# Statics and Hydrodynamics of Biaxial Nematics

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# Outline

1. Liquid crystal state is an intermediate phase between the solid state and the isotropic liquid state: anisotropic nature.
2. Three common types of liquid crystals:

   (a)nematic (orientation order, no position order);

   (b)cholesteric (orientation order, partial position order: layer structures);

   (c)smectic (orientation order, partial position order: sheet structures).

- Oseen-Frank theory (vector model): mean orientation of molecule's optical axis, $\mathbf{n} : \Omega \to \mathbb{S}^2$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, minimizes Oseen-Frank's bulk energy functionals.

**Oseen-Frank density function**:
1) $W(\mathbf{n}, \nabla \mathbf{n})$ quadratic in $\nabla \mathbf{n}$; and
2) $W(-\mathbf{n}, -\nabla \mathbf{n}) = W(Q\mathbf{n}, Q\nabla \mathbf{n} Q^t) = W(\mathbf{n}, \nabla \mathbf{n}),\ Q \in O(3) \Rightarrow$

$$\begin{aligned} 2W(\mathbf{n}, \nabla \mathbf{n}) &= k_1(\mathrm{div}\mathbf{n})^2 + k_2(\mathbf{n} \cdot \mathrm{curl}\mathbf{n} + \tau)^2 + k_3|\mathbf{n} \times \mathrm{curl}\mathbf{n}|^2 \\ &+ (k_2 + k_4)[\mathrm{tr}(\nabla \mathbf{n})^2 - (\mathrm{div}\mathbf{n})^2], \end{aligned}$$

$k_1, k_2, k_3 > 0$: splay, twist, and bending constants,
$k_2 \geq |k_4|,\ 2k_1 \geq k_2 + k_4$. ($\tau \neq 0$: cholesterics; $\tau = 0$: nematics).
Observation (Oseen): $[\mathrm{tr}(\nabla \mathbf{n})^2 - (\mathrm{div}\mathbf{n})^2]$ is null-Lagrange, i.e.,
$\int_\Omega [\mathrm{tr}(\nabla \mathbf{n})^2 - (\mathrm{div}\mathbf{n})^2]$ depends on $\mathbf{n}|_{\partial\Omega}$.

**Equilibrium configuration**: For $g : \partial\Omega \to \mathbb{S}^2$, $\exists\ \mathbf{n} \in H^1(\Omega, \mathbb{S}^2)$ minimizing the Oseen-Frank energy $\mathcal{W}(\mathbf{n}) = \int_\Omega W(\mathbf{n}, \nabla\mathbf{n})$, i.e.,

$$\mathcal{W}(\mathbf{n}) = \min\Big\{\mathcal{W}(e) :\ e \in W^{1,2}(\Omega, \mathbb{S}^2),\ e = g \text{ on } \partial\Omega\Big\}.$$

The Euler-Lagrange equation (not necessarily elliptic):

$$\frac{\delta\mathcal{W}}{\delta\mathbf{n}} := \frac{d}{dt}\Big|_{t=0}\mathcal{W}(\mathbf{n}^t) = 0,\ \ \mathbf{n}^t = \frac{\mathbf{n} + t\phi}{|\mathbf{n} + t\phi|},\ \ \phi \in C_0^\infty(\Omega, \mathbb{R}^3).$$

**Simple case**: $k_1 = k_2 = k_3 = k_2 + k_4 = 1 \Rightarrow$

$$\mathcal{W}(\mathbf{n}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{n}|^2 \Rightarrow$$

$$\frac{\delta\mathcal{W}}{\delta\mathbf{n}} = \Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n} = 0 \text{ (harmonic map to } \mathbb{S}^2\text{)}.$$

## Regularity of minimizers

1. (Hardt-Lin-Kindelerhrer, 86's) $\mathbf{n} \in C^\infty(\Omega \setminus \Sigma, \mathbb{S}^2)$, with $\dim_H(\Sigma) < 1$.

It covers the point defects observed in experiments and numerical stimulations:

$\mathbf{n}(x) = \frac{x}{|x|} : \mathbb{R}^3 \to \mathbb{S}^2$ (hedgehog).

2. (Open question) What's the optimal size estimate of the singular set $\Sigma$ for Oseen-Frank minimizers? Is $\Sigma$ a finite set?

Statistical head-to-tail symmetry or non-polarity $\Longrightarrow$ **n**: thought as *line* field instead of *vector* field, i.e. $\mathbf{n} \sim -\mathbf{n}$; and $\mathbb{S}^2$ should be replaced by $\mathbb{RP}^2$, the *real projective plane*; or **n** replaced by $\mathbf{n} \otimes \mathbf{n}$.

Line defects (disinclination or saturn ring types) and wall defects of nematic liquid crystals observed in experiments, which is beyond the scope that the Oseen-Frank theory is capable to describe.

Laudan- De Gennes model:

For simplicity, let $\rho(x, \mathbf{p}) : \Omega \times \mathbb{S}^2 \to \mathbb{R}_+$ be the probability distribution function of molecular orientation $\mathbf{p} \in \mathbb{S}^2$, $x \in \Omega$. Think **p** as the orientation of the long axis of a molecule. Then

$$\rho(x, \mathbf{p}) = \rho(x, -\mathbf{p}), \quad \int_{\mathbb{S}^2} \rho(x, \mathbf{p})\, d\sigma(\mathbf{p}) = 1.$$

The De Gennes $Q$-tensor is defined to be the traceless second moment map of $\rho$:

$$Q(x) = \int_{\mathbb{S}^2} (\mathbf{p} \otimes \mathbf{p} - \frac{1}{3}\mathbf{I}_3)\rho(x, \mathbf{p})\, d\sigma(\mathbf{p}). \quad (1)$$

$$\Downarrow$$

$$Q(x) = Q^T(x),\ \mathrm{tr}Q(x) = 0,\ \lambda_{\min}(Q(x)) > -\frac{1}{3}.$$

Landau-De Gennes energy functional is given by

$$I(Q) = \int_\Omega \psi(Q, \nabla Q), \quad (2)$$

where $\psi(Q, \nabla Q)$ enjoys frame-indifference, and is written as

$$\psi(Q, \nabla Q) = \psi(Q, 0) + (\psi(Q, \nabla Q) - \psi(Q, 0)) = \psi_B(Q) + \psi_E(Q, \nabla Q).$$

Typically

$$\psi_B(Q) = a\mathrm{tr}Q^2 - \frac{2b}{3}\mathrm{tr}Q^3 + c\mathrm{tr}Q^4,$$

where $b, c > 0$, and $a = a(T)$.

$$a < \frac{b^2}{27c} \Longrightarrow Q = s_+(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}_3) = \mathrm{argmin}_Q \psi_B(Q),$$

where $\mathbf{n} \in \mathbb{S}^2$, and

$$s_+ = \frac{b + \sqrt{b^2 - 24ac}}{4c} > 0.$$

$$\psi_E(Q, \nabla Q) = L_1|\nabla Q|^2 + L_2(\mathrm{div} Q)^2 + L_3 \mathrm{tr}(\nabla Q)^2. \qquad (3)$$

Here

$$N(\nabla Q) := (\mathrm{div} Q)^2 - \mathrm{tr}(\nabla Q)^2$$

is a null-Lagrangian.
If $L_1 = 1$ and $L_2 = L_3 = 0$, then

$$\psi_E(Q, \nabla Q) = L_1|\nabla Q|^2$$

is referred as *one constant approximation* of the Landau-De Gennes theory.

