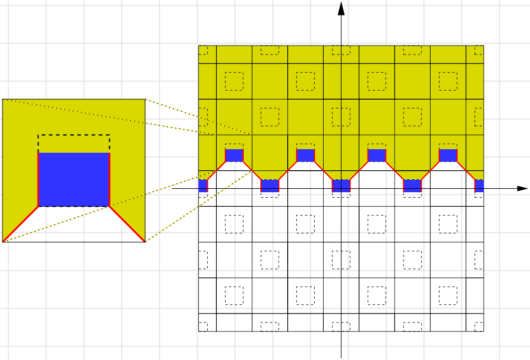
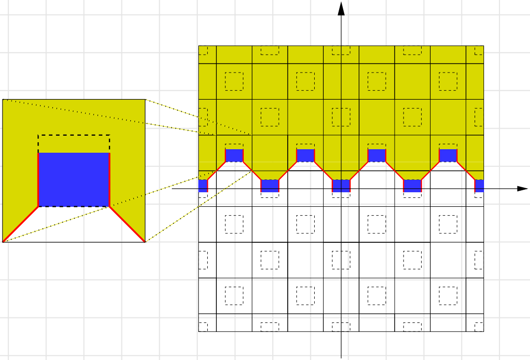


Figure: In yellow the set where u_k takes value t, in blue the set where u_k is affine, and in red the jump set S_{u_k}.

Small crack-opening: bridging mechanism

The idea is that $F_\varepsilon \sim \hat{F}_\varepsilon$ for small t.



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Small crack-opening: bridging mechanism

The idea is that $F_\varepsilon \sim \hat{F}_\varepsilon$ for small t.

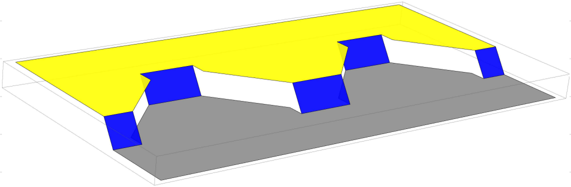


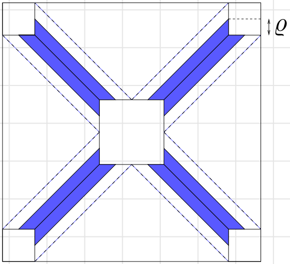
Figure: In yellow the set where u_k takes value t, in blue the set where u_k is affine, and in red the jump set S_{u_k}.

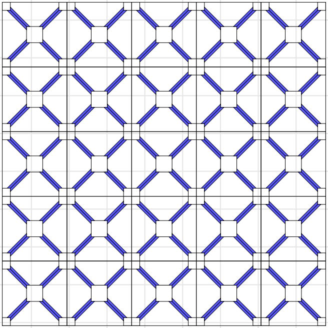
Optimality of the previous costruction (as position)

Given a small $\eta > 0$, take t so small that

$$F(u_t, Q) = g(t, e_2) \leq \frac{1}{\sqrt{2}} + \eta.$$

Here $u_t := t\chi_{(-1/2,1/2)\times(0,1/2)}$. Given a small $\varrho > 0$, define the sets T (on the left) and T_{ε_k} (on the right).





Optimality of the previous costruction

The key is that "If we want to stay close to $1/\sqrt{2}$ the jump set has to be close to the diagonal".

Theorem

Consider a recovery sequence u_k for u_t. Then

$$\mathcal{H}^1(S_{u_k} \cap T_{\varepsilon_k}) \geq \frac{1}{\sqrt{2}} - \frac{\eta}{4\varrho}.$$

Localization of the jump set

Look to the εP has a bundle of fiber l_ε^m, i.e., $\varepsilon P = \bigcup\{l_\varepsilon^m : m \in M_\varepsilon\}$. Note that the bundle undergos a sort of compression along the diagonal.

