Solving S-unit equations in Sage and Applications to Algebraic Curves

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with Alejandra Alvarado, Angelos Koutsianas, Christopher Rasmussen, Christelle Vincent, and Mckenzie West

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A Motivating Problem

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Malmskog-Rasmussen goal: Find all Picard curves defined over $\mathbb{Q}$ with good reduction at all primes except $p = 3$.

Bőrner-Bouw-Wewers: All Picard curves over $\mathbb{Q}$ have bad reduction at $p = 3$.

Reduction Properties–Why Care?

- Ihara's question

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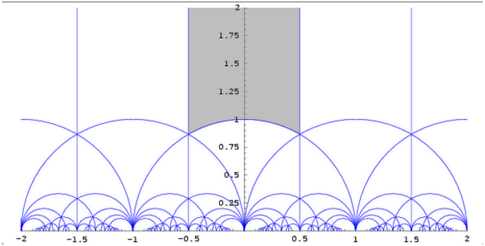

- Every quotient curve of the modular curve $\mathcal{X}_0(N)$ has good reduction except at primes dividing N.

Our Roadmap

1996 Smart–Found all genus 2 curves over $\mathbb{Q}$ with good reduction at all primes except $p = 2$.

We generalize methods, equivalence of binary forms to Picard curves.

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where $x, y \in \mathcal{O}_S^\times$, and S is a set of primes in $K/\mathbb{Q}$.
New Goal: Create self-contained functions to solve S-unit equation.

S-units

$$\begin{array}{ccccc} K & \mathbb{Z}_K & S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{t_1}, \infty_1, \ldots, \infty_{t_2}\} & \mathcal{O}_S = \mathbb{Z}_K[1/\mathfrak{p}_1, \ldots, 1/\mathfrak{p}_{t_1}] & \mathcal{O}_S^* \\ | & | & | & | & | \\ \mathbb{Q} & \mathbb{Z} & S_{\mathbb{Q}} = \{p_1, \ldots, p_s, \infty\} & \mathcal{O}_{S_{\mathbb{Q}}} = \mathbb{Z}[1/p_1, \ldots, 1/p_s] & \mathcal{O}_{S_{\mathbb{Q}}}^{\times} \end{array}$$

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$S_{\mathbb{Q}} = \{3, \infty\}$,

$$\mathcal{O}_{S_{\mathbb{Q}}}^\times = \left\{\ldots, \pm\frac{1}{9}, \pm\frac{1}{3}, \pm 1, \pm 3, \pm 9, \ldots\right\}$$

$$= \left\{(-1)^{a_1} 3^{a_2} : (a_1, a_2) \in \mathbb{Z}^2\right\}.$$

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Let $K = \mathbb{Q}(\xi)$, where $\xi^6 + 3 = 0$, so $(3) = (\xi)^6$.

$S = \{(\xi), \infty_1, \ldots, \infty_4\}$.

$$\mathcal{O}_S^\times = \left\{ \zeta_6^{a_1} \xi^{a_2} (\frac{1}{2}\xi^5 - \frac{1}{2}\xi^2 - \xi - 1)^{a_3} (\frac{1}{2}\xi^4 - \frac{1}{2}\xi^3 + \xi^2 - \frac{1}{2}\xi + \frac{1}{2})^{a_4} : (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \right\}.$$

Solving the S-Unit Equation

- 1939 Dirichlet–S-unit group is finitely generated (rank $r + s$).
- 1909-1921-1955 Thue, Siegel, Roth–There are finitely many rational numbers of bounded height within a given distance of an irrational algebraic number.
- 1966 Baker–Lower bound on linear combination of logarithms of algebraic α_i based on heights of coefficients and α_is.
- 1972-1979 Győry–Explicit bound, using Baker's method.
- 1987-1992 de Weger, Tzanakis-de Weger–Use LLL to greatly reduce bounds
- 1989 Yu–Linear forms in p-adic logarithms
- 1996-1999 Wildanger, Smart–Efficient enumeration of solutions

Big Picture

Let $\mathcal{O}_S = \langle \rho_0, \ldots, \rho_t \rangle$, where ρ_0 is a root of unity. To solve

$$x + y = 1,$$

where $x = \prod \rho_i^{a_{i,x}}$, $y = \prod \rho_i^{a_{i,y}}$, need to bound exponents and search over finite space. Three main steps:

1. Find a ridiculously large bound
2. Use LLL to greatly reduce bound
3. Somehow find all solutions in smaller search space. For us, this means sieving.

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Caveat: Need to consider prime associated to minimum absolute value of term with maximum exponent...