Although Landau-De Gennes theory provides a universal description of nematic liquid crystals, the characterization of defects turns out much more challenging mathematically: One can show that any minimizer of

$$I(Q) = \int_\Omega L_1 |\nabla Q|^2 + \psi_B(Q)$$

over $H^1(\Omega, \mathcal{S}_0)$ is smooth, in fact real analytic. However, since the defect point is reflected through the “bad" points of eigenvectors of the solutions, this makes it very had to study the size and structure of defect sets. In contrast, a defect in the Oseen-Frank theory is precisely defined to the point of discontinuity of **n**.

Recall that at any point $x \in \Omega$, there always exist $S, T \in \mathbb{R}$ and $\mathbf{n}, \mathbf{m} \in \mathbb{S}^2$ such that $\mathbf{n} \perp \mathbf{m}$, and we can decompose

$$Q(x) = S(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}_3) + T(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I}_3).$$

If either $S$ or $T$ vanishes, then $Q$ becomes the uniaxial nematics, which further reduces to:
(i) the Oseen-Frank model (where $S$ is constant in $\Omega$), or
(ii) the modified Ericksen model (where $S$ is varying in $\Omega$).
Assume $0 < S, T < 1$ are constants, and $Q(x)$ has eigenpairs:

$$\boxed{\lambda_1 = S - \tfrac{1}{3} > -\tfrac{1}{3} \text{ and } \mathbf{n};\ \lambda_2 = T - \tfrac{1}{3} \text{ and } \mathbf{m}.}$$

Govers-Vertogen (PRA, 84) derived the Landau-De Gennes energy for biaxial nematics $\mathcal{W}(\mathbf{n}, \mathbf{m}) = \int_\Omega W(\mathbf{n}, \mathbf{m}, \nabla \mathbf{n}, \nabla \mathbf{m})$ (Leslie-Laverty-Carlsson, 92; Stewart, 04; Sonnet-Virga,11):

$$\begin{aligned} & W(\mathbf{n}, \mathbf{m}, \nabla \mathbf{n}, \nabla \mathbf{m}) \quad (4) \\ = \quad & \frac{1}{2} K_{11} (\mathrm{div}\mathbf{n})^2 + \frac{1}{2} K_{12} (\mathbf{n} \cdot \mathrm{curl}\mathbf{n})^2 + \frac{1}{2} K_{13} |\mathbf{n} \times \mathrm{curl}\mathbf{n}|^2 \\ + \quad & \frac{1}{2} K_{21} (\mathrm{div}\mathbf{m})^2 + \frac{1}{2} K_{22} (\mathbf{m} \cdot \mathrm{curl}\mathbf{m})^2 + \frac{2}{3} K_{23} |\mathbf{m} \times \mathrm{curl}\mathbf{m}|^2 \\ + \quad & \frac{1}{2} K_{31} (\mathrm{div}\mathbf{l})^2 + \frac{1}{2} K_{32} (\mathbf{l} \cdot \mathrm{curl}\mathbf{l})^2 + \frac{1}{2} K_{33} |\mathbf{l} \times \mathrm{curl}\mathbf{l}|^2 \\ + \quad & \frac{1}{2} K_{41} (\mathbf{n} \cdot \mathrm{curl}\mathbf{l})^2 + \frac{1}{2} K_{42} (\mathbf{m} \cdot \mathrm{curl}\mathbf{n})^2 + \frac{1}{2} K_{43} |\mathbf{l} \times \mathrm{curl}\mathbf{m}|^2, \end{aligned}$$

where $\mathbf{l} = \mathbf{n} \times \mathbf{m}$, $K_{ij} \geq 0$ are Frank's constants.

# Two special cases of biaxial nematics

**I.** *One constant approximation*:
Assume

$$K_{3p} = K_{4p} = 0, p = 1, 2, 3; \text{ and } K_{ij} = 1, 1 \leq i, j \leq 2.$$

Then

$$\begin{aligned} W(\mathbf{n}, \mathbf{m}, \nabla\mathbf{n}, \nabla\mathbf{m}) &= \frac{1}{2}\left(|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2\right) \\ &\quad -\frac{1}{2}\left(\mathcal{N}(\mathbf{n}) + \mathcal{N}(\mathbf{m})\right), \end{aligned}$$

where

$$\mathcal{N}(\mathbf{p}) = \mathrm{tr}(\nabla\mathbf{p})^2 - (\mathrm{div}\mathbf{p})^2 \text{ for } \mathbf{p} : \Omega \to \mathbb{S}^2,$$

is a null-Lagrange.

Therefore we obtain

$$\mathcal{W}(\mathbf{n},\mathbf{m}) = E(\mathbf{n},\mathbf{m}) + \text{terms depending only on } \mathbf{n}|_{\partial\Omega}, \mathbf{m}|_{\partial\Omega},$$

where

$$E(\mathbf{n},\mathbf{m}) = \int_\Omega \frac{1}{2}\left(|\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2\right).$$

**Defintion.** A pair of maps $(\mathbf{n},\mathbf{m}) \in H^1(\Omega,\mathbb{S}^2)^{\otimes 2}$, with $\mathbf{n} \perp \mathbf{m}$ in $\Omega$, is called a harmonic biaxial map, if it is a critical point of $E(\mathbf{n},\mathbf{m})$ over the configuration space

$$\mathcal{A} = \left\{(\mathbf{n},\mathbf{m}) \in H^1(\Omega,\mathbb{S}^2)^{\otimes 2} : \mathbf{n}(x)\cdot\mathbf{m}(x) = 0 \text{ a.e. } x \in \Omega\right\}.$$

**Lemma 1.** *A pair map* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *is a harmonic biaxial map, iff*

$$\begin{cases} \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{m} = 0, \\ \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{n} = 0. \end{cases} \tag{5}$$