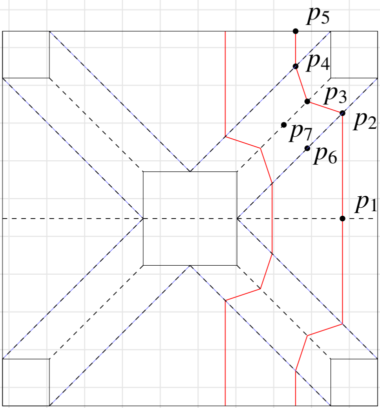
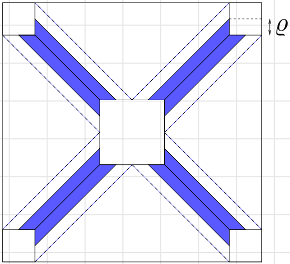
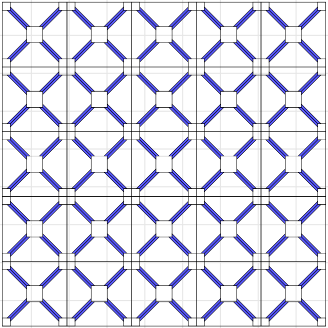


Figure: In red a couple of fibers l^m.

Localization of the jump set

The fibers have to be (asymptotically) cut.
In order to cut the bundle of fibers, the best choice is to make the cut in T_ε. Indeed, here the hard region εP is thin just $1/\sqrt{2}$. On the other hand, outside T_ε the best choice is to make the cut along the diagonal part of the boundary of T_ε itself. Indeed, here the hard region εP is thin $(1+4\varrho)/\sqrt{2}$.
Therefore, the ratio of the costs between the optimal cuts outside and inside T_ε is $1+4\varrho$.





Large crack-opening: “soft is not so soft”

Theorem

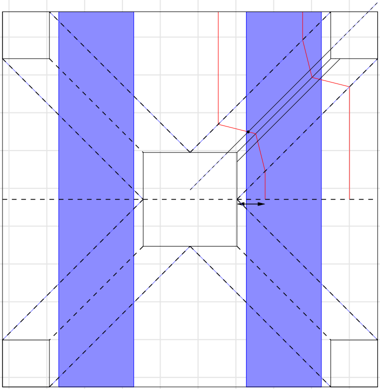
Consider a sequence u_k converging to u_t, t large. Then

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u_k, Q \setminus T_{\varepsilon_k}) \geq \frac{1}{2} - 4\varrho.$$

In particular, if u_k is a recovery sequence for u_t, $\mathcal{H}^1(S_{u_k})$ in T_{ε_k} cannot be larger than $1/\sqrt{2}$ ($> 1/2 + 4\varrho$).

Strategy: similar to the previous model.

Large crack-opening: “soft is not so soft”



Toughening phenomenon

Main Remark

Let t be small and u_k be a recovery sequence for u_t. Moreover, let $\tilde{t}$ be large and $\tilde{u}_k$ be a sequence converging to $u_{\tilde{t}}$. If $S_{\tilde{u}_k} \supset S_{u_k}$, then

$$\liminf_{k \to +\infty} F_{\varepsilon_k}(\tilde{u}_k, Q) \gtrsim \frac{1}{\sqrt{2}} + \frac{1}{2} > 1.$$

- The bridging mechanism increases the tougheness of the material: being energetically favorable when the amplitude of the crack is small, it originates a deflection of the crack path towards the soft inclusion. Because of the irreversibility of the crack process due to dissipation, this deflection persists also when the amplitude of the crack is large and a straight path should be energetically favorable with respect to the deflected one.
- This behavior cannot be captured by the Γ-limit F, since it is obtained by a minimization problem at microscopic level for any fixed amplitude of the crack.

- A bridging mechanism in the homogenisation of brittle composites with soft inclusions. Joint work with G. Lazzaroni and C. I. Zeppieri. *SIAM J. Math. Anal.*, 48 (2016).
- Toughening by crack deflection in the homogenization of brittle composites with soft inclusions. *Arch. Ration. Mech. Anal.* to appear.

http://cvgmt.sns.it/people/barchiesi/