Step 1: A Closer Look at Baker's Theorem

Theorem (Baker-Wüstholz, 1993)

Let L be a linear form in $t+1$ variables, and let $\rho_0, \ldots, \rho_t \in \overline{\mathbb{Q}} - \{0,1\}$ with linearly independent logs. Let B be the subfield of $\overline{\mathbb{Q}}$ generated by the ρ_i. If

$$\Lambda = L(\log \rho_0, \log \rho_1, \ldots, \log \rho_t) \neq 0,$$

then

$$\log |\Lambda| > -C(t, n_B) h'(L) \prod_{j=0}^{t} h'(\rho_j),$$

where the constant $C(t, n_B)$ is defined by

$$C(t, n_B) = 18(t+2)!(t+1)^{(t+2)}(32 n_B)^{(t+3)} \log\left(2(t+1)n_B\right).$$

A Simpler Look at Baker and S-Unit Solutions

Assume that

Baker-Wüstholz: If L is a linear form, $\Lambda = L(\log \rho_0, \log \rho_1, \ldots, \log \rho_t) \neq 0$, then

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Rewrite our S-unit equation:

$$x + y = 1 \Rightarrow \frac{x}{y} = \frac{1}{y} - 1 \neq 1, \text{ so}$$

$$\begin{aligned} \prod_{i=0}^{t} \rho_i^{a_i} &= \frac{1}{y} - y \neq 1. \\ \sum_{i=0}^{t} a_i \log(\rho_i) &= \Lambda \neq 0, \end{aligned}$$

where α_i are S-units generators, a_i are exponents. We want to bound a_i.

A Large Bound

Fix $\psi \colon K \hookrightarrow \mathbb{C}$. Let $H = \max\{|a_i| : 0 \leq i \leq t\}$

Ignoring all details:

$$h'(L) > C_2 \log(H) \text{ and } C_3 = \prod_{i=0}^{t} h'(\alpha_i).$$

Baker-Wustholz:

$$\log|\Lambda| > -C_1 C_2 \log(H) C_3$$
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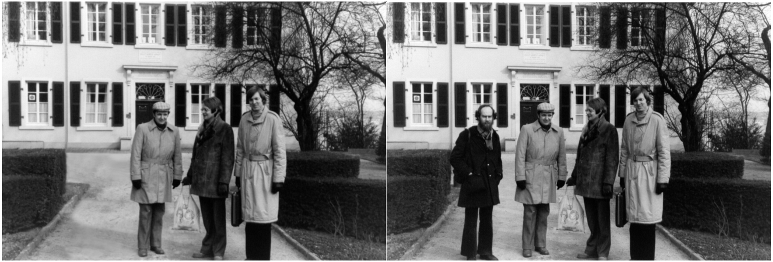
Problem: For one of our fields, $K_0 = 2.137374 \times 10^{19}$.

Step 2: LLL



LLL: lattice basis reduction algorithm devised in 1982 by Henrik Lenstra, Arjen Lenstra, and Laslo Lovász.

Applying LLL reduction to a particular lattice yields a bound $K_1 \approx \log(K_0)$. Can be repeated with new bound until there is no further improvement.

LLL in Action: Picard Curves

Need all $K/\mathbb{Q}$ with degree ≤ 4 and $\text{Disc}(K) \in \mathcal{O}_S^\times$ with $S = \{3, \infty\}$.

Field	Degree	Minimal Polynomial	K_0	K_1
M_0	1	$x - 1$	4.916825×10^9	3
M_1	2	$x^2 + x + 1$	8.018712×10^9	5
M_2	3	$x^3 - 3x + 1$	2.067269×10^{19}	217
M_3	3	$x^3 - 3$	1.957261×10^{15}	49
M_3'	3			
M_3''	3			
L_3	6	$x^6 + 3$	2.137374×10^{19}	243

All fields have class number 1.

(3) is totally ramified in all (non-trivial) extensions.

Step 3: Sieving for Solutions

A sieve:

Recall $\mathcal{O}_S^\times = \langle \rho_0, \ldots, \rho_t \rangle$, where ρ_0 is a root of unity. Say that

$$x + y = 1,$$

where

$$x = \prod \rho_i^{a_{i,x}} = \rho^{\mathbf{a_x}}, \qquad y = \prod \rho_i^{a_{i,y}} = \rho^{\mathbf{a_y}}.$$

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Let q be a prime of $\mathbb{Q}$ which splits completely in K, so

$$q\mathcal{O}_K = \mathfrak{q}_0 \ldots \mathfrak{q}_{n-1}.$$

We now consider the image of the equation $x + y = 1$ modulo $\mathfrak{q}_j$ for each j, $0 \leq j \leq n-1$, where $\overline{\alpha}$ denotes the reduction modulo $\mathfrak{q}_j$. Let

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Then we have

$$\overline{\rho}^{\mathbf{a_x}} + \overline{\rho}^{\mathbf{a_y}} = \mathbf{1}$$

for all j, which gives a set of conditions on $\mathbf{a_x}$ and $\mathbf{a_y}$ modulo $q-1$.

Sieve continued

Choosing a list of split primes $q_1, q_2, \ldots q_N$ so that

$$\mathrm{lcm}(q_1, q_2, \ldots q_N) > 2K_1,$$

can use Chinese remainder-type argument to find searchable space of potential exponent vectors in $\mathbb{Z}^{t+1}$.