*Proof.* Employing Lagrange multipliers, $(\mathbf{n}, \mathbf{m})$ is critical point of

$$E_{\lambda,\mu,\delta}(\mathbf{n}, \mathbf{m}) = E(\mathbf{n}, \mathbf{m}) + \frac{1}{2} \int_\Omega (\lambda(1 - |\mathbf{n}|^2)^2 + \mu(1 - |\mathbf{m}|^2)^2 + \delta(\mathbf{n} \cdot \mathbf{m})^2).$$

Thus we obtain $\Delta \mathbf{n} = \lambda_* \mathbf{n} + \delta_* \mathbf{m}$; $\Delta \mathbf{m} = \mu_* \mathbf{m} + \delta_* \mathbf{n}$.
Since $|\mathbf{n}| = |\mathbf{m}| = 1$ and $\mathbf{n} \cdot \mathbf{m} = 0$, it follows

$$\lambda_* = -|\nabla \mathbf{n}|^2, \ \mu_* = -|\nabla \mathbf{m}|^2, \ \delta_* = \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle.$$

In fact, $\delta_* = \langle \Delta \mathbf{n}, \mathbf{m} \rangle = \langle \Delta \mathbf{m}, \mathbf{n} \rangle$ implies $\delta_* = \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle$ and

$$\boxed{\mathrm{div} \langle \nabla \mathbf{n}, \mathbf{m} \rangle = 0.}$$

**Definition 2.** *A harmonic biaxial map* $(\mathbf{n}, \mathbf{m}) \in H^1(\Omega, \mathbb{S}^2)^{\otimes 2}$*, with* $\mathbf{n} \perp \mathbf{m}$ *in* $\Omega$*, is called stationary, if it is also critical with respect to domain variations:*

$$\frac{d}{dt}\Big|_{t=0} E(\mathbf{n}^t, \mathbf{m}^t) = 0, \ \forall\, X \in C_0^1(\Omega, \mathbb{R}^d),$$

*where* $(\mathbf{n}^t, \mathbf{m}^t)(x) = (\mathbf{n}, \mathbf{m})(x + tX(x))$*,* $x \in \Omega$*.*

**Remark.**

1) If $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ is a smooth harmonic biaxial map, then $(\mathbf{n}, \mathbf{m})$ is stationary.

2) If $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ is a minimizing harmonic biaxial map, then $(\mathbf{n}, \mathbf{m})$ is stationary.

**Lemma 2.** *If* $(\mathbf{n}, \mathbf{m})$ *is a stationary harmonic biaxial map, then*

$$
\begin{aligned}
& R^{2-d}\int_{B_R(x)}(|\nabla\mathbf{n}|^2+|\nabla\mathbf{m}|^2) \\
&= r^{2-d}\int_{B_r(x)}(|\nabla\mathbf{n}|^2+|\nabla\mathbf{m}|^2)+ \\
& 2\int_{B_R(x)\setminus B_r(x)}|y-x|^{2-d}\Big(\Big|\frac{\partial\mathbf{n}}{\partial|y-x|}\Big|^2+\Big|\frac{\partial\mathbf{m}}{\partial|y-x|}\Big|^2\Big),
\end{aligned}
\tag{6}
$$

*for* $x\in\Omega$ *and* $0<r\le R<\mathrm{dist}(x,\partial\Omega)$.

**Proposition 1.** *1) Any harmonic biaxial map* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *is smooth for* $d = 2$.
*2) Any stationary harmonic biaxial map* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *is smooth off a closed set* $\mathcal{S}(\mathbf{n}, \mathbf{m})$, *with* $\mathcal{H}^{d-2}(\mathcal{S}(\mathbf{n}, \mathbf{m})) = 0$, *for* $d \geq 3$.

*Proof*. Observe the algebraic structure:

$$\begin{aligned} -\Delta \mathbf{n}^i &= \nabla_\alpha \mathbf{n}^j \big[ (\nabla_\alpha \mathbf{n}^j \mathbf{n}^i - \nabla_\alpha \mathbf{n}^i \mathbf{n}^j) \\ &+ (\nabla_\alpha \mathbf{m}^j \mathbf{m}^i - \nabla_\alpha \mathbf{m}^i \mathbf{m}^j) \big] + \nabla_\alpha \mathbf{m}^i (\nabla_\alpha \mathbf{n}^j \mathbf{m}^j). \end{aligned} \quad (7)$$

Set

$$M_\alpha^{ij} = (\nabla_\alpha \mathbf{n}^j \mathbf{n}^i - \nabla_\alpha \mathbf{n}^i \mathbf{n}^j) + (\nabla_\alpha \mathbf{m}^j \mathbf{m}^i - \nabla_\alpha \mathbf{m}^i \mathbf{m}^j).$$

Applying (5) and (10), we can check that

$$\mathrm{div}(M^{ij}) = \nabla_\alpha M_\alpha^{ij} = 0.$$

Set $N_\alpha = \langle \nabla_\alpha \mathbf{n}^j, \mathbf{m}^j \rangle$. Then (10) $\Rightarrow \mathrm{div} N = \nabla_\alpha (N_\alpha) = 0$.

$$\Delta \mathbf{n}^i = \nabla_\alpha \mathbf{n}^j M_\alpha^{ij} + \nabla_\alpha \mathbf{m}^i N_\alpha \in \mathcal{H}^1(\mathbb{R}^d), \tag{8}$$

where $\mathcal{H}^1(\mathbb{R}^d)$ denotes the Hardy space of $\mathbb{R}^d$.

Applying Wente's or Coifman-Lions-Meyers-Semmes' compensated regularity theorem to (8) implies that $(\mathbf{n}, \mathbf{m}) \in C^0$ when $d = 2$.
The partial regularity in higher dimensions can be obtained by arguments similar to that of Evans for stationary harmonic maps to spheres for $d \geq 3$.

**II**. *Another one constant approximation*:
Assume $K_{4p} = 0, p = 1, 2, 3;$ and $K_{ij} = 1, 1 \leq i, j \leq 3 \Longrightarrow$

$$\begin{aligned} W(\mathbf{n}, \mathbf{m}, \nabla\mathbf{n}, \nabla\mathbf{m}) &= \frac{1}{2}(|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2 + |\nabla\mathbf{l}|^2) \\ &- \frac{1}{2}(\mathcal{N}(\mathbf{n}) + \mathcal{N}(\mathbf{m}) + \mathcal{N}(\mathbf{l})). \end{aligned}$$

In this case, we can consider another energy functional for a biaxial nematics $(\mathbf{n}, \mathbf{m})$:

$$\widehat{E}(\mathbf{n}, \mathbf{m}) = \frac{1}{2}\int_\Omega (|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2 + |\nabla\mathbf{l}|^2).$$

**Lemma 1'.** *A pair of maps* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *is a critical point of* $\widehat{E}$*. Then it solves*

$$\begin{cases} \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{m} + \langle \nabla \mathbf{n}, \nabla \mathbf{l} \rangle \mathbf{l} = 0, \\ \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{n} + \langle \nabla \mathbf{m}, \nabla \mathbf{l} \rangle \mathbf{l} = 0. \end{cases} \tag{9}$$

*Proof.* Think $\{\mathbf{n}, \mathbf{m}, \mathbf{l}\}$ is an orthonormal frame in $\mathbb{R}^3$, we have

$$\Delta \mathbf{n} = \langle \Delta \mathbf{n}, \mathbf{n} \rangle \mathbf{n} + \langle \Delta \mathbf{n}, \mathbf{m} \rangle \mathbf{m} + \langle \Delta \mathbf{l}, \mathbf{l} \rangle \mathbf{l}.$$

$$|\mathbf{n}| = 1 \Longrightarrow \langle \Delta \mathbf{n}, \mathbf{n} \rangle = -|\nabla \mathbf{n}|^2.$$

While

$$\langle \Delta \mathbf{n}, \mathbf{m} \rangle = -\langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \text{ and } \langle \Delta \mathbf{n}, \mathbf{l} \rangle = -\langle \nabla \mathbf{n}, \nabla \mathbf{l} \rangle$$

follows from the argument below.

Let $\mathrm{SO}(3)$ denote the special orthogonal group of $\mathbb{R}^3$. Set

$$H^1(\Omega, \mathrm{SO}(3)) = \left\{R \in H^1(\Omega, \mathbb{R}^{3\times 3}) : R^T(x)R(x) = I_3, \text{a.e. } x \in \Omega\right\}.$$

Rewrite $\{\mathbf{n}, \mathbf{m}, \mathbf{l}\}$ as $\{e_\alpha\}_{\alpha=1}^3$, $\{\mathbf{n}, \mathbf{m}, \mathbf{l}\}$ is critical of $\widehat{E}$

$$\Longrightarrow \frac{d}{dt}\Big|_{t=0} \int_\Omega \frac{1}{2}|\nabla(R^t_{\alpha\beta}(x)e_\beta(x))|^2 = 0,$$

where $R^t :\in C^1((-1,1), H^1(\Omega, \mathrm{SO}(3)))$ is such that $R^0 = I_3$. Denote the Lie algebra of $\mathrm{SO}(3)$ by $so(3)$. Then

$$V = \frac{d}{dt}|_{t=0} R^t \in H^1(\Omega, so(3)),$$

and

$$\frac{d}{dt}\Big|_{t=0}(R^t_{\alpha\beta}(x)e_\beta(x)) = V_{\alpha\beta}(x)e_\beta(x).$$

Then we obtain

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla (V_{\alpha\beta} e_\beta), \nabla e_\alpha \rangle \\ &= \int_\Omega \langle \nabla e_\alpha, \nabla e_\beta \rangle V_{\alpha\beta} + \int_\Omega \langle \langle \nabla e_\alpha, e_\beta \rangle, \nabla V_{\alpha\beta} \rangle \\ &= \int_\Omega \langle \langle \nabla e_\alpha, e_\beta \rangle, \nabla V_{\alpha\beta} \rangle \\ &= 2 \sum_{1 \le \alpha < \beta \le 3} \int_\Omega \langle \langle \nabla e_\alpha, e_\beta \rangle, \nabla V_{\alpha\beta} \rangle, \end{aligned} \tag{10}$$

as $V_{\alpha\beta} + V_{\beta\alpha} = 0$. It follows from (10) that

$$\mathrm{div}(\langle \nabla e_\alpha, e_\beta \rangle) = 0 \text{ in } \mathcal{D}'(\Omega), \ 1 \le \alpha < \beta \le 3. \tag{11}$$

**Remark.** It is not hard to see that $\mathbf{l} \in H^1(\Omega, \mathbb{S}^2)$ also solves

$$\Delta \mathbf{l} + |\nabla \mathbf{l}|^2 \mathbf{l} + \langle \nabla \mathbf{l}, \nabla \mathbf{n} \rangle \mathbf{n} + \langle \nabla \mathbf{l}, \nabla \mathbf{m} \rangle \mathbf{m} = 0. \tag{12}$$

For harmonic biaxial maps with respect to the energy functional $\widehat{E}$, the following fact also holds.

**Proposition 1'.** *1) Any biaxial map $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$, which is critical point of $\widehat{E}$, is smooth for $d = 2$.*
*2) Any biaxial map $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$, that is critical of $\widehat{E}$ with respect to both target and domain variations, is smooth off a closed set $\mathcal{S}(\mathbf{n}, \mathbf{m})$, with $\mathcal{H}^{d-2}(\mathcal{S}(\mathbf{n}, \mathbf{m})) = 0$, for $d \geq 3$.*

If $x_0 \in \mathcal{S}(\mathbf{n}, \mathbf{m})$, then there exist $r_i \to 0$ and a *tangent* harmonic biaxial map $(\phi, \psi) \in H^1(\mathbb{R}^d, \mathbb{S}^2)^{\otimes 2}$, with $\phi \perp \psi$ in $\mathbb{R}^d$, such that

$$\big(\mathbf{n}, \mathbf{m}\big)(x_0 + r_i x) \rightharpoonup (\phi, \psi)(x) \text{ in } H^1(\mathbb{R}^d) \text{ locally.}$$

However, in general, the convergence may *not be strong* in $H^1(\mathbb{R}^d)$ locally.

**Definition 3.** *A* $(\mathbf{n},\mathbf{m})\in\mathcal{A}$ *is a minimizing biaxial map, if*

$$E(\mathbf{n},\mathbf{m})\le E(\mathbf{n}',\mathbf{m}'),$$

*for all* $(\mathbf{n}',\mathbf{m}')\in\mathcal{A}$*, with* $(\mathbf{n}',\mathbf{m}')=(\mathbf{n},\mathbf{m})$ *on* $\partial\Omega$*.*

**Lemma 3.** *If* $\{(\mathbf{n}_i,\mathbf{m}_i)\}\subset\mathcal{A}$ *is a sequence of minimizing biaxial maps, and*

$$(\mathbf{n}_i,\mathbf{m}_i)\rightharpoonup(\mathbf{n},\mathbf{m})\text{ in }H^1(\Omega),$$