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Finally, check whether each exponent vector yields an actual S-unit solution.

Note: This is not the same method introduced by Wildanger and improved by Smart.

Results and Beyond

Picard curves: Implementing the above routines in Sage, we solved the S-unit equation in the above-listed fields, yielding 63 $\mathbb{Q}$-isomorphism classes of Picard curves with good reduction away from $p = 3$.

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2015/2016: Angelos Koutsianas: all elliptic curves with good reduction outside S defined over a general number field.

Note: Koutsianas also implemented S-unit solving in Sage, including both cases of LLL but avoiding sieve.

General Sage Implementation

Collaborate@ICERM January 2017

Team: Alejandra Alvarado, Angelos Koutsianas, M., Chris Rasmussen, Christelle Vincent, Mckenzie West (with moral support from Bjorn Poonen)

Implemented function to solve $x + y = 1$ for general number field K and set S.

SageTrac Ticket #22148

Computational Comparison

Smart, 1997 (paraphrased)

- The algorithm was implemented on a network of 20 SUN workstations, written in C++. Issues with load balancing and computer failure had to be navigated.
- The program took around 27 MIPS-years, or in real life, about 18 days.

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Alvarado-Koutsianas-M.-Rasmussen-Vincent-West, 2017

- General solver is approximately 3000 lines of Sage code.
- Some problems run in seconds, others in minutes, others...

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  - Incorporate bound improvements from literature
  - More general linear equations
  - Improve bound reduction using de Weger
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  - ...

Extra: More on LLL and Reducing S-Unit Exponent Bound

For $\psi_h : K \hookrightarrow \mathbb{C}$

$$\Lambda = \sum_{j=0}^{t} a_j \log(\rho_j) = \sum_{j=0}^{t} a_j \kappa_j, \tag{1}$$

where ρ_j are the generators of $\mathcal{O}_S^\times$, $\rho_0 \in \mu_w$.

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where ρ_j are the generators of $\mathcal{O}_S^\times$, $\rho_0 \in \mu_w$.
Choose $C \approx 2^{t/2}$. Define

$$\Phi_0 := \sum_{j=1}^{t} a_j [C\, \Re \kappa_j],$$

$$\Phi_1 := \sum_{j=1}^{t} a_j [C\, \Im \kappa_j] + a_0 [C \cdot \tfrac{2\pi}{w}].$$

so

$$|\Phi_0 + \sqrt{-1}\Phi_i| \leq C|\Lambda| + \frac{1}{\sqrt{2}}(2t+1)K_0$$

since $a_i \leq K_0$ for all i.

Lattice

$$\mathcal{B} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ [C\,\Re\kappa_1] & [C\,\Re\kappa_2] & \cdots & [C\,\Re\kappa_{t-1}] & [C\,\Re\kappa_t] & 0 \\ [C\,\Im\kappa_1] & [C\,\Im\kappa_2] & \cdots & [C\,\Im\kappa_{t-1}] & [C\,\Im\kappa_t] & [C\cdot\frac{2\pi}{w}] \end{pmatrix}.$$

Let $\mathcal{L} = L(\mathcal{B}^T)$. Then $\mathbf{a} = (a_1, a_2, \dots, a_{t-1}, \Phi_0, \Phi_1) \in \mathcal{L}$.

Reduction

$$\mathbf{a} = (a_1, a_2, \ldots, a_{t-1}, \Phi_0, \Phi_1) \in \mathcal{L}$$

The Euclidean length of any nonzero lattice element in $\mathcal{L}$ is bounded below by $B := 2^{-t/2}\|\mathbf{b}_1\|$, where $\mathbf{b}_1$ is the shortest vector in the LLL-reduced basis for $\mathcal{L}$.

$$B^2 \leq |\mathbf{a}|^2 = \sum_{i=1}^{t-1} a_i^2 + \Phi_0^2 + \Phi_1^2 \leq C^2|\Lambda|^2 + \frac{1}{2}(2t+1)^2 K_0^2$$

Smart: $|\Lambda| < C_0 e^{-C_1 H}$, by a geometric argument. *holds for some embedding–have to calculate constants for all and take worst constant

$$B^2 \leq C^2(C_0 e^{-C_1 H})^2 + \frac{1}{2}(2t+1)^2 K_0^2$$

Define

$$S_{\mathcal{L}} := \left(B^2 - (t-1)K_0^2\right)^{1/2}, \qquad T_{\mathcal{L}} := \frac{1}{2}\left(w + 2 + \sqrt{2}\right) tK_0.$$

If $B^2 > T_{\mathcal{L}}^2 + (t-1)K_0^2$, then every solution to the S-unit equation satisfies

$$H \leq K_1 := C_6\left(\log(CC_4) - \log(S_{\mathcal{L}} - T_{\mathcal{L}})\right).$$

$$K_1 \sim \log(K_0)$$