*for some* $(\mathbf{n},\mathbf{m})\in\mathcal{A}$*. Then* $(\mathbf{n},\mathbf{m})$ *is a minimizing biaxial map, and*

$$(\mathbf{n}_i,\mathbf{m}_i)\to(\mathbf{n},\mathbf{m})\text{ in }H^1(\Omega).$$

**Proposition 3.** *If* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *is a minimizing biaxial map. Then* $\mathcal{S}(\mathbf{n}, \mathbf{m})$ *has Hausdorff dimension at most* $(d-3)$ *for* $d \geq 4$*, and is discrete when* $d = 3$*.*

*Proof*. For $x_0 \in \mathcal{S}(\mathbf{n}, \mathbf{m})$, it follows from Lemma 3 that there exists $r_i \to 0$, and a tangent minimizing biaxial map $(\phi, \psi) \in \mathcal{A}$ such that

$$(\mathbf{n}, \mathbf{m})(x_0 + r_i \cdot) \to (\phi, \psi) \text{ in } H^1(\mathbb{R}^d) \text{ locally.}$$

(6) $\Longrightarrow$ $(\phi, \psi)$ is homogeneous of degree zero, i.e.

$$(\phi, \psi)(x) = (\phi, \psi)(x/|x|).$$

For $d = 3$, if there exists $\{x_i\} \subset \mathcal{S}(\mathbf{n}, \mathbf{m})$ such that $x_i \to x_0 \in \Omega$. Let $r_i = |x_i - x_0| \to 0$ and $(\mathbf{n}_i, \mathbf{m}_i)(x) = (\mathbf{n}, \mathbf{m})(x_0 + r_i x)$. Then

$$(\mathbf{n}_i, \mathbf{m}_i) \to (\phi, \psi) \text{ in } H^1(\mathbb{R}^d) \text{ locally}$$

for a tangent minimizing biaxial map $(\phi, \psi)$ at $x_0$. It is clear that

$$\mathcal{S}(\phi, \psi) \supset \{0, y_0\}, \text{ with } y_0 = \lim_{i \to \infty} \frac{x_i - x_0}{|x_i - x_0|} \in \mathbb{S}^{d-1}.$$

Hence $\mathcal{S}(\phi, \psi) \supset [0, y_0]$, and $\mathcal{H}^1(\mathcal{S}(\phi, \psi)) > 0$.
For $d \geq 4$, Almgren-Federer dimension reduction argument. $\square$

# Nothing is new about two special cases

For the case I), if we define

$$N = \left\{u = (u_1, u_2) \in \mathbb{S}^2 \times \mathbb{S}^2 : \ u_1 \cdot u_2 = 0\right\}.$$

Then $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ is a harmonic biaxial map $\Longleftrightarrow$
$u = (\mathbf{n}, \mathbf{m}) \in H^1(\Omega, N)$ is a harmonic map from $\Omega$ to $N$, i.e.

$$\Delta u + A(u)(\nabla u, \nabla u) = 0,$$

where $A$ is a second fundamental form of $N \subset \mathbb{R}^6$. Hence Proposition 1 follows from Helein and Bethuel's theories.

$$\boxed{N \approx \mathbb{RP}^3 \approx SO(3) \Rightarrow \Pi_1(N) = \mathbb{Z}_2}$$

For II), $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ is a harmonic biaxial map $\Longleftrightarrow$

$u = (\mathbf{n}, \mathbf{m}, \mathbf{l}) \in H^1(\Omega, SO(3))$ is harmonic from $\Omega$ to $SO(3)$.

For general biaxial energy functional (4), since there is no energy monotonicity available, we can only prove a weaker partial regularity theorem.

**Theorem 1.** *If* $(\mathbf{n}, \mathbf{m}) \in \mathcal{A}$ *minimizes the general biaxial nematics energy* (4), *then* $\mathcal{H}^{d-2}(\mathcal{S}(\mathbf{n}, \mathbf{m})) = 0$.

**Remark.** For uniaxial nematics in the Oseen-Frank theory, Hardt-Kindelerhrer-Lin proved a stronger theorem that asserts the defect set has Hausdorff dimension strictly smaller than 1.

**Definition 4.** *A pair* $(\phi, \psi) \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)^{\otimes 2}$ *is called a biaxial bubble, if* $(\phi, \psi) \neq$ *constant, satisfies the harmonic biaxial map equation in* $\mathbb{R}^2$*, and* $E(\phi, \psi) < +\infty$*.*

**Lemma 3** (gap phenomena).

$$\begin{aligned} c &\equiv \inf\left\{E(\phi,\psi) : (\phi,\psi) : \mathbb{S}^2 \to \mathbb{S}^2 \text{ biaxial bubble}\right\} \\ &= \inf\left\{E(u) : u : \mathbb{S}^2 \to N \text{ nontrivial harmonic}\right\} \\ &> 0. \end{aligned} \tag{13}$$

**Question.** *Estimating c in term of topological invariant of N?*

**Hydrodynamics** (Ericksen and Leslie, 1958-1968): Assume liquid is homogeneous (density $\rho = 1$), $u : \Omega \to \mathbb{R}^n$ is the fluid velocity. Based on:
i) Conservation of linear momentum,
ii) Conservation of angular momentum,
iii) Incompressibility of the fluid,
the Ericksen-Leslie (EL) system takes the form:

$$u_t + u \cdot \nabla u = -\nabla P + \nabla \cdot \sigma \tag{14}$$

$$\nabla \cdot u = 0 \tag{15}$$

$$\mathbf{n} \times \left(\frac{\delta W}{\delta \mathbf{n}} - \gamma_1 N - \gamma_2 A\mathbf{n}\right) = 0 \tag{16}$$

$\sigma$ is modeled by the phenomenological relation: $\sigma = \sigma^L + \sigma^E$.

i) $\sigma^E$ – the elastic (Ericksen) stress: $\sigma^E = -\frac{\partial W}{\partial(\nabla \mathbf{n})} \cdot (\nabla \mathbf{n})^t$.

ii) $\sigma^L$ – the viscous (Leslie) stress ($\alpha_i$'s - Leslie constants):

$$\sigma^L(u, \mathbf{n}) = \begin{cases} \alpha_1(\mathbf{n} \otimes \mathbf{n} : A)\mathbf{n} \otimes \mathbf{n} + \alpha_2 \mathbf{n} \otimes N + \alpha_3 N \otimes \mathbf{n} \\ +\alpha_4 A + \alpha_5 \mathbf{n} \otimes (A \cdot \mathbf{n}) + \alpha_6 (A \cdot \mathbf{n}) \otimes \mathbf{n}, \end{cases}$$

$$A = \frac{\nabla u + (\nabla u)^t}{2}, N = \mathbf{n}_t + u \cdot \nabla \mathbf{n} + \omega \cdot \mathbf{n},\ \omega = \frac{\nabla u - (\nabla u)^t}{2}.$$

iii) $\frac{\delta W}{\delta \mathbf{n}}$ – $1^{\text{st}}$-order variation of $W$.

$$\gamma_1 = \alpha_3 - \alpha_2,\ \gamma_2 = \alpha_6 - \alpha_5 \tag{17}$$

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5 \ \text{(Parodi's condition)} \tag{18}$$

## One constant approx. of W:

$$W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}|\nabla\mathbf{n}|^2 \Rightarrow$$

$$\frac{\delta W}{\delta\mathbf{n}} = (\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n}), \ \frac{\partial W}{\partial(\nabla\mathbf{n})} = \nabla\mathbf{n},$$

$$\sigma^E = -\frac{\partial W}{\partial(\nabla\mathbf{n})}\cdot(\nabla\mathbf{n})^t = -\nabla\mathbf{n}\odot\nabla\mathbf{n} = -(\nabla_i\mathbf{n}\cdot\nabla_j\mathbf{n})_{1\le i,j\le d}.$$

(EL) system can be written as

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = -\nabla\cdot(\nabla\mathbf{n}\odot\nabla\mathbf{n}) + \nabla\cdot(\sigma^L(u,\mathbf{n})) \\ \nabla\cdot u = 0 \\ N + \frac{\gamma_2}{\gamma_1}A\mathbf{n} = \frac{1}{|\gamma_1|}(\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n}) + \frac{\gamma_2}{\gamma_1}(\mathbf{n}^tA\mathbf{n})\mathbf{n}. \end{cases} \tag{19}$$

Consider $\Omega = \mathbb{R}^n$. Set $\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^n}(|u|^2 + |\nabla \mathbf{n}|^2)$.

Energy law under Parodi (Lin-Liu, 99; Wang-Zhang-Zhang,13):

$$\frac{d}{dt}\mathcal{E}(t) + \int_{\mathbb{R}^n}\left[\alpha_4|\nabla u|^2 + \frac{2}{|\gamma_1|}|\Delta \mathbf{n} + |\nabla \mathbf{n}|^2\mathbf{n}|^2\right] \tag{20}$$

$$= -2\int_{\mathbb{R}^n}\left[(\alpha_1 - \frac{\gamma_2^2}{\gamma_1})|A : \mathbf{n}\otimes\mathbf{n}|^2 + (\alpha_5 + \alpha_6 + \frac{\gamma_2^2}{\gamma_1})|A\cdot\mathbf{n}|^2\right]$$

$$\leq 0,$$

provided $\alpha_1, \cdots, \alpha_6$ satisfy

$$\gamma_1 < 0,\ \alpha_1 - \frac{\gamma_2^2}{\gamma_1} \geq 0,\ \alpha_4 > 0,\ \alpha_5 + \alpha_6 \geq -\frac{\gamma_2^2}{\gamma_1}. \tag{21}$$

## Basic Questions:

(A) Establish, in dimensions $n = 2, 3$, the global existence of Leray-Hopf type weak solutions to EL (6) under general initial-boundary conditions.

(B) (Partial) regularity and uniqueness issues of suitable weak solutions of EL (6).

(C) Global or local well-posedness of the EL (6) for rough initial data belonging to the largest possible function spaces.

**Notation**. A Leray-Hopf's weak solution refers to any weak solution in the energy space that satisfies a weak form of the energy dissipation inequality (7).

**Simplified EL system** (Lin, 89):

i) Neglect stretching, rigid rotation, and interacting Leslie:
$\sigma^L(u, \mathbf{n}) = \alpha_4 A \quad N = \mathbf{n}_t + u \cdot \nabla \mathbf{n}.$

ii) $\gamma_1 = -1, \ \gamma_2 = 0.$

$$\begin{cases} u_t + u \cdot \nabla u - \alpha_4 \triangle u + \nabla P = -\nabla \cdot (\nabla \mathbf{n} \odot \nabla \mathbf{n}) \\ \nabla \cdot u = 0 \\ \mathbf{n}_t + u \cdot \nabla \mathbf{n} = \underbrace{\Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n}}. \end{cases} \tag{22}$$

**Remark**. (26) is a strongly coupling system between Navier-Stokes equation (NSE) and harmonic heat flow:

i) $\mathbf{n}$ constant $\Rightarrow$ NSE.

ii) $u = 0 \Rightarrow$ harmonic heat flow to $\mathbb{S}^2$:

$$\mathbf{n}_t = \triangle \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n}, \ \text{and} \ \underbrace{\Delta \mathbf{n} \cdot \nabla \mathbf{n} = 0} \ \text{(stationarity)}.$$

Sufficiently regular (e.g. strong) solutions $(u, \mathbf{n})$ to (26) enjoy

**Energy Dissipation Inequality**

$$\frac{d}{dt}\underbrace{\int_\Omega (|u|^2 + |\nabla \mathbf{n}|^2)} \le -2\int_\Omega (\mu|\nabla u|^2 + |\Delta \mathbf{n} + |\nabla \mathbf{n}|^2\mathbf{n}|^2) \quad (23)$$

Let $u : \Omega \times [0, T) \to \mathbb{R}^n$, $\mathbf{n}, \mathbf{m} : \Omega \times [0, T) \to \mathbb{S}^2$, with $\mathbf{n} \cdot \mathbf{m} = 0$ in $\Omega \times [0, T)$, satisfy the simplified hydrodynamic equation of biaxial nematic liquid crystals:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P - \Delta u \\ = -\mathrm{div}\big(\nabla \mathbf{n} \odot \nabla \mathbf{n} + \nabla \mathbf{m} \odot \nabla \mathbf{m}\big), \\ \mathrm{div} u = 0, \\ \mathbf{n}_t + u \cdot \nabla \mathbf{n} - \Delta \mathbf{n} = |\nabla \mathbf{n}|^2 \mathbf{n} + (\nabla \mathbf{n} \cdot \nabla \mathbf{m})\mathbf{m}, \\ \mathbf{m}_t + u \cdot \nabla \mathbf{m} - \Delta \mathbf{m} = |\nabla \mathbf{m}|^2 \mathbf{m} + (\nabla \mathbf{n} \cdot \nabla \mathbf{m})\mathbf{n}. \end{cases} \tag{24}$$

Here $P : \Omega \times [0, T) \to \mathbb{R}$ represents a pressure function.

**Lemma 5.** *Assume* $(u, \mathbf{n}, \mathbf{m})$, *with* $|\mathbf{n}| = |\mathbf{m}| = 1$ *and* $\mathbf{n} \cdot \mathbf{m} = 0$ *in* $\Omega \times (0, T)$, *are smooth solutions to (28) along with*

$$(u, \mathbf{n}, \mathbf{m}) = (0, \mathbf{n}_0, \mathbf{m}_0) \text{ on } \partial\Omega \times (0, T). \quad (25)$$

*Then*

$$\frac{d}{dt}E(u, \mathbf{n}, \mathbf{m}) = \frac{d}{dt}\frac{1}{2}\int_\Omega (|u|^2 + |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2)$$
$$= -\int_\Omega \left(|\mathbf{n}_t + u \cdot \nabla \mathbf{n}|^2 + |\mathbf{m}_t + u \cdot \nabla \mathbf{m}|^2\right). \quad (26)$$

**Theorem 2.** *For* $u_0 \in \mathbf{H}$, $\mathbf{n}_0, \mathbf{m}_0 \in H^1(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\partial\Omega, \mathbb{S}^2)$, *with* $\mathbf{n}_0 \cdot \mathbf{m}_0 = 0$ *in* $\Omega$, *there exists a global weak solution* $u \in L^\infty(\mathbb{R}_+, \mathbf{H}) \cap L^2(\mathbb{R}_+, \mathbf{J})$ *and* $\mathbf{n}, \mathbf{m} \in L^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2))$ *to* (24), *with the initial data* $(u_0, \mathbf{n}_0, \mathbf{m}_0)$. *Moreover*
*(i) There exists* $L \in \mathbb{N}$ *depending only on* $E(u_0, \mathbf{n}_0, \mathbf{m}_0)$ *and* $0 < T_1 < ... < T_L < \infty$ *such that*

$$(u, \mathbf{n}, \mathbf{m}) \in C^\infty(\Omega \times ((0, \infty) \setminus \{T_i\}_{i=1}^L)) \cap C_\beta^{2,1}(\overline{\Omega} \times ((0, \infty) \setminus \{T_i\}_{i=1}^L)).$$

*(ii) Each singular time* $T_i$, $1 \leq i \leq L$, *can be characterized by*

$$\liminf_{t \uparrow T_i^-} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_r(x)} (|u|^2 + |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2)(y, t) \geq c.$$

(iii) $E(u_0, \mathbf{n}_0, \mathbf{m}_0) \leq c \Longrightarrow (u, \mathbf{n}, \mathbf{m}) \in C^\infty(\Omega \times (0, \infty))$.

(iv) The solution $(u, \mathbf{n}, \mathbf{m})$ is unique in the class $\mathcal{C}$ of all weak solutions $u : \Omega \times [0, \infty) \to \mathbb{R}^2$ and $(\mathbf{n}, \mathbf{m}) : [0, \infty) \to \mathcal{A}$ that satisfy:

- $E(u, \mathbf{n}, \mathbf{m})(t)$ is monotone decreasing for $t \geq 0$.
- $\nabla u,\ \mathbf{n}_t + u \cdot \nabla \mathbf{n},\ \mathbf{m}_t + u \cdot \nabla \mathbf{m} \in L^2([0, T), L^2(\Omega)),\ T > 0$.

**Remark.** 1) For Ericksen-Leslie system, previously the uniqueness is obtained under the stronger assumption on the director $\mathbf{n}$ (Lin-Wang, Xu-Zhang, Li-Titi):

$$\mathbf{n}_t, \nabla^2 \mathbf{n} \in L^2([0, T), L^2(\Omega)).$$

2) For the harmonic heat flow, the uniqueness is obtained among weak solutions whose energy is monotone decreasing by A. Freire, L. Wang, Colding-Minicozzi II.

Step 1. $\exists\ (u_0^k, \mathbf{n}_0^k, \mathbf{m}_0^k) \in C_0^{2,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2)^{\otimes 2}$, with

$$\mathrm{div} u_0^k = 0,\ \mathbf{n}_0^k \cdot \mathbf{m}_0^k = 0,\ (u_0^k, \mathbf{n}_0^k, \mathbf{m}_0^k)\big|_{\partial\Omega} = (0, \mathbf{n}_0, \mathbf{m}_0)$$

such that

$$(u_0^k, \mathbf{n}_0^k, \mathbf{m}_0^k) \to (u_0, \mathbf{n}_0, \mathbf{m}_0) \text{ in } L^2(\Omega) \times H^1(\Omega)^{\otimes 2}.$$

Step 2. $\exists T_k > 0$ and smooth solutions $(u_k, \mathbf{n}_k, \mathbf{m}_k)$ of (24) in $\Omega \times [0, T_k)$, with initial-boundary condition $(u_0^k, \mathbf{n}_0^k, \mathbf{m}_0^k)$.
This can be done by the fixed point theorem in suitably chosen spaces, except that we need to verify that the obtained solution satisfies the constraint condition $|\mathbf{n}| = |\mathbf{m}| = 1$, and $\mathbf{n} \cdot \mathbf{m} = 0$.

**Lemma.** *For* $T > 0$, *if* $\mathbf{n}_0, \mathbf{m}_0 \in C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2)$ *satisfies* $\mathbf{n}_0 \cdot \mathbf{m}_0 = 0$, *and if* $\mathbf{n}, \mathbf{m} \in C^\infty(\Omega \times [0,T), \mathbb{R}^3) \cap C^{2,1}_\alpha(\overline{\Omega} \times [0,T), \mathbb{R}^3)$ *solves*

$$\begin{cases} \mathbf{n}_t + u \cdot \nabla \mathbf{n} - \Delta \mathbf{n} = |\nabla \mathbf{n}|^2 \mathbf{n} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{m}, \\ \mathbf{m}_t + u \cdot \nabla \mathbf{m} - \Delta \mathbf{m} = |\nabla \mathbf{m}|^2 \mathbf{m} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{n} \end{cases} \tag{27}$$

*for some* $u \in C^1_0(\Omega \times (0,T), \mathbb{R}^2)$ *with* $\mathrm{div} u = 0$ *in* $\Omega$, *and*

$$(\mathbf{n}, \mathbf{m}) = (\mathbf{n}_0, \mathbf{m}_0) \text{ on } \partial_p(\Omega \times (0,T)). \tag{28}$$

*Then* $|\mathbf{n}| = |\mathbf{m}| = 1$ *and* $\mathbf{n} \cdot \mathbf{m} = 0$ *in* $\Omega \times (0,T)$.

**Proof.** It follows from a Gronwall type inequality to the quantity:

$$\phi(t) = \int_\Omega \left[ (1 - |\mathbf{n}|^2)^2 + (1 - |\mathbf{m}|^2)^2 + (\mathbf{n} \cdot \mathbf{m})^2 \right](t).$$

$$(\mathbf{n}\cdot\mathbf{m})_t + u\cdot\nabla(\mathbf{n}\cdot\mathbf{m}) - \Delta(\mathbf{n}\cdot\mathbf{m}) + 2\langle\nabla\mathbf{n},\nabla\mathbf{m}\rangle$$
$$= (|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2)\mathbf{n}\cdot\mathbf{m} + \langle\nabla\mathbf{n},\nabla\mathbf{m}\rangle(|\mathbf{n}|^2 + |\mathbf{m}|^2).$$

$$\Downarrow$$

$$\frac{d}{dt}\int_\Omega(\mathbf{n}\cdot\mathbf{m})^2 + 2\int_\Omega|\nabla(\mathbf{n}\cdot\mathbf{m})|^2$$
$$= 2\int_\Omega(|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2)(\mathbf{n}\cdot\mathbf{m})^2$$
$$+ \int_\Omega\langle\nabla\mathbf{n},\nabla\mathbf{m}\rangle(|\mathbf{n}|^2 + |\mathbf{m}|^2 - 2)(\mathbf{n}\cdot\mathbf{m})$$
$$\le C\int_\Omega\left[(|\mathbf{n}|^2-1)^2 + (|\mathbf{m}|^2-1)^2 + (\mathbf{n}\cdot\mathbf{m})^2\right].$$

$$(|\mathbf{n}|^2 - 1)_t + u \cdot \nabla(|\mathbf{n}|^2 - 1) - \Delta(|\mathbf{n}|^2 - 1)$$
$$= 2|\nabla \mathbf{n}|^2(|\mathbf{n}|^2 - 1) + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{n} \cdot \mathbf{m}.$$

$$\Downarrow$$

$$\frac{d}{dt}\int_\Omega (|\mathbf{n}|^2 - 1)^2 + \int_\Omega |\nabla(|\mathbf{n}|^2 - 1)|^2$$
$$\leq C \int_\Omega \left[(|\mathbf{n}^2 - 1)^2 + (\mathbf{n} \cdot \mathbf{m})^2\right].$$

Similarly, we have

$$\frac{d}{dt}\int_\Omega (|\mathbf{m}|^2 - 1)^2 + \int_\Omega |\nabla(|\mathbf{m}|^2 - 1)|^2$$
$$\leq C \int_\Omega \left[(|\mathbf{m}^2 - 1)^2 + (\mathbf{n} \cdot \mathbf{m})^2\right].$$

Step 3. Lower bounds on $T_k$: For $\epsilon_0 > 0$, $\exists R_0 > 0$ such that

$$\mathcal{E}(2R_0) = \sup_k \max_{x \in \overline{\Omega}} E(u_0^k, \mathbf{n}_0^k, \mathbf{m}_0^k; \Omega \cap B_{2R_0}(x)) \leq \epsilon_0^2/2.$$

Global and local energy inequalities $\Longrightarrow \exists \theta_0 \in (0,1)$ such that

$$E(u^k(t), \mathbf{n}^k(t), \mathbf{m}^k(t); \Omega \cap B_{R_0}(x)) \leq \epsilon_0^2, \ \forall t \in [0, \theta_0 R_0^2], \qquad (29)$$

for $x \in \overline{\Omega}$ and $k \geq 1$.
This, combined with the estimate on the pressure $P^k$ and Ladyzhenskaya's inequality, implies

$$\max_{x \in \overline{\Omega}} \int_{B_{R_0}(x) \times [0, \theta_0 R^2]} |u^k|^4 + |\nabla \mathbf{n}^k|^4 + |\nabla \mathbf{m}^k|^4 \leq \epsilon_0^2,$$

$$\max_{x \in \overline{\Omega}} \int_{B_{R_0}(x) \times [0, \theta_0 R^2]} |\nabla^2 \mathbf{n}^k|^2 + |\nabla^2 \mathbf{m}^k|^2 \leq C_0.$$

$$\Longrightarrow \sup_{(x,t)\in\overline{\Omega}\times[\delta,\theta_0 R_0^2]} (|u^k|+|\nabla \mathbf{n}^k|+|\nabla \mathbf{m}^k|)(x,t) \le C(\epsilon_0,\delta),\ \forall k\ge 1.$$

Step 4. Sketch of the uniqueness: Under the assumptions, we will show that $\mathbf{n}, \mathbf{m} \in L^2([0,T], H^2(\Omega))$.

**Lemma.** $\exists\ \epsilon_0 > 0$ *such that if* $(\mathbf{n}, \mathbf{m}) \in H^1(B_1, \mathbb{S}^2)^{\otimes 2}$, *with* $\mathbf{n}\cdot\mathbf{m} = 0$ *in* $B_1$, *satisfies*

$$\begin{cases} \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} = -\langle \nabla \mathbf{n}, \nabla \mathbf{m}\rangle \mathbf{n} + (\mathbf{n}_t + u\cdot\nabla \mathbf{n}), \\ \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} = -\langle \nabla \mathbf{n}, \nabla \mathbf{m}\rangle \mathbf{m} + (\mathbf{m}_t + u\cdot\nabla \mathbf{m}), \end{cases} \quad (30)$$

*and* $\int_{B_1} |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2 \le \epsilon_0^2$, *then* $(n, m) \in H^2(B_{\frac{1}{2}})$, *and*

$$\int_{B_{\frac{1}{2}}} |\nabla^2 \mathbf{n}|^2 + |\nabla^2 \mathbf{m}|^2 \lesssim \int_{B_1} |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2 + \int_{B_1} |\mathbf{n}_t + u \cdot \nabla \mathbf{n}|^2 + |\mathbf{m}_t + u \cdot \nabla \mathbf{m}|^2. \quad (31)$$

Ideas:

$$\mathrm{div}\langle \nabla \mathbf{n}, \mathbf{m}\rangle = \langle \mathbf{n}_t + u \cdot \nabla \mathbf{n}, \mathbf{m}\rangle \Longrightarrow$$

$$\Delta \mathbf{n}^i = -\mathbf{n}^j_\alpha W^{ij}_\alpha - \mathbf{m}^j_\alpha L_\alpha + (\mathbf{n}_t + u \cdot \nabla \mathbf{n})^j,$$

where $W^{ij}_\alpha$ and $L_\alpha$ satsify:

$$|\partial_\alpha W^{ij}_\alpha| \le |\mathbf{n}_t + u \cdot \nabla \mathbf{n}| + |\mathbf{m}_t + u \cdot \nabla \mathbf{m}| \in L^2(B_1),$$

$$|\partial_\alpha L_\alpha| \le |\mathbf{n}_t + u \cdot \nabla \mathbf{n}| \in L^2(B_1).$$